σ-algebras

Lemma 5.12 : Let $(\Omega_1, \mathcal{A}_1), (\Omega_2, \mathcal{A}_2)$ be measurable spaces and $\mathcal{B}_2$ a family of sets in $\mathcal{A}_2$ such that $\mathcal{A}_2 = \sigma(\mathcal{B}_2) :=$ smallest σ-algebra which contains all sets from $\mathcal{B}_2$.
Consider a function $f: \Omega_1 \to \Omega_2$ such that

$$\{\omega : f(\omega) \in B\} \in \mathcal{A}_1$$

for all $B \in \mathcal{B}_2$.
Then f is $\mathcal{A}_1, \mathcal{A}_2$-measurable, that is

$$\{\omega : f(\omega) \in B\} \in \mathcal{A}_1$$

for all $B \in \mathcal{A}_2$

→ Prop. 5.13 : X is a random measure iff $X(A)$ is a r.v. for all $A \in \mathcal{B}(E)$

Proof : We can choose in L. 5.12 $f = X$, $\Omega_1 = \Omega$, $\Omega_2 = M_+(E)$, $\mathcal{A}_1 = \mathcal{F}$, $\mathcal{A}_2 = \mathcal{B}(M_+(E)) \overset{\text{Def}}{=} \mathcal{M}_+(E)$ and

$$\mathcal{B}_2 = \{\{\mu \in M_+(E) : \mu(G) \in B\} : G \in \mathcal{B}(E), B \in \mathcal{B}([0,\infty])\}$$

We know from Prop. 5.4, (5.9) that $\mathcal{A}_2 = \sigma(\mathcal{B}_2)$

$\overset{\text{L.5.12}}{\Longrightarrow}$ X is a random measure iff

$$\{\omega : \underbrace{X(\omega, A)}_{= X(A)} \in B\} = \{\omega : X(\omega) \in \{\mu \in M_+(E) : \mu(A) \in B\}\}$$

$\in \mathcal{A}_1 = \mathcal{F}$ for all $B \in \mathcal{B}([0,\infty])$

$\overset{\text{Def.}}{\Longleftrightarrow}$ $X(A)$ is a (real valued) r.v. $\Longrightarrow$ proof.

Ex. : The point process of exceedances $N_n : \Omega \to M_+(E)$ for $E = (0,1]$ in Ex. 5.1 is a random meas., since (by Prop. 5.13)

$$N_n(A) = \sum_{i=1}^{n} \varepsilon_{n-1,i}(A) \cdot \mathbb{1}_{\{X_i > u_n\}}$$

is a r.v. for all $A \in \mathcal{B}((0,1])$

In view of applications to extreme value theory we also want to study the weak (or vague) convergence of random measures $\Xi_n \xrightarrow{v} \Xi$, that is the weak convergence of the prob. meas. P_{Ξ_n}, P_Ξ on $S = M_+(E)$ defined by

$$P_{\Xi_n}(A) = P(\{\Xi_n \in A\}), \quad P_\Xi(A) = P(\{\Xi \in A\}), \quad A \in \mathcal{M}_+(E) \tag{5.10}$$

So

$$\Xi_n \xrightarrow{v} \Xi$$

$$\Longleftrightarrow \quad E[f(\Xi_n)] = \int_S f(\gamma) P_{\Xi_n}(d\gamma) \xrightarrow[n\to\infty]{} \int_S f(\gamma) P_\Xi(d\gamma) \quad (= E[f(\Xi)] \text{ see Ex. 2.3.9})$$

for all $f \in C_b(S)$ (space of bounded cont. functions).

$$\Longleftrightarrow \quad E[f(\Xi_n)] \xrightarrow[n\to\infty]{} E[f(\Xi)] \tag{5.11}$$

for all $f \in C_b(S)$.

To this end, we need the following result from probability theory, which shows the link between weak convergence of r.v.'s and "usual" convergence of r.v.'s in the almost everywhere sense.

Th. 5.14 (Skorohod's theorem)
Let (S,d) be a complete separable metric space and $\Xi_n, \Xi : \Omega \to S$ be r.v.'s. Then $\Xi_n \xrightarrow[n\to\infty]{v} \Xi$ iff there exist r.v.'s $\Xi_n^*, \Xi^* : \widetilde{\Omega} \to S$ on the prob. space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P}) = ([0,1], \mathcal{B}([0,1]), \lambda)$ (Lebesgue-meas.) such that

$$\Xi_n \overset{d}{=} \Xi_n^*, \quad \Xi \overset{d}{=} \Xi^*$$

(i.e. $P_{\Xi_n} = P_{\Xi_n^*}$, $P_\Xi = P_{\Xi^*}$ (see (5.10)))

for all $n \geq 1$ and

$$\Xi_n^* \xrightarrow[n\to\infty]{} \Xi^*$$

with prob. 1, that is a.e.

Proof: See exercises for $S = \mathbb{R}$

An important direct consequence of Th. 5.14 in connection with dominated convergence is the following result:

Th. 5.15 (Continuous mapping theorem)

Let $(S_1, d_1), (S_2, d_2)$ complete, separable metric spaces and $X_n, X : \Omega \to S_1$, $n \geq 1$ r.v.'s such that $X_n \xrightarrow{V} X$ (i.e. $E[f(X_n)] \xrightarrow[n\to\infty]{} E[f(X)]$ $\forall f \in C_b(S_1)$).

Assume $h : S_1 \to S_2$ is a measurable function ($\in \mathcal{B}(S_1)$) such that

$$P(X \in \{s_1 \in S_1 : h \text{ discontinuous in } s_1\}) = 0$$

Then

$$h(X_n) \xrightarrow[n\to\infty]{V} h(X).$$

It turns out that weak convergence of r.v.'s given by random measures can be characterized by so-called Laplace functionals, which play a similar role as characteristic functions for r.v.'s on $\mathbb{R}^d$:

Def. 5.16 (Laplace functional)

The Laplace functional Ψ_Q of a prob. measure Q on $M_+(E)$ is defined as

$$\Psi_Q(f) = \int_{M_+(E)} \exp\Big(-\underbrace{\int_E f(x)\,\mu(dx)}_{\stackrel{\text{def}}{=} \mu(f)}\Big)\, Q(d\mu)$$

for all $f \in C_0^+(E)$ (space of continuous $f \geq 0$ which vanish outside a compact set)

The Laplace functional Ψ_X of a random measure $X : \Omega \to M_+(E)$ is defined as

$$\Psi_X(f) = E[\exp(-X(f))] = \int_{M_+(E)} \exp(-\mu(f))\, P_X(d\mu)$$

for all $f \in C_0^+(E)$.

Recall

$$X(f) \stackrel{\text{def}}{=} \int_E f(y)\, X(\omega, dy)$$

Ex. 5.17 : $X = N_n$ point process of exceedances (Ex. 4.1)

$E = (0,1]$

$$\Rightarrow X(f) = \int_0^1 f(x)\, dN_n(\omega, dx)$$

$$= \int_0^1 f(x)\, d\Big(\sum_{i=1}^{n0} \varepsilon_{n^{-1}\cdot i} \mathbb{1}_{\{X_i > u_n\}}\Big) \quad \leftarrow \text{Dirac measure in } n^{-1}\cdot i$$

(linearity in $d(\cdot)$)

$$= \sum_{i=1}^{n} \mathbb{1}_{\{X_i > u_n\}} \int_0^1 f(x)\, \varepsilon_{n^{-1}\cdot i}(dx)$$

$$= \sum_{i=1}^{n} \mathbb{1}_{\{X_i > u_n\}} f(n^{-1}\cdot i)$$

$$\Rightarrow \Psi_{N_n}(f) = E\Big[\exp\Big(-\sum_{i=1}^{n} \mathbb{1}_{\{X_i > u_n\}} f(n^{-1}\cdot i)\Big)\Big]$$

$$\overset{X_i \text{ i.i.d.}}{=} \prod_{i=1}^{n} \big(1 - \bar{F}(u_n)\big(1 - e^{-f(i/n)}\big)\big) \quad \text{(exercise)}$$

Th. 5.18 (Characterization of weak convergence by Ψ)

Let P_n, P, $n \geq 1$ be prob. meas. on $M_+(E)$ and X_n, X random measures.

Then

(i) $P_n \xrightarrow[n\to\infty]{v} P \iff \Psi_{P_n}(f) \xrightarrow[n\to\infty]{} \Psi_P(f)$ for all $f \in C_0^+(E)$,

(ii) $X_n \xrightarrow[n\to\infty]{v} X \iff \Psi_{X_n}(f) \xrightarrow[n\to\infty]{} \Psi_X(f)$ for all $f \in C_0^+(E)$.

Rem. 5.19 : It follows from Th. 5.18 (ii) for rand. meas. X and Y that

$$E[f(X)] = E[f(Y)] \text{ for all } f \in C_b(M_+(E)) \quad (*)$$

$$\iff \Psi_X(f) = \Psi_Y(f) \text{ for all } f \in C_0^+(E)$$

However, $(*)$ is equivalent to

$$X \overset{d}{=} Y \quad \text{(see exercises)}$$

For the proof of Th. 5.18 we need the following auxiliary result:

L. 5.20: Let X_n, $n \geq 1$ be a sequence of rand. measures. Then the family of prob. measures P_{X_n}, $n \geq 1$ on $\mathcal{B}(M_+(E))$ is tight (Def. 5.7) iff

(94) the family of prob. measures P_{Y_n}, $n \geq 1$ on $\mathcal{B}(\mathbb{R})$ is tight with $Y_n = X_n(f) = \int_E f(y) X(\omega, dy)$ for all $f \in C_0^+(E)$.

Proof

"$\Leftarrow$": Let $X_n(f)$, $n \geq 1$ be tight in $\mathbb{R}$ for all $f \in C_0^+(E)$. Let us now choose $g_i \in C_0^+(E)$, $i \geq 1$ s.t. $g_i(x) \uparrow 1$ for all $x \in \mathbb{R}$.
It is known from basic mathematical analysis that this convergence must be necessarily uniform on compact sets of E (see also Exerc. 9, Prob. 3).
Since any compact set in $\mathbb{R}$ must be contained in an interval $[-c_i, c_i]$, we can find by tightness c_i, $i \geq 1$ such that

$$P(X_n(f) > c_i) \leq \varepsilon / 2^{i+1}$$

for all i for some $\varepsilon > 0$.

Claim: $M := \bigcap_{i \geq 1} \{\mu \in M_+(E) : \mu(g_i) \leq c_i\}$ (*)
is relatively compact in $M_+(E)$, i.e. the closure of M, that is $\overline{M}$ is compact in $M_+(E)$

Reason: Let $f \in C_0^+(E)$. Define $K_0 := \sup_{x \in E} |f(x)| + 1$
We know that for all $\delta > 0$ there exists a $i_0 = i_0(\delta)$ s.t. for all $i \geq i_0$:

$$1 - \delta \leq g_i(x) \leq 1 + \delta$$

uniformly in x

$\Rightarrow$ $\forall i \geq i_0$: $K_0 \cdot (1 - \delta) \leq K_0 \, g_i(x)$
uniformly in x

δ small $\Rightarrow$ $f(x) \leq \sup_{x \in E} |f(x)| \leq K_0 (1 - \delta) \leq K_0 \cdot g_i(x)$
for all $i \geq i_0$ uniformly in x

$\Rightarrow$ $\sup_{\mu \in M} \mu(f) \overset{\text{Def.}}{=} \sup_{\mu \in M} \int_E f(y) \mu(dy) \leq \sup_{\mu \in M} \int_E K_0 g_{i_0}(y) \mu(dy)$
$\overset{(*)}{\leq} K_0 \, c_{i_0} < \infty$.

$\overset{\text{Th. 5.6}}{\Longrightarrow}$ $\overline{M}$ is compact in $M_+(E)$ $\qquad M_+(E)\setminus M$

On the other hand,

$$P_{X_n}(\overline{M}) = P(X_n \notin \overline{M}) \overset{M\subseteq\overline{M}}{\leq} P(X_n \notin M) = P(X_n \in M^c)$$

$$= P\Big(X_n \in \bigcup_{i\geq 1} \{\mu \in M_+(E) : \mu(g_i) > c_i\}\Big)$$

$$= P\Big(\bigcup_{i\geq 1} \{X(g_i) > c_i\}\Big) \overset{\text{properties of measures}}{\leq} \sum_{i\geq 1} \varepsilon/2^{i+1} = \varepsilon$$

$\overset{\text{definition of tightness (Def. 5.7)}}{\Longrightarrow}$ $P_{X_n}, n\geq 1$ is tight

"$\Longleftarrow$": simple exercise. $\Longrightarrow$ proof.

Proof of Th. 5.18:

"$\Longrightarrow$": Let $P_{X_n} \xrightarrow[n\to\infty]{v} P_X$.

We know from the definition of weak convergence in $M_+(E)$ that the function $h: M_+(E) \to \mathbb{R}$ given by $h(\mu) = \mu(f)$ for given $f \in C_0^+(E)$ is continuous

$\overset{\text{continuous mapping theorem (Th. 5.15)}}{\Longrightarrow}$ $\underbrace{P_{h(X_n)}}_{=X_n(f)} \xrightarrow[n\to\infty]{v} \underbrace{P_{h(X)}}_{=X(f)}$

On the other hand, it follows from Skorohod's theorem (Th. 4.14) that there ex. $Y_n^*, n\geq 1, Y^*$ on $(\Omega, \mathcal{F}, P) = ([0,1], \mathcal{B}([0,1]), \lambda)$ such that

$$X_n(f) \overset{d}{=} Y_n^*, \quad X(f) \overset{d}{=} Y^*$$

and

$$Y_n^* \xrightarrow[n\to\infty]{} Y^* \quad \text{with prob. 1}$$

$$\Longrightarrow \quad \Psi_{X_n}(f) \overset{\text{Def.}}{=} E[\exp(-X_n(f))] \overset{X_n(f)\overset{d}{=}Y_n^*}{=} E[\exp(-Y_n^*)]$$

$$\overset{\text{dom. convergence}}{\xrightarrow[n\to\infty]{}} E[\exp(-Y^*)] \overset{X(f)\overset{d}{=}Y^*}{=} E[\exp(-X(f))] \overset{\text{Def}}{=} \Psi_X(f)$$

"$\Longleftarrow$": Suppose $\Psi_{X_n}(f) \xrightarrow[n\to\infty]{} \Psi_X(f)$ for all $f \in C_0^+(E)$

We have that

$\lambda f \in C_0^+(E), \lambda > 0$ and $X_n(\lambda f) = \lambda X_n(f), \lambda > 0$

So

$$E[\exp(-\lambda X_n(f))] = E[\exp(X_n(\lambda f))] \overset{\text{Def}}{=} \psi_{X_n}(\lambda f)$$

$$\xrightarrow[n\to\infty]{} \psi_X(\lambda \cdot f) = E[\exp(-\lambda X(f))],$$

that is the usual Laplace transforms of real valued r.v.'s $X_n(f)$ converge to that of $X(f)$

Probability theory $\Longrightarrow$

$$X_n(f) \xrightarrow[n\to\infty]{d} X(f) \text{ or equivalently}$$

$$X_n(f) \xrightarrow[n\to\infty]{v} X(f)$$

$\Longleftrightarrow$

$$P_{X_n(f)} \xrightarrow[n\to\infty]{v} P_{X(f)} \quad \text{in } M_+(\mathbb{R})$$

Def. $\Longleftrightarrow$

$$\underbrace{P_{X_n(f)}(g)}_{\text{Def.}} \xrightarrow[n\to\infty]{} \underbrace{P_{X(f)}(g)}_{\text{Def.}} \quad \text{for all } g \in C_0^+(\mathbb{R})$$

$$= \int_{\mathbb{R}} g(y)\, P_{X_n(f)}(dy) \qquad = \int_{\mathbb{R}} g(y)\, P_{X(f)}(dy)$$

$\Longrightarrow$

$$\sup_{\mu\in M} \mu(g) < \infty \quad \text{for all } g \in C_0^+(\mathbb{R}),$$

where $M := \{\mu \in M_+(\mathbb{R}) : \mu = P_{X_n(f)} \text{ for some } n \geq 1\}$

Th. 5.6 $\Longrightarrow$ M relatively compact, i.e. $\overline{M}$ compact in $M_+(\mathbb{R})$

Th. 5.10 $\Longleftrightarrow$ The family $\Pi := M = \{P_{X_n(f)}, n \geq 1\}$ is tight for all f

L. 5.20 $\Longleftrightarrow$ $\widetilde{\Pi} := \{P_{X_n}, n \geq 1\}$ is tight

Th. 5.10 $\Longleftrightarrow$ the closure of $\widetilde{\Pi}$ is compact (in $\mathcal{Z}$)

Def. of compactness in metric spaces $\Longrightarrow$ For all subsequences $n_k, k \geq 1$ there ex. a subseq. $m_\ell, \ell \geq 1$ and a prob. meas. P^* s.t.

$$P_{X_{m_\ell}} \xrightarrow[\ell\to\infty]{v} P^* \qquad (+)$$

"$\Longrightarrow$"

$$\psi_{X_{m_\ell}}(f) \xrightarrow[\ell\to\infty]{} \psi_{P^*}(f) \quad \text{for all } f \in C_0^+(E)$$

On the other hand, by assumption we know that

$$\psi_{P_{X_n}}(f) \xrightarrow[n\to\infty]{} \psi_{P_X}(f) = \psi_X(f)$$

$$\| \qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$\psi_{X_n}(f) \qquad\qquad\qquad\qquad\qquad\qquad \psi_X(f)$$

$\Rightarrow \quad \psi_{P^*}(f) = \psi_{P_{X'}}(f)$ for all $f \in C_0^+(E)$

Rem. 5.19 $\Longleftrightarrow \quad P^* = P_{X'}$

Since (+) is true for all subsequences, we get that

$$P_{X'_n} \xrightarrow[n\to\infty]{v} P_{X'}.$$

$\Rightarrow$ proof.

In fact, later on we want to apply Th. 5.18 to the point process N_n (Ex. 5.1) of exceedances and show that

$$N_n \xrightarrow[n\to\infty]{v} N,$$

where N is a so-called <u>Poisson random measure</u>

To clarify the latter notion, we need to consider a <u>special case of random measures</u>, namely <u>point processes</u>.

For this purpose denote by $M_p(E)$ the subspace of $M_+(E)$ of all measures μ of the form

$$\mu = \sum_{i\geq 1} \varepsilon_{x_i}, \qquad (5.12)$$

(ε_{x_i}: Dirac measure in x_i)

where $x_i, i \geq 1$ is a sequence of (not necessarily distinct) points in E

Measures μ from $M_p(E) \subseteq M_+(E)$ are called <u>point measures</u>.

$\longrightarrow$ <u>properties</u>:

(i) For $\mu \in M_p(E)$ the set $S_\mu := \{x \in E : \mu(\{x\}) \neq 0\}$ is closed (since μ is a Radon measure by assumption) and given by the distinct points of $\{x_i : i \geq 1\}$.

(ii) $M_p(E)$ is a <u>closed subset</u> of $M_+(E)$, i.e. if $\mu_n \in M_p(E)$, $n \geq 1$ and $\mu_n \xrightarrow[n\to\infty]{v} \mu \in M_+(E)$, then $\mu \in M_p(E)$. (5.13)

(iii) Since $\mathcal{B}(M_p(E)) = M_p(E) \cap \mathcal{B}(M_+(E))$ (Borel-σ-algebra)

it also follows from (5.9) in Prop. 5.4 that

$$\mathcal{M}_p(E) := \mathcal{B}(M_p(E)) = \text{smallest } \sigma\text{-algebra containing all sets of the form } \{\mu \in M_p(E) : \mu(G) \in B\},\ G \in \mathcal{B}(E),\ B \in \mathcal{B}([0,\infty]) \tag{5.14}$$

→ **Def. 5.21** (Point process)

A random measure $X : \Omega \to M_+(E)$ is called <u>point process</u> if $X(\omega) \in M_p(E)$ for all $\omega \in \Omega$

Rem. 5.22 : Because of Prop. 5.13 we see that X is a point process iff $X(\omega) \in M_p(E)$ for all ω and $X(A)$ is a real valued r.v. for all $A \in \mathcal{B}(E)$

Ex. : $N_n = \sum_{i=1}^{n} \varepsilon_{n^{-1}\cdot i}\, 1_{\{X_i > u_n\}}$ in Ex. 4.1 is a point process.

In view of convergence of N_n we know want to study a <u>special class of point processes</u>, namely that of <u>Poisson processes</u>.

Def. 5.23 (Poisson process)

Let $\mu \in M_+(E)$. Then a point process $X : \Omega \to M_p(E)$ is called a Poisson process or a Poisson random measure (PRM(μ)) with mean measure μ if

(i) for all $A \in \mathcal{B}(E)$ and non-negative integers k we have that

$$P(X(A) = k) = \begin{cases} \exp(-\mu(A)) \dfrac{(\mu(A))^k}{k!}, & \text{if } \mu(A) < \infty \\ 0, & \text{if } \mu(A) = \infty \end{cases}$$

(i.e. $X(A) \sim \text{Pois}(\mu(A))$ if $\mu(A) < \infty$)

and

(ii) for all mutually disjoint $A_1, \dots, A_k \in \mathcal{B}(E)$

$X(A_1), \dots, X(A_k)$

are independent real valued r.v.'s

Th. 5.24 (Laplace functional of PRM(μ))

A point process X is a PRM(μ) iff

$$\Psi_X(f) = \exp\left(-\int_E (1-e^{-f(y)})\,\mu(dy)\right) \qquad (5.15)$$

for all $f \in C_0^+(E)$ (or all meas. $f \geq 0$)

Th. 5.25 (Existence of PRM(μ))

A PRM(μ) exists and is uniquely determined by its Laplace functional.

Proof of Th. 5.24:

"$\Rightarrow$" Let X be a PRM(μ) and let $f = c \cdot \mathbb{1}_F$ for $c>0$, $F \in \mathcal{B}(E)$, w.l.o.g. $\mu(F) < \infty$.

$\int_E c\cdot \mathbb{1}_F(y) X(\omega, dy) = c \cdot X(F)$, $= E$

$$\Rightarrow \Psi_X(f) = E[\exp(-X(f))] \qquad = 1$$

$$= E[\exp(-c\cdot X(F))] = E\Big[\Big(\sum_{k\geq 0} \mathbb{1}_{\{X(F)=k\}}\Big) e^{-c\cdot X(F)}\Big]$$

$$\overset{\text{mon. conv.}}{=} \sum_{k\geq 0} E[\mathbb{1}_{\{X(F)=k\}} e^{-c\cdot \underbrace{X(F)}_{=k}}] = \sum_{k\geq 0} E[\mathbb{1}_{\{X(F)=k\}} \underbrace{e^{-c\cdot k}}_{=(e^{-c})^k}]$$

$$= \sum_{k\geq 0} (e^{-c})^k \underbrace{E[\mathbb{1}_{\{X(F)=k\}}]}_{=P(X(F)=k)} \overset{\text{Def. 5.2.3, (i)}}{=}$$

$$= \int_E \underbrace{(e^{-c}-1)\cdot \mathbb{1}_F(y)}_{=(e^{-c\mathbb{1}_F(y)}-1)} \mu(dy)$$

$$= \sum_{k\geq 0} (e^{-c})^k \frac{(\mu(F))^k}{k!} \exp(-\mu(F)) = \exp\big(\underbrace{(e^{-c}-1)}\,\mu(F)\big)$$

$$= \exp\left(-\int_E (1-e^{-c\,\mathbb{1}_F(y)})\,\mu(dy)\right) \Rightarrow (5.15) \text{ satisfied for } f = c\cdot \mathbb{1}_F \quad (*)$$

Assume now $c_i \geq 0$, $i = 1,\dots,K$, $F_1,\dots,F_K \in \mathcal{B}(E)$ disjoint (w.l.o.g. $\mu(F_i) < \infty$ for all i)

$\overset{\text{Def. 5.2.3}}{\Rightarrow}$ $X(F_1), \dots, X(F_K)$ indep. r.v.'s

(100) Let $f = \sum_{i=1}^{K} c_i \mathbb{1}_{F_i}$.

$$\Rightarrow \Psi_X(f) = E[\exp(-X(f))] = E\Big[\exp\Big(-\sum_{i=1}^{K} c_i X(F_i)\Big)\Big] = \prod_{i=1}^{K} \exp(c_i X(F_i))$$

$$\overset{\text{indep. r.v.'s}}{=} \prod_{i=1}^{K} E[\exp(c_i X(F_i))] \overset{(*)}{=} \prod_{i=1}^{K} \exp\Big(-\int_E (1-e^{-c_i \mathbb{1}_{F_i}(y)})\,\mu(dy)\Big)$$

$$= \exp\Big(-\int_E \underbrace{\sum_{i=1}^{K} (1-e^{-c_i \mathbb{1}_{F_i}(y)})}_{= (1-e^{-\sum_{i=1}^{K} c_i \mathbb{1}_{F_i}(y)})} \mu(dy)\Big)$$

$$= \exp\Big(-\int_E (1-e^{-f(y)})\,\mu(dy)\Big)$$

$\Rightarrow$ (5.15) for $f = \sum_{i=1}^{n} c_i \mathbb{1}_{F_i}$

Consider now a general meas. $f \geq 0$

$\Rightarrow \exists\ f_n = \sum_{i=1}^{K_n} c_i^{(n)} \mathbb{1}_{F_i^{(n)}}$ with $c_i^{(n)} \geq 0$, $F_1^{(n)}, \dots, F_{K_n}^{(n)}$ disjoint, $n \geq 1$ such that

$$0 \leq f_n(x) \uparrow f(x)$$

for $n \to \infty$ for all x.

$$\overset{\text{mon. conv.}}{\Longrightarrow} \quad X(f_n) \overset{\text{Def.}}{=} \int_E f_n(y)\, X(\omega, dy) \uparrow \int_E f(y)\, X(\omega, dy) = X(f)$$

for $n \to \infty$ and for all ω

$$e^{-X(f_n)} \leq 1 \Longrightarrow \Psi_X(f_n) = E[\exp(-X(f_n))] \xrightarrow[n\to\infty]{} E[\exp(-X(f))] = \Psi_X(f)$$

dom. conv.

and

$$\Psi_X(f_n) = \exp\Big(-\int_E (1-e^{-f_n(y)})\,\mu(dy)\Big)$$

$$\xrightarrow{n\to\infty} \exp\Big(-\int_E (1-e^{-f(y)})\,\mu(dy)\Big)$$

$\Rightarrow$ (5.15) for all meas. $f \geq 0$.

"$\Leftarrow$": Let X be a point process with Ψ_X of the form (5.15)

Set $f = \lambda \mathbb{1}_F$, $\lambda \geq 0$, $F \in \mathcal{B}(E)$

$$\overset{(5.15)}{\Rightarrow} \quad E[e^{-X(f)}] = \underbrace{E[e^{-\lambda X(F)}]}_{\text{classical Laplace transform of the r.v. } X(F) \geq 0} = \exp(-(1-e^{-\lambda})\mu(F))$$

$\Rightarrow$ $X(F)$ has a classical Laplace transform of a

Poisson rand. variable with parameter $\mu(F)$
$\Rightarrow$ (i) in Def. 5.23 is satisfied (since classical Laplace transforms of r.v. $Y \geq 0$ uniquely determine the distr. of Y)

Let now $F_1, \dots, F_K \in \mathcal{D}(E)$ be disjoint
Set $f = \sum_{i=1}^{K} \lambda_i 1_{F_i}$, $\lambda_i \geq 0$, $i = 1, \dots, K$

$$\Rightarrow \underbrace{E[e^{-N(f)}]}_{= E[e^{-\lambda_1 N(F_1) + \dots + \lambda_K N(F_K)}]} \overset{(5.15)}{=} \exp\left(-\int_E \underbrace{\left(1 - \exp\left(\sum_{i=1}^{K} \lambda_i 1_{F_i}(y)\right)\right)}_{= \sum_{i=1}^{K} 1_{F_i}(y)(1 - \exp(-\lambda_i))} \mu(dy)\right)$$

$$= \exp\left(-\sum_{i=1}^{K} (1 - e^{-\lambda_i}) \underbrace{\int_E 1_{F_i}(y)\mu(dy)}_{= \mu(F_i)}\right)$$

$$= \prod_{i=1}^{K} \exp\left((1 - e^{-\lambda_i})\mu(F_i)\right) = \prod_{i=1}^{K} E[e^{-\lambda_i N(F_i)}]$$

$\Rightarrow$ $N(F_1), \dots, N(F_K)$ indep. $\Rightarrow$ (ii) in Def. 5.23 holds $\Rightarrow$ proof.

Rem. 5.26: Recall: (i) $X, Y \geq 0$ r.v.'s with the same classical Laplace transform $\Psi_X(\lambda) \overset{\text{def}}{=} E[e^{-\lambda X}] = E[e^{-\lambda Y}] = \Psi_Y(\lambda)$ for all $\lambda \geq 0$
$\Leftrightarrow$ $X \overset{d}{=} Y$

(ii) $X_1, \dots, X_n \geq 0$ r.v.'s $\Rightarrow$ $X_1, \dots, X_n$ indep. $\Leftrightarrow$
$$\underbrace{E[e^{-\lambda_1 X_1 + \dots + \lambda_n X_n}]}_{\overset{\text{def}}{=} \Psi_{X_1, \dots, X_n}(\lambda_1, \dots, \lambda_n)} = \prod_{i=1}^{n} \underbrace{E[e^{-\lambda_i X_i}]}_{= \Psi_{X_i}(\lambda_i)} \text{ for all } \lambda_1, \dots, \lambda_n \geq 0$$

$\Psi_{X_1, \dots, X_n}$: classical Laplace transform of the rand. vector $(X_1, \dots, X_n)$

Proof of Th. 5.25 (Existence of PRM(μ))

1. case: $\mu(E) < \infty$ $\Rightarrow$ $\mu = c \cdot \nu$ for $c > 0$, where ν is a prob. meas.

Let $T, X_1, X_2, X_3, \dots$ be indep. r.v.'s such that $X_i, i \geq 1$ is a sequence of i.i.d. r.v.'s with distr. ν (i.e. $F_{X_i}(x) = \nu((-\infty, x])$ for all x) and T is a Poisson rand. variable with parameter c.

(102) Define $X'^*: \Omega \to M_+(E)$ by (Dirac measure in $X_i(\omega)$)

$$X'^*(\omega) = \begin{cases} \sum_{i=1}^{\hat{\tau}(\omega)} \varepsilon_{X_i(\omega)}, & \text{if } \hat{\tau}(\omega) > 0 \\ 0, & \text{if } \hat{\tau}(\omega) = 0 \end{cases}$$

Prop. 5.13 (5.12) $\Rightarrow$ X'^* is a random meas. with values in $M_p(E)$

$\Rightarrow$ X'^* point process

Let $F \in \mathcal{B}(E)$. We want to show that $X'^*(F)$ is Poisson distributed. For $k \geq 1$ we have that

$$P(X'^*(F)=k) = P\Big(\sum_{i=1}^{\hat{\tau}} \varepsilon_{X_i}(F) = k\Big)$$

$\varepsilon_{X_i}(F) = \mathbb{1}_F(X_i)$

$$= P\Big(\sum_{i=1}^{\hat{\tau}} \mathbb{1}_F(X_i) = k,\ \hat{\tau} = k, k+1, \dots\Big) = \sum_{j \geq k} P\Big(\sum_{i=1}^{\hat{\tau}} \mathbb{1}_F(X_i) = k, \hat{\tau} = j\Big)$$

$= \Big\{\sum_{i=1}^{j} \mathbb{1}_F(X_i) = k, \hat{\tau} = j\Big\}$

$\overset{\hat{\tau}, X_i, i\geq 1 \text{ indep.}}{=} \sum_{j \geq k} P\Big(\sum_{i=1}^{j} \mathbb{1}_F(X_i) = k\Big) P(\hat{\tau} = j)$

$= e^{-c} \frac{c^j}{j!}$

$\sum_{i=1}^{j} \mathbb{1}_F(X_i)$: binomial r.v. with $p = V(F)$

$$= \sum_{j \geq k} \binom{j}{k} (V(F))^k (1-V(F))^{j-k} e^{-c} \frac{c^j}{j!}$$

$$= \sum_{j \geq k} \frac{(c\cdot(1-V(F)))^{j-k}}{(j-k)!} \frac{e^{-c}(c\cdot V(F))^k}{k!} = e^{c(1-V(F))} \frac{e^{-c}(c\cdot V(F))^k}{k!}$$

$$= e^{-cV(F)} \cdot \frac{(c\cdot V(F))^k}{k!} \Rightarrow X'^*(F) \text{ Poisson distr. with parameter } c\cdot V(F) = \mu(F)$$

Let $F_0, \dots, F_K \in \mathcal{B}(E)$ disjoint s.t. $E = \bigcup_{i=0}^{K} F_i$ and let $n \geq 1$, $n_i \in \mathbb{N}_0$, $i = 0, \dots, K$ s.t. $\sum_{i=0}^{K} n_i = n$

We want to show that $X'^*(F_0), \dots, X'^*(F_K)$ are indep. r.v.'s:

$E = \bigcup_{i=0}^{K} F_i$, F_i disjoint

$$P(X'^*(F_0)=n_0, \dots, X'^*(F_K)=n_K)$$

$$P\Big(\sum_{i=1}^{\hat{\tau}=n} \mathbb{1}_{F_0}(X_i) = n_0, \dots, \sum_{i=1}^{\hat{\tau}=n} \mathbb{1}_{F_K}(X_i) = n_K,\ \hat{\tau} = n\Big)$$

$\overset{\hat{\tau}, X_i, i\geq 1 \text{ indep.}}{=}$ $P\Big(\sum_{i=1}^{n} \mathbb{1}_{F_0}(X_i) = n_0, \dots, \sum_{i=1}^{n} \mathbb{1}_{F_K}(X_i) = n_K\Big) \cdot P(\hat{\tau} = n)$

multinomial probability

$$\overset{\text{induction}}{=} \frac{n!}{\prod_{i=0}^{K} n_i!} \prod_{i=0}^{K} (V(F_i))^{n_i} \frac{e^{-c} c^n}{n!} = \prod_{i=0}^{K} \frac{V(F_i)^n}{n_i!} \cdot e^{-c} \cdot c^n$$

Since $1 = \sum_{i=0}^{K} V(F_i)$ and $n = \sum_{i=0}^{K} n_i$, we see that

$$\prod_{i=0}^{K} \frac{(V(F_i))^{n_i}}{n_i!} e^{-c} c^n = \prod_{i=0}^{K} \left(\frac{(V(F_i) \cdot c)^{n_i}}{n_i!} e^{-c \cdot V(F_i)} \right)$$

$$= \prod_{i=0}^{K} P(X^*(F_i) = n_i) \qquad (*)$$

Let $F_1, \dots, F_K \in \mathcal{B}(E)$ be disjoint and $F_0 := E \setminus (\bigcup_{i=1}^{K} F_i)$

$\Rightarrow E = \bigcup_{i=0}^{K} F_i$

$\Rightarrow$ for all $n_1, \dots, n_K \in \mathbb{N}_0$:

$$P(X^*(F_1) = n_1, \dots, X^*(F_K) = n_K) =$$

$$P(X^*(F_0) = n_0 = 0,1,2,\dots, X^*(F_1) = n_1, \dots, X^*(F_K) = n_K)$$

$$= \sum_{n_0 \geq 0} P(X^*(F_0) = n_0, X^*(F_1) = n_1, \dots, X^*(F_K) = n_K)$$

$$\overset{(*)}{=} \sum_{n_0 \geq 0} P(X^*(F_0) = n_0) \prod_{i=1}^{K} P(X^*(F_i) = n_i)$$

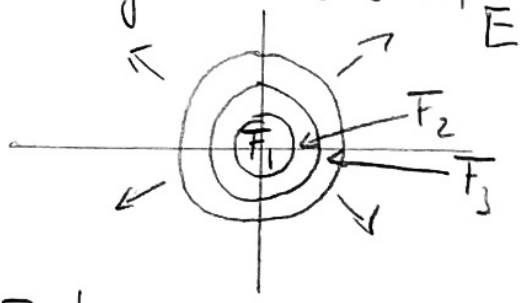
$$= \prod_{i=1}^{K} P(X^*(F_i) = n_i) \Rightarrow X^*(F_1), \dots, X^*(F_K) \text{ indep.}$$

$\Rightarrow X^* \sim PRM(\mu)$, if $\mu(E) < \infty$.

2. case: $\mu(E) = \infty$

Let F_K, $K \geq 1$ be relatively compact disjoint sets in E s.t.

$$E = \bigcup_{K \geq 1} F_K$$



$\mu \in M_+(E)$

$\Rightarrow \mu(F_K) < \infty$ for all $K \geq 1$

Define the finite measures μ_K on E by

$$\mu_K(A) = \mu(A \cap F_K)$$

Consider now (see 1. case) indep. $PRM(\mu_K)$ X_K^*, $K \geq 1$

(104) Define $X^*: \Omega \longrightarrow M_+(E)$ by

$$X^*(\omega) = \sum_{k\geq 1} X_k^*$$

Prop. 5.13, (5.12), F_i disjoint $\Rightarrow$ X^* rand. meas. with values in $M_p(E)$

$\Rightarrow$ X^* point process

Let $f \in C_0^+(E)$.

$$\Rightarrow \quad \Psi_{X^*}(f) = E[\exp(-X^*(f))]$$

$$= E[\exp(-\sum_{k\geq 1} \underbrace{X_k^*(f)}_{\geq 0})] \overset{\text{dom. convergence}}{=}$$

$$= \lim_{m\to\infty} E[\exp(-\sum_{k=1}^{m} X_k^*(f))] \quad \overset{X_k^*,\, k\geq 1 \text{ indep.}}{=} \quad \overset{\mu \mapsto \mu(f) \text{ continuous}}{=}$$

$$= \lim_{m\to\infty} \prod_{k=1}^{m} E[\exp(-X_k^*(f))]$$

$X_k^* \overset{\text{Th. 5.24}}{=} PRM(\mu_k)$,

$$\lim_{m\to\infty} \prod_{k=1}^{m} \exp\left(-\int_E (1-e^{-f(y)})\mu_k(dy)\right)$$

$$= \lim_{m\to\infty} \exp\left(-\sum_{k=1}^{m} \int_E (1-e^{-f(y)})\mu_k(dy)\right)$$

$$= \exp\left(-\sum_{k\geq 1} \int_E (1-e^{-f(y)}) 1_{F_k}(y)\mu(dy)\right)$$

$$\overset{\text{mon. conv.}}{=} \exp\left(-\int_E (1-e^{-f(y)}) \underbrace{\left(\sum_{k\geq 1} 1_{F_k}(y)\right)}_{=1} \mu(dy)\right)$$

$$= \exp\left(-\int_E (1-e^{-f(y)})\mu(dy)\right)$$

Th. 5.24 $\Rightarrow$ $X^* \sim PRM(\mu)$ $\Rightarrow$ proof.

In view of important applications in e.g. insurance mathematics or mathematical finance we now want to discuss two methods, which enable us to construct from a given Poisson random measure a new one, namely either

(i) through mapping of the points of a Poisson process

or

(ii) via "marking" of the points of a Poisson process

The first method is matter of the following result:

Th. 5.27 (1. method)
Let $E_1 \subseteq \mathbb{R}^{d_1}$, $E_2 \subseteq \mathbb{R}^{d_2}$ (or more generally E_1, E_2 are locally compact spaces with countable bases) and let $T: E_1 \longrightarrow E_2$ be a $\mathcal{B}(E_1), \mathcal{B}(E_2)$-measurable function, that is

$$\{x \in E_1 : T(x) \in B\} \in \mathcal{B}(E_1)$$

for all $B \in \mathcal{B}(E_2)$.
Assume X' is a $PRM(\mu)$, $\mu \in M_+(E_1)$.
For $\mu \in M_+(E_1)$ define the (image) measure μ_T on $\mathcal{B}(E_2)$ by

$$\mu_T(B) = \mu(\{x \in E_1 : T(x) \in B\}),$$

$B \in \mathcal{B}(E_2)$
Suppose that $\mu_T \in M_+(E_2)$ for all $\mu \in M_+(E_2)$.
Then X'_T is a $PRM(\mu_T)$.
So, if

$$X' = \sum_{i \geq 1} \varepsilon_{x_i},$$