$$n \cdot P_{X'_{n,j}}(B) \xrightarrow[n\to\infty]{} \mu^*(B) \quad \text{for all } B \text{ with } \overline{B} \text{ compact and } \mu^*(\partial B) = 0 \qquad (*)$$

However, $n \cdot P_{X'_{n,j}}(B) = n \cdot P(X'_j > u_n)\, \delta_1(B)$ for $B \subset \mathcal{B}(E^*)$

$$\Rightarrow \mu^*(B) = \tilde{\tau} \cdot \delta_1(B) \text{ for all } B \text{ as in } (*)$$

(δ_1: Dirac measure in 1)

$$\Rightarrow \quad (*) \iff$$

$$n \overline{F}_{X'_1}(u_n) \xrightarrow[n\to\infty]{} \tilde{\tau}$$

→ **Prop. 5.41** (Weak convergence of N_n)

Let N_n be a point process of exceedances w.r.t. a sequence of i.i.d. r.v.'s $X'_j, j \geq 1$ and let $\tilde{\tau} > 0$.

Then the following statements are equivalent

(i) $n \overline{F}_{X'_1}(u_n) \xrightarrow[n\to\infty]{} \tilde{\tau}$

(ii) $N_n \xrightarrow[n\to\infty]{v} N$ in $M_p((0,1])$, where N is a $PRM(\tilde{\tau} \cdot \lambda)$

(iii) $P(M_n \leq u_n) \xrightarrow[n\to\infty]{} e^{-\tilde{\tau}}$ for $M_n = \max(X'_1, \dots, X'_n)$

(iv) $P(X'_{(n-k+1)} \leq u_n) \xrightarrow[n\to\infty]{} e^{-\tilde{\tau}} \sum_{i=0}^{k-1} \frac{\tilde{\tau}^i}{i!}$ for all $k \geq 1$

(k-th largest order statistics; $X'_{(1)} \leq \dots \leq X'_{(n)}$ order statistics)

Proof:

(i) $\iff$ (ii): follows from Ex. 5.40

(ii) $\Rightarrow$ (iii): Using an approximation argument as in (5.20) we see that

$$N_n((0,1]) \xrightarrow[n\to\infty]{d} N((0,1]) \sim Pois(\tilde{\tau})$$

Th. 5.5 (ii) $\overset{\text{for } E=\mathbb{N}_0}{\Longrightarrow}$ (iii)

$$P(M_n \le u_n) \overset{(5.1)}{=} P(N_n((0,1]=0) \xrightarrow[n\to\infty]{} P(N((0,1]=0) = e^{-\tau}$$

(iii) $\Rightarrow$ (i): $P(M_n \le u_n) = (\overline{F}(u_n))^n$

$$= e^{n \log(1-\overline{F}(u_n))}$$

We know from (5.21) that

$$\log(1+z) = z\cdot(1+\varepsilon(z)),\ |z| \le \tfrac{1}{2},\ |\varepsilon(z)| \le |z|$$

$$\Rightarrow e^{n\log(1-\overline{F}(u_n))} \overset{\substack{n \text{ large enough},\\ \overline{F}(u_n)\to 0}}{=} e^{-n\overline{F}(u_n)(1+\varepsilon(-\overline{F}(u_n)))}$$

$$\xrightarrow[n\to\infty]{} e^{-\tau} \text{ by assumption}$$

$$\Rightarrow n\overline{F}(u_n)\underbrace{(1+\varepsilon(-\overline{F}(u_n)))}_{\xrightarrow[n\to\infty]{} 1} \xrightarrow[n\to\infty]{} \tau$$

$\Rightarrow$ (i)

(iv) $\Rightarrow$ (i): We know from (5.2) that

$$\{X_{(n-k+1)} \le u_n\} = \{N_n((0,1]) < k\}$$

We also observe that $N_n((0,1]) \sim \mathrm{Bin}(n, \overline{F}(u_n))$

$$\Rightarrow P(X_{(n-k+1)} \le u_n) = \sum_{i=0}^{k-1} \binom{n}{i} (\overline{F}(u_n))^i (\overline{F}(u_n))^{n-i},\ k \ge 1$$

$$\overset{k=1}{\Rightarrow} (\overline{F}(u_n))^n \xrightarrow[n\to\infty]{} e^{-\tau} \Rightarrow \overline{F}(u_n) \xrightarrow[n\to\infty]{} 1$$

(iv)

$$\Rightarrow \sum_{i=0}^{k-1} \frac{n!}{(n-i)!\,n^i} \cdot \frac{1}{i!} (n\overline{F}(u_n))^i \underbrace{(\overline{F}(u_n))^{n-i}}_{= (\overline{F}(u_n))^n (\overline{F}(u_n))^{-i}}$$

$$\overset{n\ge n_0}{\le} (1+\varepsilon)\underbrace{(\overline{F}(u_n))^n}_{e^{n\cdot\log(1-\overline{F}(u_n))}} \sum_{i=0}^{k-1} \frac{(n\overline{F}(u_n))^i}{i!}$$

$$= e^{-n\overline{F}(u_n)(1+\varepsilon(-\overline{F}(u_n)))} \sum_{i=0}^{k-1} \frac{(n\overline{F}(u_n))^i}{i!}$$

$$\overset{\substack{n\ge\hat{n}_0,\\ (5.21)}}{=} (1+\varepsilon)\cdot e^{-n\overline{F}(u_n)(1+\varepsilon(-\overline{F}(u_n)))} \sum_{i=0}^{k-1} \frac{(n\overline{F}(u_n))^i}{i!}$$

$$\overset{n\ge n^*}{\le} (1+\varepsilon^*)\, e^{-n\overline{F}(u_n)} \sum_{i=0}^{k-1} \frac{(n\overline{F}(u_n))^i}{i!}$$

Similarly, we can show a lower estimate with the factor $(1-\varepsilon^*)$ for $n \ge n^*(\varepsilon^*)$, $\varepsilon^* > 0$ arbitrary

(121) $\Rightarrow P(X_{(n-K+1)} \le u_n) \sim e^{-n\bar{F}(u_n)} \sum_{i=0}^{K-1} \frac{(n\bar{F}(u_n))^i}{i!}$ (*)

for $n \to \infty$

On the other hand, the function f_K given by

$$f_K(x) = e^{-x} \sum_{i=0}^{K-1} \frac{x^i}{i!}, \quad x > 0$$

is continuous and strictly decreasing

$\Rightarrow f_K^{-1}$ exists and is continuous

From (*) it follows that

$$f_K(n\bar{F}(u_n)) \xrightarrow[n\to\infty]{} f_K(\tau)$$

f_K^{-1} cont. $\Rightarrow$ $n\bar{F}(u_n) \xrightarrow[n\to\infty]{} \tau \quad \Rightarrow$ (i)

(i) $\Rightarrow$ (iv): From $n\bar{F}(u_n) \xrightarrow[n\to\infty]{} \tau$ it follows that

$$f_K(n\bar{F}(u_n)) \xrightarrow[n\to\infty]{} f_K(\tau)$$

$\overset{(*)}{\Rightarrow}$ $P(X_{(n-K+1)} \le u_n) \xrightarrow[n\to\infty]{} f_K(\tau) \Rightarrow$ (iv)

$\Rightarrow$ proof.

Prop. 5.41 $\Rightarrow$ Corollary 5.42: Let X_i, $i \ge 1$ be an i.i.d. sequence with common distribution F, $G \in \{\Lambda, \Phi_\alpha, \Psi_\alpha\}$ and $a_n > 0$, $b_n \in \mathbb{R}$, $n \ge 1$.

Then the following statements are equivalent:

(i) $n\bar{F}(a_n x + b_n) \xrightarrow[n\to\infty]{} -\log G(x)$

for all x s.t. $0 < G(x) < 1$.

(ii) The point process of exceedances

$$N_{n,x} := \sum_{i=1}^{n} \varepsilon_{n^{-1}i} \mathbb{1}\{X_i > a_n x + b_n\}, \quad n \ge 1$$

converges weakly to a $PRM(-\log G(x) \cdot \lambda)$ N on $E = (0,1]$ for all x s.t. $0 < G(x) < 1$.

(iii) $a_n^{-1}(M_n - b_n) \xrightarrow[n\to\infty]{d} G$

(iv) $\lim_{n\to\infty} P(a_n^{-1}(X_{(n-k+1)} - b_n) \le x) = Z(x)$, $k \ge 1$ for all x s.t. $0 < G(x) < 1$, where Z defines a distribution given by

$$Z(x) = G(x) \sum_{i=0}^{k-1} \frac{(-\log G(x))^i}{i!}$$

Proof: Direct consequence of Prop. 5.41 for $u_n = a_n x + b_n$ and $\tau = -\log G(x)$ and Th. 3.8 (Fisher-Tippett-Gnedenko).

Finally, in view of important applications in connection with dependent stationary sequences we also mention some sufficient criteria for the weak convergence of simple point processes, that is of point processes $\xi : \Omega \to M_p(E)$ s.t.

$$P(\xi(\{x\}) \le 1 \text{ for all } x \in E) = 1$$

Th. 5.43 (Sufficient criteria for the weak convergence of simple point processes; Kallenberg 1983)

Let ξ be a simple point process on E and $\mathcal{J}$ a basis of relatively compact open sets which is closed under finite unions and intersections. Assume that

$$P(\xi(\partial J) = 0) = 1$$

for all $J \in \mathcal{J}$.

Further, if ξ_n, $n \ge 1$ is a sequence of point processes on E s.t. for all $J \in \mathcal{J}$

$$\lim_{n\to\infty} P(\xi_n(J) = 0) = P(\xi(J) = 0)$$

and

$$\lim_{n\to\infty} E[\xi_n(J)] = E[\xi(J)] < \infty,$$

then

$$\xi_n \xrightarrow[n\to\infty]{V} \xi \text{ in } M_p(E)$$

Rem. 5.44 : If E is Euclidean, then $\mathcal{I}$ typically is a family of sets which can be represented as a finite union of bounded rectangles.

Ex. 5.45 : The point process of exceedances and the renewal process are examples of simple point processes.