Def. 2.1.5 (Ruin, ruin time, ruin probability)

(i) The event

$$\text{Ruin} := \{U(t) < 0 \text{ for some } t > 0\}$$

is called *ruin*

→ event that U ever falls below 0

(ii) The r.v.

$$T := \inf\{t > 0 : U(t) < 0\} \in [0, \infty] \quad (\inf \emptyset \overset{\text{def}}{=} \infty)$$

is called *ruin time*

→ time when ruin occurs

(iii) The *probability of ruin* is def. as

$$\Psi(u) := \underbrace{P(\text{Ruin} \mid U(0) = u)}_{= P(\text{Ruin})} = P(T < \infty)$$

Note that

$$\text{Ruin} = \bigcup_{t \geq 0} \{U(t) < 0\} = \{\inf_{t \geq 0} U(t) < 0\}$$

$$= \{T < \infty\}$$

→ *main objective* of insurance business is to keep $\Psi(u)$ small for sufficiently large initial capital u

worst case →

Th. 2.1.6 (Ruin with prob. 1)

Let $E[W_1], E[X_1] < \infty$ and

$$E[X_1] - c \cdot E[W_1] \geq 0 \qquad (2.1.2)$$

Then

$$P(\text{Ruin} \mid U(0) = u) = 1$$

for all $u > 0$.

Ex. 2.1.7 : Recall the premium $p(t)$ based on the equivalence principle, that is

$$p(t) = p_{Net}(t) = E[S(t)]$$

$N(t)$ indep. of the i.i.d. X_i, $i \geq 1$

$$\Rightarrow \quad E[S(t)] = E[N(t)] \cdot E[X_1] \qquad \text{(see exerc.)}$$

(19)

Suppose $N(t)$ hom. Poiss. proc. with intensity $\lambda > 0$,

then $E[S(t)] = \lambda \cdot t \cdot E[X_1] = \frac{E[X_1]}{E[W_1]} \cdot t$,

since $E[W_1] = \frac{1}{\lambda}$ (see (2.1.1)).

$\Rightarrow$ $c = \frac{E[X_1]}{E[W_1]}$ $\Rightarrow$ (2.1.2) $= 0$ $\Rightarrow$

P_{Net} leads to ruin for all $u > 0$

$\Rightarrow$ P_{Net} unwise calculation principle

Proof of Th. 2.1.6 : idea : SLLN

Rewrite

$$\text{Ruin} = \left\{ \inf_{t>0} U(t) < 0 \right\} \underset{\text{on } [T_n, T_{n+1})}{\overset{U(t)\uparrow}{=}} \left\{ \inf_{n \geq 0} U(T_n) < 0 \right\}$$

$$\overset{\text{def. of } U}{=} \left\{ \inf_{n \geq 1} [u + p(T_n) - S(T_n)] < 0 \right\} \overset{S(t) = \sum_{i \geq 1} X_i 1_{[0,t]}(T_i)}{=}$$

$$= \left\{ \inf_{n \geq 1} [u + \underbrace{(c \cdot T_n - \sum_{i=1}^{n} X_i)}_{= -(Z_1 + \ldots + Z_n) =: -S_n}] < 0 \right\}$$

with $Z_i := X_i - c \cdot W_i$ (since $T_n = \sum_{i=1}^{n} W_i$)

$$\Rightarrow \quad \psi(u) = P\left(\inf_{n \geq 1} (-S_n) < -u \right) = P\left(\sup_{n \geq 1} S_n > u \right)$$

$Z_i, i \geq 1$ i.i.d.

SLLN

$$\Rightarrow \quad \frac{S_n}{n} = \frac{Z_1 + \ldots + Z_n}{n} \xrightarrow[n \to \infty]{} E[Z_1] = E[X_1] - c E[W_1] \overset{(2.1.3)}{\geq} 0$$

1. case : $E[Z_1] > 0 \overset{(2.1.3)}{\Rightarrow} S_n \xrightarrow[n \to \infty]{} \infty$ with prob. 1

$$\Rightarrow \quad \psi(u) = P\left(\sup_{n \geq 1} S_n > u \right) = 1 \quad \forall u$$

2. case: $E[Z_1] = 0$. See e.g. Spitzer (1976): Principles of Random Walk

$\Rightarrow$ proof.

What happens if $E[X_1] - cE[W_1] < 0$?

→ ruin may still occur, but not necessarily with prob. 1

→ Def. 2.1.7 (Net profit condition)

$S(t)$ satisfies the net profit condition (NPC) if

$$E[X_1] - cE[W_1] < 0 \qquad (2.1.4)$$

2.2. Lundberg's inequality

Assumption: $S(t)$ is described by the renewal model (Def. 2.1.1) s.t. NPC holds (Def. 2.1.7)

as well as

the small claim condition is satisfied, that is

$$m_{X_1}(h) := E[e^{h \cdot X_1}] < \infty, \; h \in (-h_0, h_0) \text{ for some } h_0 > 0$$

(moment generating function) (2.2.1)

→ Markov's inequality for $h \in (0, h_0)$:

$$P(X_1 > x) \leq e^{-hx} m_{X_1}(h) \text{ for all } x > 0$$

⇒ tail $P(X_1 > x)$ decays to 0 exponentially fast

→ not realistic in many applications where claims are rather heavy-tailed

Def. 2.2.1 (Adjustment or Lundberg coefficient)

Suppose that $m_{Z_1}(h) < \infty$ on $(-h_0, h_0)$, $h_0 > 0$ for $Z_1 := X_1 - cW_1$ and that there ex. a unique $r > 0$ s.t.

$$m_{Z_1}(r) = E[e^{r(X_1 - cW_1)}] = 1 \qquad (2.2.2)$$

Then r is called the adjustment or Lundberg coefficient.

Rem. 2.2.2: Sufficient condition for (2.2.2):

$\exists \; 0 < h_1 \leq \infty : m_{Z_1}(h) < \infty$ for $h < h_1 \wedge \lim_{h \uparrow h_1} m_{Z_1}(h) = \infty$

(see exercises)

Th. 2.2.3 (Lundberg inequality)
Consider the renewal model with NPC (Def. 2.1.7). Suppose that the adjustment coefficient r in Def. 2.2.1 exists. Then for all $u > 0$:

$$\psi(u) \leq e^{-ru} \qquad (2.2.3)$$

Rem. 2.2.4: The inequality (2.2.3) implies that large initial capital u (depending on the magnitude of the adjustment coeff.) leads to a small ruin probability.
$\longrightarrow$ Roughly speaking: Principally, there will be no danger of ruin, if the small claim condition holds and if there is large initial capital.

Proof of Th. 2.2.3: Proof by induction. $S_k \overset{\text{def}}{=} Z_1 + \ldots + Z_k$

Def. $\psi_n(u) = P(\max_{1 \leq k \leq n} S_k > u) = P(\exists k \in \{1, \ldots, n\} : S_k > u)$

We know from the proof of Th. 2.1.6 that

$$\psi(u) = P(\exists k \geq 1 : S_k > u)$$

property (iii) of prob. meas. $\Longrightarrow$

$$\psi_n(u) \uparrow \psi(u) \text{ for } n \longrightarrow \infty$$

$\Longrightarrow$ sufficient to show that

$$\psi_n(u) \leq e^{-r \cdot u} \text{ for all } n \geq 1 \text{ and } u > 0 \qquad (2.2.4)$$

$n = 1$:

$$\psi_1(u) = P(Z_1 > u) \overset{\text{Markov ineq. for } f(y) = e^{ry}}{\leq} e^{-ru} \underbrace{M_{Z_1}(r)}_{\overset{(2.2.2)}{=} 1} = e^{-ru}$$

Assume that (2.2.4) holds for $n = k \geq 1$. Then

$$\psi_{k+1}(u) = P(\max_{1 \leq n \leq k+1} S_n > u) = \underbrace{P(Z_1 > u, \max_{1 \leq n \leq k+1} S_n > u)}_{= P(Z_1 > u)}$$
$$+ P(Z_1 \leq u, \max_{1 \leq n \leq k+1} S_n > u)$$
$$= P(Z_1 > u) + P(\max_{1 \leq n \leq k+1} \underbrace{S_n}_{= Z_1 + (S_n - Z_1)} > u, Z_1 \leq u)$$

(22) $= P(Z_1 > u) + P\Big(\max_{2 \le n \le \kappa+1} (Z_1 + (S_n - Z_1)) > u \,,\, Z_1 \le u \Big)$ (*)

w.l.o.g. let $\kappa = 1$. Then $S_n - Z_1 = Z_2$

$$(*) = P(Z_1 > u) + P(Z_1 + (S_n - Z_1) > u \,,\, Z_1 \le u)$$

Z_i i.i.d. dobbel-forventning $\quad \underbrace{P(Z_1 > u)}_{=: P_1} + \underbrace{E\big[P(x + Z_2 > u \,,\, x \le u)\big|_{x=Z_1} \big]}_{=: P_2}$

As for P_2 we see that

$$P_2 = E\big[P(Z_2 > u - x \,,\, x \le u)\big|_{x=Z_1} \big]$$

$$= E\Big[\mathbb{1}_{(-\infty,u]}(x) \cdot \underbrace{P(Z_2 > u-x)}_{\overset{\text{def}}{=} P(Z_1 > u-x) \overset{\text{def}}{=} \psi_1(u-x)} \Big|_{x=Z_1} \Big] \overset{\text{induction assumption for } n=\kappa=1}{\le}$$

$$\le E\big[\mathbb{1}_{(-\infty,u]}(x)\, e^{r(u-x)}\big|_{x=Z_1} \big] = E\big[\mathbb{1}_{(-\infty,u]}(Z_1)\, e^{r(Z_1-u)} \big] \quad (**)$$

On the other hand, $P(A) = E[\mathbb{1}_A]$,

$$P_1 \overset{\text{def}}{=} P(Z_1 > u) \overset{A = \{Z_1 > u\}}{=} E\big[\mathbb{1}_{(u,\infty)}(Z_1) \big]$$

$$\le E\Big[\mathbb{1}_{(u,\infty)}(Z_1)\, \underbrace{e^{r(Z_1-u)}}_{\ge 0} \Big] \le E\big[\mathbb{1}_{(u,\infty)}(Z_1)\, e^{r(Z_1-u)} \big]$$

$(**), (*) \Rightarrow$

$$\psi_2(u) = P_1 + P_2 = E\Big[\underbrace{\big(\mathbb{1}_{(-\infty,u]}(Z_1) + \mathbb{1}_{(u,\infty)}(Z_1) \big)}_{=1} e^{r(Z_1-u)} \Big]$$

$$= e^{-ru}\, \underbrace{E\big[e^{rZ_1} \big]}_{= m_{Z_1}(r) = 1} = e^{-ru} \Rightarrow \text{ind. for } n = \kappa+1 = 2$$

$\Rightarrow$ proof.

2.3 Exact asymptotics for $\psi(u)$ for small claims

→ objective: bounds for $\psi(u)$ from above and below in the case of the Cramér-Lundberg model, that is in the case, when $N(t), t \geq 0$ in the renewal model is described by a hom. Poisson proc. with an intensity $\lambda > 0$.

→ one of the most important results in risk theory:

Th. 2.3.1 ((Cramér's ruin bound)
Consider the Cramér-Lundberg model ((0.1)
Suppose that the following conditions are satisfied:

(i) NPC in (2.1.4)

(ii) The claim size distr. F_{X_1} has a prob. density.

(iii) $m_{X_1}(h) \overset{\text{def.}}{=} E[e^{hX_1}] < \infty$, $h \in (-h_0, h_0)$.

(iv) The adjustment coefficient τ (Def. 2.2.1) exists in $(0, h_0)$.

Then there exists a $C > 0$ s.t.

$$\lim_{u \to \infty} e^{\tau u} \psi(u) = C.$$

Moreover, the constant C is given by

$$C = \left[\frac{\tau}{\rho E[X_1]} \int_0^\infty x e^{\tau x} P(X_1 > x)\, dx \right]^{-1},$$

where $\rho > 0$ is the safety loading w.r.t. a premium income defined by

$$p(t) = (1+\rho) E[S(t)] = \underbrace{(1+\rho)\frac{E[X_1]}{E[W_1]}}_{= c \text{ premium rate}} \cdot t$$

↑ satisfies the NPC

Before we proceed with the proof of Th. 2.3.1 we need some notions and auxiliary results

Recall a measure m on a σ-algebra $\mathcal{F}$ is a function

$$m : \mathcal{F} \longrightarrow [0,\infty] \quad \text{s.t.}$$

(i) $m(\emptyset) = 0$

(ii) σ-additivity: $A_i \in \mathcal{F}$ for all $i \geq 1$, $A_i \cap A_j = \emptyset$ for $i \neq j$

Then

$$m\left(\bigcup_{i\geq 1} A_i\right) = \sum_{i\geq 1} m(A_i)$$

Ex. 2.3.2 : $m = P$ prob. meas.

Ex. 2.3.3 : Let $V : \mathbb{R} \longrightarrow [0,\infty)$ be a non-decreasing and right-continuous function. Then there exists a unique measure m on $\mathcal{B}(\mathbb{R})$ (Borel-σ-algebra) s.t.

$$m((a,b]) = V(b) - V(a)$$

for all $a < b$.

$\longrightarrow$ Ex: $V(t) = t \Rightarrow \lambda \overset{\text{def.}}{=} \mu$ Lebesgue-Borel-meas.

Ex. 2.3.4 : $V = F_X$, F_X distr. function of a r.v. X

$\Rightarrow$ V non-decreasing and right-continuous $\overset{\text{Ex. 2.3.3}}{\Longrightarrow}$

$\exists$ unique (probability) measure m s.t.

$$m((a,b]) = F_X(b) - F_X(a) \quad \text{for all } a < b$$

Ex. 2.3.5 (Renewal measure)

$V(t) = E[N(t)] + 1$, $N(t)$, $t \geq 0$ renewal process with $0 < E[W_1] = \frac{1}{\lambda} < \infty \Rightarrow$ V non-decreasing and right-continuous $\overset{\text{Ex. 2.3.3}}{\Longrightarrow}$

$\exists$ unique measure m on $[0,\infty) \cap \mathcal{B}(\mathbb{R}) \overset{\text{def}}{=} \{A : A = [0,\infty) \cap B \text{ for a } B \in \mathcal{B}(\mathbb{R})\}$ s.t.

$$m((a,b]) = V(b) - V(a) \quad \text{for all } 0 \leq a < b$$

$\longrightarrow$ m renewal measure

Rem. 2.3.6 (Integral w.r.t. m)
Just as in Def. 2.9 one can define an integral w.r.t. m

$$\int_\Omega g(\omega)\, m(d\omega)$$

for $\mathcal{F}$-measurable functions $g : \Omega \to \mathbb{R}$, i.e. for functions g s.t. $\{\omega \in \Omega : g(\omega) < \alpha\} \in \mathcal{F}$ for all $\alpha \in \mathbb{R}$.

Ex 2.3.7 (Lebesgue-Stieltjes integral)
$V \geq 0$, right-continuous, non-decreasing, g $\mathcal{F}$-measurable
$\to$ Lebesgue-Stieltjes integral (L-S.-int.): (from Ex. 2.3.3)

$$\int_{-\infty}^{\infty} g(s)\, dV(s) \overset{\text{def.}}{=} \int_\Omega g(\omega)\, m(d\omega),$$

if the integral exists ($\Omega \overset{\text{def}}{=} (-\infty, \infty)$, $\mathcal{F} \overset{\text{def}}{=} \mathcal{B}(\mathbb{R})$)

Ex. 2.3.8 : $V(t) = \int_{-\infty}^{t} f(u)\, du$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$, f, g $\mathcal{F}$-meas., $f \geq 0$. Then

$$\int_{-\infty}^{\infty} g(s)\, dV(s) = \int_{-\infty}^{\infty} g(s) f(s)\, ds,$$

if the last integral exists.

Ex. 2.3.9 :

$$E[f(X)] \overset{\text{def}}{=} \int_\Omega f(X(\omega))\, P(d\omega)$$

$$= \int_{\mathbb{R}} f(y)\, dF_X(y)$$

We need the following central result from renewal theory:

Th. 2.3.10 (Smith's key renewal theorem)

Let $\lambda^{-1} = E[W_1] < \infty$ and $V(t) = E[N(t)] + 1$ the renewal function. Assume that $F \overset{def}{=} F_{W_1}$ has a prob. density.

(i) If $h \geq 0$, non-increasing with $\int_0^\infty h(s)ds < \infty$ or the difference of such functions. Then

$$\lim_{t\to\infty} \underbrace{\int_0^t h(t-x)\,dV(x)}_{\text{L-S-int. (Def. 2.3.7)}} = \lambda \cdot \int_0^\infty h(x)\,dx$$

(ii) Consider the equation (Ex. 2.3.8)

$$g(t) = h(t) + \int_0^t g(t-x)\,dF(x), \quad t \geq 0,$$

(renewal equation)

where h is bounded on every finite interval. Then the unique solution (in the class of bounded funct. on finite intervals) is

$$g(t) = \int_0^t h(t-x)\,dV(x).$$

So, if in addition, $h \geq 0$, non-decreasing and $\int_0^\infty h(s)ds < \infty$, then

$$\lim_{t\to\infty} g(t) = \lambda \int_0^\infty h(x)\,dx$$

Proof: See Asmussen, S.: Applied Probability and Queues (1987).

In the sequel define

$$\varphi(u) = 1 - \psi(u) \quad \text{(non-ruin probability)}$$

and

$$\overline{F}_X = 1 - F_X \quad \text{(tail distribution)}$$

for r.v's X.

Lemma 2.3.11 (Fundamental integral eq. for non-ruin probability)

Assume the Cramér-Lundberg model for $S(t)$ s.t. NPC and $E[X_1] < \infty$ hold. Further, suppose that F_{X_1} has a prob density.

(27) Then the non-ruin probability $\varphi(u)$ satisfies the integral eq.

$$\varphi(u) = \varphi(0) + \frac{1}{(1+\delta)E[X_1]} \int_0^u \underbrace{F_{X_1}(y)}_{=P(X_1 \le y)} \varphi(u-y)\,dy \quad (2.3.1)$$

Rem: Recall that $P(t) = (1+\delta)E[S(t)]$, δ safety loading

Rem 2.3.12: $\varphi(0) = \delta/(1+\delta)$ (see exercises)

Proof of L. 2.3.11: Recall that

$$\psi(u) = P(\sup_{n\ge 1} S_n > u) \overset{\text{def.}}{=} 1 - \varphi(u)$$

for $S_n = \sum_{i=1}^n Z_i$, $Z_i \overset{\text{def}}{=} X_i - cW_i$ i.i.d. (see the proof of Th. 2.1.6)

$$\Rightarrow \varphi(u) = P(\sup_{n\ge 1} S_n \le u) = P(S_n \le u \text{ for all } (n \ge 1)) \quad (*)$$

$$\overset{=S_1}{=} P(Z_1 \le u,\ S_n - Z_1 \le u - Z_1 \text{ for all } (n \ge 2))$$

$P(A) = E[1_A]$, $1_{A\cap B} = 1_A \cdot 1_B$, $X_1 - cW_1 = Z_1$

$$= E\left[1_{\{Z_1 \le u\}}\, 1_{\{S_n - Z_1 \le u - Z_1\ \forall n\ge 2\}}\right]$$

(X_1, W_1) indep. of $S_n - Z_1, n \ge 2$

$$= E_{X_1,W_1}\left[E\left[1_{\{X_1 - cW_1 \le u\}} \cdot 1_{\{S_n - Z_1 \le u - Z_1\ \forall n\ge 2\}}\right]\right]$$

dobbel-forventning

$$= E\left[\left\{1_{\{x - cw \le u\}} E\left[1_{\{S_n - Z_1 \le u - x + cw\ \forall n\ge 2\}}\right]\right\}\Big|_{\substack{x = X_1 \\ w = W_1}}\right]$$

dobbel-forventning, Ex. 2.3.9

$$= \int_0^\infty \int_0^\infty \underbrace{1_{\{x - cw \le u\}}}_{= 1_{\{x \le u + cw\}}} P(S_n - Z_1 \le u - x + cw\ \forall n \ge 2) \underbrace{\lambda e^{-\lambda w} dw}_{= dF_{W_1}(w)} \cdot dF_{X_1}(x)$$

$\sum_{i=2}^n Z_i \overset{d}{=} \sum_{j=1}^{n-1} Z_j = S_{n-1}$

Ex. 2.3.8

$$= \int_0^\infty \int_0^{u+cw} \underbrace{P(S_n - Z_1 \le u - x + cw\ \forall n \ge 2)}_{\text{prop. (iii) of prob. meas., } Z_i, i\ge 1 \text{ i.i.d}} dF_{X_1}(x)\, \lambda e^{-\lambda w} dw$$

$$\overset{(*)}{=} \varphi(u - x + cw)$$

$z = u + cw$

$$\Rightarrow \varphi(u) = \frac{\lambda}{c} e^{u\lambda/c} \int_u^\infty e^{-\lambda z/c} \underbrace{\int_0^z \varphi(z-x)\, dF_{X_1}(x)}_{\overset{\text{def}}{=} g(z)}\, dz \quad (**)$$

(28)

$\Longrightarrow$ g continuous

(**) $\Longrightarrow$ F_{X_1} density / g continuous

fund. th. of calculus $\Longrightarrow$ $\int_u^\infty e^{-\lambda z/c} g(z)\,dz$ diff. in u

(**) $\Longrightarrow$ prod. rule g diff. with

$$g'(u) = \frac{\lambda}{c} g(u) - \frac{\lambda}{c} \int_0^u g(u-x)\, dF_{X_1}(x)$$

$\int_0^t (\cdot)\,du$ $\Longrightarrow$ on both sides

$$g(t) - g(0) - \frac{\lambda}{c}\int_0^t g(u)\,du = -\frac{\lambda}{c}\int_0^t \left(\int_0^u g(u-x)\, dF_{X_1}(x)\right) du$$

$$\overset{\text{Ex. 2.3.8}}{=} \int_0^u g(u-x)\cdot F'_{X_1}(x)\,dx$$

integration by parts

$$-\frac{\lambda}{c}\int_0^t \Big[\underbrace{g(u-x)F_{X_1}(x)\Big|_{x=0}^u}_{=g(0)F_{X_1}(u)} + \int_0^u g'(u-x)F_{X_1}(x)\,dx\Big]\,du$$

(since $F_{X_1}(0)=0$ because of $X_1>0$ a.e.)

$$\Rightarrow g(t)-g(0) = \frac{\lambda}{c}\int_0^t g(u)\,du - \frac{\lambda}{c}g(0)\int_0^t F_{X_1}(u)\,du - \frac{\lambda}{c}\int_0^t\int_0^u g'(u-x)F_{X_1}(x)\,dx\,du$$

$$= \int_0^t \mathbb{1}^{(u)}_{[0,t]} \int_0^u \mathbb{1}^{(x)}_{[0,u]}\, g'(u-x) F_{X_1}(x)\,dx\,du$$

$$= \mathbb{1}^{(u)}_{[x,\infty)}$$

Tubini

$$= \int_0^t \int_0^\infty \mathbb{1}^{(u)}_{[x,t]}\, g'(u-x) F_{X_1}(x)\,du\,dx$$

$$= \int_0^t F_{X_1}(x) \int_x^t g'(u-x)\,du\,dx$$

$$= g(t-x)-g(0)$$

substitution

$$= \frac{\lambda}{c}\int_0^t g(t-u)\,du - \frac{\lambda}{c}\int_0^t F_{X_1}(x)\, g(t-x)\,dx$$

$\overline{F}_{X_1} = 1 - F_{X_1}$

$$\frac{\lambda}{c}\int_0^t \overline{F}_{X_1}(x)\cdot g(t-x)\,dx$$

$$= \underbrace{\lambda\cdot t}_{= E[N(t)]\cdot E[X_1]}$$

Since $\underbrace{p(t)}_{=c\cdot t} = (1+s)\,\underbrace{E[S(t)]}$

we see that

$$\frac{\lambda}{c} = \frac{1}{(1+s)}\cdot\frac{1}{E[X_1]}.$$

$\Rightarrow$ Proof.

Proof of Th. 2.3.1 (Cramér's ruin bound):

idea: Application of Smith's key renewal th. (Th. 2.3.10) to the previous Lemma

$\longrightarrow$ reformulation of (2.3.1) in terms of a renewal equation:

It follows from (2.3.1) and Rem. 2.3.12 that

$$\underbrace{\psi(u)}_{1-\psi(u)} = \frac{\delta}{(1+\delta)} + \frac{1}{1+\delta}\int_0^u \frac{1}{E[X_1]}\cdot \overline{F}_{X_1}(y)\,\underbrace{\psi(u-y)}_{=1-\psi(u-y)}\,dy$$

$$\Longrightarrow \quad \psi(u) = q\,\overline{F}_{X_1,I}(u) + \int_0^u \psi(u-x)\cdot\frac{q}{E[X_1]}\,\overline{F}_{X_1}(x)\,dx \qquad (2.3.2),$$

for $q \overset{\text{def}}{=} \frac{1}{1+\delta}$ and $\overline{F}_{X_1,I}(u) \overset{\text{def}}{=} 1 - F_{X_1,I}(u)$, where

$$F_{X_1,I}(x) \overset{\text{def.}}{=} \frac{1}{E[X_1]}\int_0^x \overline{F}_{X_1}(y)\,dy$$

(integrated tail distribution)

Recall: $E[X_1] = \int_0^\infty \overline{F}_{X_1}(y)\,dy$

Define

$$F^{(r)}(x) = \frac{q}{E[X_1]}\int_0^x e^{ry}\,\overline{F}_{X_1}(y)\,dy \qquad (2.3.3)$$

(Esscher transform of $F = F_{X_1}$)

integrat by parts $\Longrightarrow$ $\lim_{x\to\infty} F^{(r)}(x) = 1 \Longrightarrow F^{(r)}$ distr. (exerc.)

$\Longrightarrow$ $1 = m_Z(r) = m_{X_1}(r)\cdot m_{-W_1}(r)$

multipl. with e^{ru} on both sides of (2.3.2)

$$\underbrace{e^{ru}\psi(u)}_{\overset{\text{def}}{=} g(u)} = \underbrace{q\,e^{ru}\,\overline{F}_{X_1,I}(u)}_{\overset{\text{def}}{=} h(u)} + \int_0^u e^{r(u-x)}\psi(u-x)\cdot\frac{q\,e^{rx}}{E[X_1]}\,\overline{F}_{X_1}(x)\,dx$$

$\overset{\text{Ex. 2.3.9}}{=}$ $\overset{(2.3.3)}{=}$ $\int_0^u e^{r(u-x)}\psi(u-x)\,dF^{(r)}(x)$

$\Longrightarrow$ $g(u) = h(u) + \int_0^u g(u-x)\,dF^{(r)}(x)$ renewal equation

(h: bounded on finite intervals; g: diff. of non-increasing functions)

$$h(u) = \int_u^\infty e^{rx}\cdot q\,d\overline{F}_{X_1,I}(x) \overset{\text{int by parts}}{=} -r\int_u^\infty q\,\overline{F}_{X_1,I}(x)\,e^{rx}\,dx$$

$\overset{\text{Th. 2.3.10}}{\Longrightarrow}$ $g(u) \overset{\text{def}}{=} e^{ru}\psi(u) = \int_0^u h(u-x)\,d\underbrace{V(x)}_{=E[N(t)]+1}$ (renewal function associated with $F^{(r)}$)

and

$$\lim_{u\to\infty} e^{ru}\psi(u) = \tilde{\lambda}\cdot\int_0^\infty h(x)\,dx, \qquad (2.3.4)$$

where $\widehat{\lambda} = \left(\int_0^\infty x \, dF^{(\tau)}(x) \right)^{-1}$

$= E[Y]$, Y with distr. $F^{(\tau)}$

Further calculation shows that

(2.3.4) $= C$ as in Th. 2.3.1

$\Rightarrow$ proof.

Ex. 2.3.13 ($X_1 \sim \text{Exp}(\gamma)$)

If $X_1 \sim \text{Exp}(\gamma)$, $\gamma > 0$ in Th. 2.3.1, then

$$e^{\tau u} \psi(u) = \frac{1}{1+\beta} \qquad (2.3.5)$$

See exercises.

2.4 Exact asymptotics for $\psi(u)$ for large claims

The small claim condition (2.2.1) may be reasonable in connection with claims in water insurance. Examples of claim size distributions satisfying (2.2.1) are

$1 - F(x)$ ⟶ (i) $\overline{F}(x) = e^{-\lambda x}$, $x \geq 0$, $\lambda > 0$
(exponential distr.)

prob. density (ii) $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$, $x \geq 0$, $\alpha, \beta > 0$ (2.4.1)
(Gamma distr.)

(iii) $\overline{F}(x) = e^{-cx^\tau}$, $x \geq 0$, $c > 0$, $\tau \geq 1$ (2.4.2)
(Weibull distr.)

⟶ the tail distributions decay to zero exponentially fast for $x \to \infty$

However, claim size data in fire insurance or windstorm insurance indicate that the true claim size distr. F_{X_1} does not fulfill (2.2.1), but is rather heavy-tailed, that is subexponential

Examples of such distr are exponential transformations of r.v.'s of the type (2.2.1), thus e.g. the normally, exponentially or Gamma-distr. r.v.'s:

(i) $f(x) = \frac{1}{\sqrt{2\pi}\beta \cdot x} e^{-(\log x - \mu)^2/(2\beta^2)}$, $x \geq 0, \mu \in \mathbb{R}, \beta > 0$ (2.4.3)

(log-normal distr.)

(ii) $\bar{F}(x) = \left(\frac{\kappa}{x+\kappa}\right)^{\alpha}$, $x \geq 0, \alpha, \kappa > 0$ (2.4.4)

(Pareto-distr.)

(iii) $f(x) = \frac{\kappa^{\beta}}{\Gamma(\beta)} (\log x)^{\beta-1} x^{-\alpha-1}$, $x \geq 0, \alpha, \beta > 0$ (2.4.5)

(log-Gamma-distr.)

Other examples are

(iv) $\bar{F}(x) = \left(\frac{\kappa}{x+\kappa^{\tau}}\right)^{\alpha}$, $x \geq 0, \alpha, \kappa, \tau > 0$ (2.4.6)

(Burr-distr.)

(v) $\bar{F}(x) = e^{-cx^{\tau}}$, $x \geq 0, c > 0, 0 < \tau < 1$ (2.4.7)

(Weibull-distr.)

→ objective: Calculation of the exact asymptotics of $\psi(u)$ in the Cramér-Lundberg model for large claims, that is for subexponential claim size distr.

In doing so, we have to study subexponential distributions.

For this purpose, we need some notions and auxiliary results from the theory of regularly varying funtions

Def. 2.4.1 (slowly varying functions)

A (measurable) function $L: (0,\infty) \to (0,\infty)$ is called slowly varying (at infinity), if for all $c > 0$:

$$\lim_{x \to \infty} \frac{L(cx)}{L(x)} = 1 \qquad (2.4.8)$$

→ Th. 2.4.2 (Representation th. for slowly varying functions)

Let L be a slowly varying function. Then there ex. positive funct. c and ε s.t.