then

$$X_T = \sum_{i \geq 1} \varepsilon_{T(X_i)}.$$

Proof: Exercise

Ex. 5.28: Let T_i, $i \geq 1$ be the arrival times of claims w.r.t. Cramér-Lundberg model

$\Rightarrow$ the claim number process $N(t)$ is a Poisson proc. with parameter $\lambda > 0$ and

$$N(t) = \sum_{i \geq 1} 1_{[0,t]}(T_i) = \sum_{i \geq 1} \varepsilon_{T_i}([0,t]) = X'([0,t]),$$

where $X' : \Omega \longrightarrow M_p(E_1)$, $E_1 = [0,\infty)$ given by

$$X' = \sum_{i \geq 1} \varepsilon_{T_i}$$

is a PRM(μ) with $\mu = \lambda \cdot m$ (m: Lebesgue-measure) (see Exercises)

If e.g. $T : E_1 \to E_2$ has a continuous inverse, then

$$X_T = \sum_{i \geq 1} \varepsilon_{T(T_i)}$$

is a PRM(μ_T) with $\mu_T = \lambda m_T$

Here T can be used to capture seasonal effects in fire insurance.

As for the 2. method ("marked point processes") we need some notions and an auxiliary result:

Def. 5.29 (Monotone vector space)

A monotone vector space $\mathcal{H}$ is a class of bounded functions $f : \Omega \longrightarrow \mathbb{R}$ such that

(i) $\mathcal{H}$ is a vector space over $\mathbb{R}$.

(ii) $1_\Omega \in \mathcal{H}$ (i.e. constant functions are in $\mathcal{H}$).

(iii) If $f_n \in \mathcal{H}$, $n \geq 1$ with $0 \leq f_n(x) \uparrow f(x)$ for $n \to \infty$ for all $x \in \Omega$ and if f is bounded, then $f \in \mathcal{H}$.

Def 5.30 (Multiplicative class)
A class $\mathcal{M}$ of functions $f: \Omega \to \mathbb{R}$ is called multiplicative, if $f, g \in \mathcal{M}$ entails that $f \cdot g \in \mathcal{M}$.

Th 5.31 (Monotone class theorem)
Let $\mathcal{H}$ be a monotone vector space and $\mathcal{M} \subseteq \mathcal{H}$ be multiplicative.
Then $\mathcal{H}$ contains all bounded $\mathcal{A}$-meas. functions $f: \Omega \to \mathbb{R}$, where

$\mathcal{A} := \sigma(\mathcal{M}) :=$ smallest σ-algebra which contains all sets of the form

$$\{x \in \Omega : f(x) \in L\}, \quad L \in \mathcal{B}(\mathbb{R}), \ f \in \mathcal{M}$$

Proof: See e.g. Dellacherie, Meyer: Probabilities and Potential. North Holland (1978).

Def 5.32 (Transition function)
Let $\mathcal{A}_i$ be a σ-algebra on Ω_i, $i = 1, 2$. Then a function $K: \Omega_1 \times \mathcal{A}_2 \to [0,1]$ is called a transition function from Ω_1 to Ω_2, if

(i) for all $F_2 \in \mathcal{A}_2$ $K(\cdot, F_2)$ is $\mathcal{A}_1$-meas.

(ii) for all $x \in \Omega_1$ $K(x, \cdot)$ is a probability measure on $\mathcal{A}_2$

Th. 5.33 (2. method: marked point processes)
Let E_1, E_2 as in Th. 5.27.
Suppose

$$\chi = \sum_{i \geq 1} \varepsilon_{x_i}$$

is a PRM(μ), $\mu \in M_+(E_1)$.
Let $K: E_1 \times \mathcal{B}(E_2) \to [0,1]$ be a transition function from E_1 to E_2.

Assume r.v.'s $\eta_i : \Omega \to E_2$, $i \geq 1$, which are conditionally independent given $X_i, i \geq 1$:

$$P(\eta_i \in F_2 \mid X_j, j \geq 1, \eta_\alpha, \alpha \neq i) = K(X_i, F_2) \quad (5.16)$$

for all i and $F_2 \in \mathcal{B}(E_2)$.

Then (X_i "marked" by η_i)

$$X^* := \sum_{i \geq 1} \varepsilon_{(X_i, \eta_i)}$$

is a PRM(μ^*) on $E_1 \times E_2$ with $\mu^* \in M_+(E_1 \times E_2)$ given by

$$\mu^*(B) = \int_{E_1 \times E_2} 1_B(x,y)\, K(x, dy)\, \mu(dx), \quad B \in \mathcal{B}(E_1) \otimes \mathcal{B}(E_2)$$

Ex. 5.34 (claim number process marked by claim delays)

Let X be the Poisson random measure associated with the claim number process $N(t), t \geq 0$ in Ex. 5.28. Assume the claims, which arrive at $X_i = T_i$, $i \geq 1$ are reported to the insurance with delays η_i. Suppose that $\eta_i, i \geq 1$ are i.i.d. with common law $F = F_{\eta_1}$ and that $\eta_i, i \geq 1$ is indep. of $X_i, i \geq 1$. Then it follows from Th. 5.33 that

$$X^* = \sum_{i \geq 1} \varepsilon_{(X_i, \eta_i)}$$

is a PRM(μ^*) with $K = F$ and μ^* given by

$$\mu^*(B) = \int_{[0,\infty) \times [0,\infty)} 1_B(x,y)\, dF(y)\, \lambda\, m(dx)$$

for $B \in \mathcal{B}([0,\infty) \times [0,\infty))$

So, if e.g. η_i is uniformly distr. on $[0,1]$, then we can define $T : [0,\infty) \times [0,\infty) \to [0,\infty)$, $(x,y) \mapsto x+y$ and obtain from Th. 5.27 that

$$X_T^* = \sum_{i \geq 1} \varepsilon_{T(X_i, \eta_i)} = \sum_{i \geq 1} \varepsilon_{X_i + \eta_i}$$

is a PRM(ν) with $\nu = (\mu^*)_T$

The proof Th. 5.33 is based on the following two auxiliary results:

L. 5.35 : Assume that (5.16) holds. Then

$$E[f(X_i,Y_i) \mid X_j, j\geq 1] = \int_{E_2} f(X_i,y)\, K(X_i,dy) \text{ a.e. } (5.17)$$

for all bounded meas. $f : E_1 \times E_2 \longrightarrow \mathbb{R}$.

Proof : Denote by $\mathcal{H}$ the class of all bounded meas. $f : E_1 \times E_2 \longrightarrow \mathbb{R}$, satisfying the equation (5.17).

Using dominated convergence, we directly see that $\mathcal{H}$ is a monotone vector space (Def. 5.29)

Consider now the function f given by

$f(x,y) = 1_A(x) \cdot 1_B(y)$ for $A \in \mathcal{B}(E_1)$, $B \in \mathcal{B}(E_2)$

$$\Rightarrow E[f(X_i,Y_i) \mid X_n, n\geq 1]$$

$$\overset{\text{lower prop. of cond. expect.}}{=} E[E[\underbrace{f(X_i,Y_i)}_{= 1_A(X_i)\cdot 1_B(Y_i)} \mid X_n, n\geq 1, Y_\alpha, \alpha \neq i] \mid X_n, n\geq 1]$$

$$= E[1_A(X_i) \cdot \underbrace{E[1_B(Y_i) \mid X_n, n\geq 1, Y_\alpha, \alpha\neq i]}_{= P(Y_i \in B \mid X_n, n\geq 1, Y_\alpha, \alpha \neq i)} \mid X_n, n\geq 1]$$

$$\overset{(5.16)}{=} E[1_A(X_i)\cdot K(X_i,B) \mid X_n, n\geq 1]$$

$$= 1_A(X_i)\, K(X_i,B) = \underbrace{1_A(X_i)}_{E_2} \cdot \int 1_B(y)\, K(X_i,dy)$$

$$= \int_{E_2} 1_A(X_i)\cdot 1_B(y)\, K(X_i,dy) = \int_{E_2} f(X_i,y)\, K(X_i,dy)$$

Recall : X, Y r.v., Y $\mathcal{A}$-meas. $\Rightarrow$

$$E[Y\cdot X \mid \mathcal{A}] = Y \cdot E[X \mid \mathcal{A}]$$

(110) $\Rightarrow f = 1_A \cdot 1_B \in \mathcal{H}$ for all $A \in \mathcal{B}(E_1)$, $B \in \mathcal{B}(E_2)$

$\Rightarrow$ the set $\mathcal{M}$ of all finite linear combinations of such f is contained in $\mathcal{H}$

The class $\mathcal{M}$ is obviously multiplicative (Def. 5.27)

Th. 5.28 $\Rightarrow$ all bounded $\mathcal{B}(\mathcal{M})$-meas. functions $f: E_1 \times E_2 \to \mathbb{R}$ are contained in $\mathcal{H}$

However $\mathcal{B}(E_1) \otimes \mathcal{B}(E_2) = \mathcal{B}(\mathcal{M})$.

$\Rightarrow$ proof.

L. 5.36 : Assume that (5.16) holds.

If $g: E_1 \times E_2 \to [0,1]$ is meas. then

$$E\Big[\prod_{i\geq 1} g(X_i, Y_i) \,\Big|\, X_n, n \geq 1\Big] = \prod_{i\geq 1} E[g(X_i, Y_i) \,|\, X_n, n \geq 1] \quad \text{a.e.}$$

Proof : Very similar to the proof of L. 5.35.

Proof of Th. 5.33 (Marked point processes)

Let $f: E_1 \times E_2 \to [0, \infty)$ be measurable.

$$\Rightarrow \Psi_{X^*}(f) \overset{\text{Def}}{=} E[\exp(-X^*(f))]$$

$$\overset{\text{Def}}{=} \int_{E_1 \times E_2} f(x,y) \underbrace{X^*(\omega, dx, dy)}_{= \sum_{i\geq 1} \varepsilon_{(X_i, Y_i)}(dx, dy)}$$

$$= E\Big[\exp\Big(-\sum_{i\geq 1} \underbrace{\int_{E_1 \times E_2} f(x,y)\, d\varepsilon_{(X_i,Y_i)}(dx,dy)}_{= f(X_i, Y_i)}\Big)\Big]$$

$$= E\Big[\exp\Big(-\sum_{i\geq 1} f(X_i, Y_i)\Big)\Big]$$

$$= E\Big[\prod_{i\geq 1} \exp(-f(X_i, Y_i))\Big]$$

$$= E\Big[E\Big[\prod_{i\geq 1} \exp(-f(X_i, Y_i)) \,\Big|\, X_n, n \geq 1\Big]\Big]$$

$$\overset{\text{L. 5.36}}{=} E\Big[\prod_{i\geq 1} E[\exp(-f(X_i, Y_i)) \,|\, X_n, n \geq 1]\Big]$$

Recall:

$$\int_E g(z)\, \varepsilon_x(dz) = g(x)$$

and

$$E[E[X \,|\, \mathcal{M}]] = E[X]$$

(111) $\overset{(5.17)}{=} E\left[\prod_{i\geq 1} \int_{E_2} e^{-f(X_i,y)} K(X_i, dy)\right]$

Set $\theta(X_i) = \int_{E_2} e^{-f(X_i,y)} K(X_i, dy) \Rightarrow 0 < \theta(X_i) \leq 1$

$$\Rightarrow \Psi_{X^*}(f) = E\left[\prod_{i\geq 1} \theta(X_i)\right] = E\left[\exp\left(-\sum_{i\geq 1}(-\log(\theta(X_i)))\right)\right]$$

$$= E\left[\exp\left(-\int_{E_1}(-\log(\theta(x)))X(\omega, dx)\right)\right]$$

$$= \Psi_X(-\log\theta)$$

$X \sim PRM(\mu)$

$\overset{\text{Th. 5.24}}{\Rightarrow} \Psi_{X^*}(f) = \exp\left(-\int_{E_1}(1-e^{-(-\log\theta(x))})\mu(dx)\right)$

$$= \exp\left(-\int_{E_1}(1-\theta(x))\mu(dx)\right)$$

$$= \exp\left(-\int_{E_1}\left(1-\int_{E_2} e^{-f(x,y)}K(x,dy)\right)\mu(dx)\right)$$

$K(x_1,\cdot)$ prob. meas. $\exp\left(-\int_{E_1\times E_2}(1-e^{-f(x,y)})K(x,dy)\mu(dx)\right)$

Monotone class Theorem (Th. 5.31) $\exp\left(-\int_{E_1\times E_2}(1-e^{-f(x,y)})\mu^*(dx,dy)\right)$,

where $\mu^*(B) = \int_{E_1\times E_2} 1_B(x,y)K(x,dy)\mu(dx)$

for $B \in \mathcal{B}(E_1)\times\mathcal{B}(E_2)$. $\Rightarrow$ proof.

After having studied Poisson random measures we can now discuss the link between the convergence of extremal events and that of point processes, which is in fact matter of the next result

Prop. 5.37. Let $X_{n,j} \in E$, $j \geq 1$ be a sequence of i.i.d. r.v.'s for all $n \geq 1$ and let μ be a Radon measure on E (i.e. $\mu(K) < \infty$ for all compact $K \subset E$)
Consider the point processes

$$\xi_n := \sum_{j \geq 1} \varepsilon_{(jn^{-1}, X_{n,j})}, \quad n \geq 1$$

and a PRM($\lambda \times \mu$) ξ on $\hat{E} = [0,\infty) \times E$ (λ: Lebesgue-meas.)
Then

$$\xi_n \xrightarrow[n\to\infty]{v} \xi$$

iff

$$n \cdot P_{X_{n,1}} \xrightarrow[n\to\infty]{v} \mu \quad \text{in } M_+(E) \tag{5.18}$$

Here $P_{X_{n,1}}$ is the (image) measure (of $X_{n,1}$) defined by $P_{X_{n,1}}(B) = P(X_{n,1} \in B)$, $B \in \mathcal{B}(E)$.

Rem 5.38: Prop. 5.37 can be regarded as a version of Prop. 3.2.11 (Poisson approximation) in the setting of point processes.

Proof of Prop. 5.37:

(i) Let us first show the following: Let N be a PRM(μ). Then

$$N_n := \sum_{j=1}^{n} \varepsilon_{X_{n,j}} \xrightarrow{v} N \quad \text{in } M_p(E) \tag{5.19}$$

iff

(5.18) holds

(13) "$\Leftarrow$" : Let $f \in C_0^+(E)$. Then

$$\Psi_{N_n}(f) \overset{\text{Def}}{=} E[\exp(-N_n(f))] = E[\exp(-\int_E f(y) N_n(\omega, dy))]$$

$$\overset{\int g(y)d\varepsilon_{x_i} = g(x_i)}{=} E[\exp(-\sum_{j=1}^{n} f(X_{n,j}))] \overset{X_{n,j}, j\geq 1 \text{ i.i.d.}}{=} (E[\exp(-f(X_{n,1}))])^n$$

$$\overset{\text{Ex. 2.3.9}}{=} \left(\int_E \exp(-f(x)) P_{X_{n,1}}(dx)\right)^n \overset{\int_E 1 P_{X_{n,1}}(dx) = 1}{=} \left(1 - \int_E (1-\exp(-f(x))) P_{X_{n,1}}(dx)\right)^n$$

Def. 5.2 and cond. (5.18)

$$\overset{n\to\infty}{\longrightarrow} \int_E (1-e^{-f(x)})\mu(dx)$$

$$= \left(1 - \frac{\int_E (1-\exp(-f(x)))(n \cdot P_{X_{n,1}}(dx))}{n}\right)^n$$

$(1-\frac{x}{n})^n$ converges uniformly on compact intervals

Th. 5.18 $\Longrightarrow$

$$\exp(-\int_E (1-e^{-f(x)})\mu(dx)) \overset{\text{Th. 5.24}}{=} \Psi_N(f)$$

$$N_n \xrightarrow[n\to\infty]{v} N$$

"$\Longrightarrow$" : Assume that N is a PRM(μ) and $N_n \xrightarrow[n\to\infty]{v} N$.

Th. 5.18, Th. 5.24 $\Longrightarrow$

$$\left(1 - \frac{\int_E (1-\exp(-f(x)))(n P_{X_{n,1}}(dx))}{n}\right)^n \xrightarrow[n\to\infty]{} \exp(-\int_E (1-e^{-f(x)})\mu(dx))$$

$=: a_n$

Suppose there ex. a subseq. $n_k, k\geq 1$ s.t. $a_{n_k} \xrightarrow[k\to\infty]{} \infty$

Let $b > 0$ s.t. $a_{n_k} > b$ for all $k \geq k_0$

$(1-\frac{x}{n_k})^{n_k}$ decreasing in $x \leq n_k$ for all k

$$\Longrightarrow \left(1-\frac{b}{n_k}\right)^{n_k} \geq \left(1-\frac{a_{n_k}}{n_k}\right)^{n_k} \quad (\text{since } \tfrac{a_{n_k}}{n_k} \leq 1)$$

$$\overset{k\to\infty}{\Longrightarrow} e^{-b} \geq \underbrace{\lim_{k\to\infty}\left(1-\frac{a_{n_k}}{n_k}\right)^{n_k}}_{= \exp(-\int_E (1-e^{-f(x)})\mu(dx))} \quad \text{for all } b > 0$$

$\Longrightarrow$ ↯ (contradiction)

$\Longrightarrow$ $a_n, n\geq 1$ is bounded $\Longrightarrow$ $a_n, n\geq 1$ has a convergent subseq. w.l.o.g. say $a_n \xrightarrow[n\to\infty]{} a$

$$\Longrightarrow \left(1-\frac{a_n}{n}\right)^n \xrightarrow[n\to\infty]{} e^{-a} = \exp(-\int_E (1-e^{-f(x)})\mu(dx))$$

$$\Longrightarrow a = \int_E (1-e^{-f(x)})\mu(dx)$$

(14) $\Rightarrow \int_E (1-\exp(-f(x)))\,(n P_{X_{n,1}}(dx)) \xrightarrow[n\to\infty]{} \int_E (1-\exp(-f(x)))\,\mu(dx)$

Let $g \in C_0^+(E)$ with $\sup_{x\in E} g(x) < 1$ and choose $f(x) = -\log(1-g(x)) \in C_0^+(E)$

$\Rightarrow 1-\exp(-f(x)) = g(x)$ for all x

$\Rightarrow \int_E g(x)\,(n P_{X_{n,1}}(dx)) \xrightarrow[n\to\infty]{} \int_E g(x)\,\mu(dx)$

for all $g \in C_0^+(E)$ with $\sup_{x\in E} g(x) < 1$

If $b := \sup_{x\in E} g(x) > 1$ then set $g^*(x) = \frac{g(x)}{b+1} < 1$

$\Rightarrow \int_E \underbrace{g^*(x)}_{\frac{g(x)}{b+1}}\,(n P_{X_{n,1}}(dx)) \xrightarrow[n\to\infty]{} \int_E g^*(x)\,\mu(dx)$

$\cdot(b+1)$ on both sides $\Rightarrow$

$\int_E g(x)\,(n P_{X_{n,1}}(dx)) \xrightarrow[n\to\infty]{} \int_E g(x)\,\mu(dx)$

Def. 5.2 $\Rightarrow$ $n P_{X_{n,1}} \xrightarrow[n\to\infty]{v} \mu \Rightarrow$ (5.18) holds.

(ii) "$\Rightarrow$" Th 5.18: Assume that $\zeta_n \xrightarrow[n\to\infty]{v} \zeta$.

$\Rightarrow \Psi_{\zeta_n}(f) \xrightarrow[n\to\infty]{} \Psi_\zeta(f)$ for all $f \in C_0^+(\overbrace{[0,\infty)\times E}^{\widehat{E}})$

Choose $h_\ell^-, h_\ell^+ \in C_0^+([0,\infty))$ s.t. $h_\ell^-(s) \uparrow \mathbb{1}_{[0,1]}(s)$, $h_\ell^+(s) \downarrow \mathbb{1}_{[0,1]}(s)$ for all s and $g \in C_0^+(E)$ ($\in C_0^+([0,\infty)\times E)$, $\in C_0^+([0,\infty)\times E)$, $\in C_0^+([0,\infty)\times E)$)

$\Rightarrow \Psi_{\zeta_n}(h_\ell^- \cdot g) \le \Psi_{\zeta_n}(\mathbb{1}_{[0,1]} \cdot g) \le \Psi_{\zeta_n}(h_\ell^+ \cdot g)$

for ℓ, n

$n\to\infty \Rightarrow \Psi_\zeta(h_\ell^- \cdot g) \le \overline{\lim_{n\to\infty}} \Psi_{\zeta_n}(\mathbb{1}_{[0,1]} g) \le \underline{\lim_{n\to\infty}} \Psi_{\zeta_n}(\mathbb{1}_{[0,1]} \cdot g) \le \Psi_\zeta(h_\ell^+ \cdot g)$

for all ℓ

$\ell\to\infty \Rightarrow$ dom. conv. $\Psi_\zeta(\mathbb{1}_{[0,1]} \cdot g) \le \overline{\lim_{n\to\infty}} \Psi_{\zeta_n}(\mathbb{1}_{[0,1]} \cdot g) \le \underline{\lim_{n\to\infty}} \Psi_{\zeta_n}(\mathbb{1}_{[0,1]} g) \le \Psi_\zeta(\mathbb{1}_{[0,1]} g)$

$\Rightarrow \lim_{n\to\infty} \Psi_{\zeta_n}(\mathbb{1}_{[0,1]} \cdot g) = \Psi_\zeta(\mathbb{1}_{[0,1]} \cdot g)$ (5.20)

However

$$\Psi_{\xi_n}(\mathbb{1}_{[0,1]}\cdot g) = E\Big[\exp\Big(-\int_{[0,\infty)\times E} \mathbb{1}_{[0,1]}(s)\, g(x)\, \xi_n(\omega, ds, dx)\Big)\Big]$$

$\frac{k}{n} \in [0,1]$

$$= E\Big[\exp\Big(-\sum_{k=1}^{n} g(X_{n,k})\Big)\Big] = \Psi_{N_n}(g)$$

$\int f(y)\, d\varepsilon_{x_i} = f(x_i)$

On the other hand,

$$\Psi_{\xi}(\mathbb{1}_{[0,1]} g) = \exp\Big(-\int_0^1 \int_E (1-e^{-g(x)})\, \mu(dx)\, ds\Big)$$

$$= \exp\Big(\int_E (1-e^{-g(x)})\, \mu(dx)\Big) = \Psi_N(g)$$

(5.19) $\Rightarrow$ $n P_{X_{n,1}} \xrightarrow[n\to\infty]{} \mu$

"$\Leftarrow$" : Assume that $n P_{X_{n,1}} \xrightarrow[n\to\infty]{} \mu$

We have for $f \in C_0^+([0,\infty)\times E)$ that

$$\Psi_{\xi_n}(f) = E\Big[\exp\Big(-\int_{[0,\infty)\times E} f(s,x)\, \xi_n(\omega, ds, dx)\Big)\Big]$$

$$= E\Big[\exp\Big(-\sum_k f\Big(\tfrac{k}{n}, X_{n,k}\Big)\Big)\Big] \overset{X_{n,k},\, k\geq 1}{\underset{\text{i.i.d}}{=}}$$

$$\prod_{k\geq 1}\Big(1 - \int_E (1-e^{-f(\frac{k}{n},x)})\, P_{X_{n,1}}(dx)\Big)$$

So, if $\Psi_{\xi_n} \xrightarrow[n\to\infty]{} \Psi_\xi$ then equivalently

$$-\log \Psi_{\xi_n}(f) = -\sum_{k\geq 1} \log\Big(1 - \int_E (1-e^{-f(\frac{k}{n},x)})\, P_{X_{n,1}}(dx)\Big) \xrightarrow[n\to\infty]{} \log \Psi_\xi(f) \qquad (+)$$

Define λ_n by

$$\lambda_n = \sum_{k\geq 1} \varepsilon_{\frac{k}{n}} \times P_{X_{n,1}}$$

$$\Rightarrow \int_{[0,\infty)\times E} f(s,x)\, \lambda_n(ds, dx) = \int_E \Big(\sum_{k\geq 1} f\Big(\tfrac{k}{n}, x\Big)\cdot \tfrac{1}{n}\Big)\, (n P_{X_{n,1}}(dx)) \qquad (*)$$

There ex. a compact $K \subset E$ and $a, b \geq 0$ s.t. f vanishes outside $[a,b]\times K$

w.l.o.g let $a=0$ and $b=1$

Since $[a,b]\times K$ is compact in $[0,\infty)\times E$, f is uniformly continuous on $[a,b]\times K$

$\Rightarrow$ the function $\hat{f}: [0,1] \to C(K)$;
$s \mapsto (x \mapsto f(s,x))$ is continuous

Riemann-sum $\Rightarrow$ $\sum_{k=1}^{n} f(\frac{k}{n}, x)\frac{1}{n} \xrightarrow[n\to\infty]{} \int_0^1 f(s,x)\,ds$ uniformly in $x \in K$

$$\Rightarrow \left| \int_E \left(\sum_{k=1}^{n} f\left(\tfrac{k}{n}, x\right) \cdot \tfrac{1}{n} \right) (n P_{X_{n,1}}(dx)) - \int_{[0,\infty)\times E} f(s,x)\,\lambda(ds)\times\mu(dx) \right|$$

$$\leq \left| \int_E \left(\sum_{k=1}^{n} f\left(\tfrac{k}{n}, x\right) \cdot \tfrac{1}{n} - \int_0^1 f(s,x)\,ds \right) n P_{X_{n,1}}(dx) \right|$$

$$+ \left| \int_E \int_0^1 f(s,x)\,ds\,(n P_{X_{n,1}}(dx)) - \int_{[0,\infty)\times E} f(s,x)\,\lambda(ds)\times\mu(dx) \right| \quad (**)$$

Now let $g \in C_0^+(E)$ s.t. $g(x) = 1$ for all $x \in K$

$$\Rightarrow (**) \leq \underbrace{\sup_{x\in K} \left| \sum_{k=1}^{n} f\left(\tfrac{k}{n}, x\right)\cdot\tfrac{1}{n} - \int_0^1 f(s,x)\,dx \right|}_{\xrightarrow{C_0^+(E)\ni\psi,\ n\to\infty} 0} \underbrace{\int_E g(x)\, n P_{X_{n,1}}(dx)}_{\xrightarrow[n\to\infty]{(5.18)} \int_E g(x)\mu(dx)}$$

$$+ \left| \int_E \left(\int_0^1 f(s,x)\,ds \right) (n P_{X_{n,1}}(dx)) - \int_E \left(\int_0^1 f(s,x)\,ds \right) \mu(dx) \right|$$

Lebesgue meas. $\xrightarrow[n\to\infty]{(5.18)} 0$

$$\Rightarrow \quad \lambda_n \xrightarrow{V} \lambda \times \mu$$

So, in particular

$$\sum_{k\geq 1} \int_E \left(1 - e^{-f(\frac{k}{n}, x)}\right) P_{X_{n,1}}(dx) \overset{\text{Def.}}{=}$$

$$\int_{[0,\infty)\times E} \underbrace{\left(1 - e^{-f(s,x)}\right)}_{\in C_0^+([0,\infty)\times E)} \lambda_n(ds, dx) \xrightarrow{n\to\infty} \int_0^\infty \int_E \left(1 - e^{-f(s,x)}\right) ds\,\mu(dx) \quad (++)$$

On the other hand, we know that f vanishes outside $[a,b] \times K$ for a compact set $K \subset E$.
Let $g \in C_0^+(E)$ s.t. $g(x) = 1$ for all $x \in K$

(17) Then

$$\sup_{k\geq 1} \int_E \left(1-e^{-f\left(\frac{k}{n},x\right)}\right) P_{X'_{n,1}}(dx)$$

$$= \sup_{k\geq 1} \int_E \underbrace{\left(1-e^{-f\left(\frac{k}{n},x\right)}\right)}_{\leq 1} g(x)\, P_{X'_{n,1}}(dx)$$

$$\leq \int_E g(x)\, P_{X'_{n,1}}(dx) = \frac{1}{n} \underbrace{\int_E g(x)\,\left(n\, P_{X'_{n,1}}(dx)\right)}_{\xrightarrow[n\to\infty]{(5.18)} \int_E g(x)\,\mu(dx)} \xrightarrow[n\to\infty]{} 0 \qquad (+++)$$

Taylor's expansion gives

$$\log(1+z) - z = z\,\varepsilon(z), \quad |z| \leq \frac{1}{2}, \qquad (5.21)$$

where $|\varepsilon(z)| \leq |z|$

$$\Rightarrow \left| -\log \Psi_{\xi_n}(f) - \sum_{k\geq 1} \int_E \left(1-e^{-f\left(\frac{k}{n},x\right)}\right) P_{X'_{n,1}}(dx) \right|$$

$$\overset{(+)}{=} \left| \sum_{k\geq 1} \left\{ \log\left(1 - \underbrace{\int_E \left(1-e^{-f\left(\frac{k}{n},x\right)}\right) P_{X'_{n,1}}(dx)}_{=z}\right) + \int_E \left(1-e^{-f\left(\frac{k}{n},x\right)}\right) P_{X'_{n,1}}(dx) \right\} \right|$$

$$\overset{\substack{n \text{ large enough,} \\ (+++)}}{\leq} \sum_{k\geq 1} \left| \int_E \left(1-e^{-f\left(\frac{k}{n},x\right)}\right) P_{X'_{n,1}}(dx) \right|^2$$

$$\leq \underbrace{\left\{ \sup_{k\geq 1} \int_E \left(1-e^{-f\left(\frac{k}{n},x\right)}\right) P_{X'_{n,1}}(dx) \right\}}_{\xrightarrow[n\to\infty]{(+++)} 0} \cdot \underbrace{\sum_{k\geq 1} \int_E \left(1-e^{-f\left(\frac{k}{n},x\right)}\right) P_{X'_{n,1}}(dx)}_{\xrightarrow[n\to\infty]{(++)} \int_0^\infty \int_E \left(1-e^{-f(s,x)}\right) ds\,\mu(dx)}$$

$$\xrightarrow[n\to\infty]{} 0$$

$$\overset{(++)}{\Rightarrow} \Psi_{\xi_n}(f) \longrightarrow \Psi_\xi(f) \quad \text{for all } f \in C_0^+([0,\infty)\times E),$$

where ξ is a PRM$(\lambda\times\mu)$

$\Rightarrow$ proof.

Rem 5.39 : The proof of Prop. 5.37 shows that the results there also hold, if ξ_n is given by a restriction

$$\xi_n = \sum_{j \geq 1} \varepsilon_{(j^{n-1}, X'_{n,j})} \big|_{(0,1] \times E^*}$$

on $(0,1] \times E^* \subseteq [0,\infty) \times E$ (when $C_0^+((0,1] \times E^*) \subseteq C_0^+([0,\infty) \times E)$).

Ex 5.40 (point process of exceedances N_n)

Recall that

$$N_n = \sum_{j=1}^{n} \varepsilon_{j^{n-1}} \mathbb{1}_{\{X'_j > u_n\}}.$$

Consider

$$\xi_n = \sum_{j \geq 1} \varepsilon_{(j^{n-1}, X'_{n,j})} \big|_{(0,1] \times E^*}$$

on $(0,1] \times E^*$ for $E^* = (0,2)$ and

$$X'_{n,j} = \mathbb{1}_{\{X'_j > u_n\}} \quad \text{i.i.d.}$$

Let $h \in C_0^+((0,1] \times E^*)$

$$\Rightarrow \Psi_{\xi_n}(h(\cdot,\cdot)) \overset{\text{def.}}{=} E\Big[\exp\Big(-\int h(s,x)\, d\Big(\underbrace{\sum_{j \geq 1} \varepsilon_{(j^{n-1}, X'_{n,j})}}_{=\xi_n}\Big)\Big]$$

$$= E\Big[\exp\Big(-\sum_{j \geq 1} h(j^{n-1}, X'_{n,j})\, \mathbb{1}_{(0,1] \times (0,2)}(j^{n-1}, X'_{n,j})\Big)\Big]$$

$$= E\Big[\exp\Big(-\sum_{j=1}^{n} h(j^{n-1}, 1)\, \mathbb{1}_{\{X'_j > u_n\}}\Big)\Big] \overset{\text{def.}}{=} \Psi_{N_n}(h(\cdot,1))$$

$$\Rightarrow \Psi_{\xi_n}(h(\cdot,\cdot)) = \Psi_{N_n}(h(\cdot,1)) \quad \text{for all } h \in C_0^+((0,1] \times E^*)$$

On the other hand, we know from Prop. 5.37 and Rem. 5.39 that

$$\xi_n \xrightarrow[n \to \infty]{v} \xi$$

$$\Longleftrightarrow$$

$$n \cdot P_{X'_{n,j}} \xrightarrow{v} \mu^* \text{ on } E^*$$

$$\overset{\text{Th. 5.5}}{\Longleftrightarrow}$$

(119)

$$n \cdot P_{X'_{n,j}}(B) \xrightarrow[n\to\infty]{} \mu^*(B) \quad \text{for all } B \text{ with } \overline{B} \text{ compact}$$

and $\mu^*(\partial B) = 0$ (*)

However, $n \cdot P_{X'_{n,j}}(B) = n \cdot P(X_j > u_n)\, \delta_1(B)$ for $B \subset \mathcal{B}(E^*)$

$\Rightarrow \mu^*(B) = \tau \cdot \delta_1(B)$ for all B as in (*)

(δ_1: Dirac measure on 1)

$\Rightarrow$ (*) $\Longleftrightarrow$

$$n \overline{F}_{X_1}(u_n) \xrightarrow[n\to\infty]{} \tau$$