$$L(x) = c(x)\exp\left\{\int_{x_0}^{x}\frac{\varepsilon(t)}{t}\,dt\right\},\ x \geq x_0 \qquad (2.4.9)$$

for some $x_0 > 0$, where $\varepsilon(t) \xrightarrow[t\to\infty]{} 0$ and $c(t) \xrightarrow[t\to\infty]{} c_0$ for some constant $c_0 > 0$.

Rem. 2.4.3 : It follows from (2.4.9) that

$$\lim_{x\to\infty}\frac{L(x)}{x^{\delta}} = 0 \quad \text{and} \quad \lim_{x\to\infty} x^{\delta}L(x) = \infty \qquad (2.4.10)$$

for all $\delta > 0$.

$\longrightarrow$ L "small" compared to the power function x^{δ}

$\longrightarrow$ Def. 2.4.4 (Regularly varying functions and r.v.'s)

Let L be slowly varying. Then

(i) for any $\delta \in \mathbb{R}$, a function $f : (0,\infty) \to (0,\infty)$ given by

$$f(x) = x^{\delta}L(x),\ x > 0 \qquad (2.4.11)$$

is called regularly varying with index δ

(ii) a positive r.v. X is said to be regularly varying with index $\alpha \geq 0$, if

$$\overline{F}_X(x) \overset{\text{def}}{=} P(X > x) = x^{-\alpha}L(x),\ x > 0 \qquad (2.4.12)$$

Rem 2.4.5 (alternative definition)

$$\lim_{x\to\infty}\frac{f(cx)}{f(x)} = c^{\delta} \text{ for all } c > 0 \qquad (2.4.13)$$

Rem : (i) (2.4.12) $\longrightarrow$ "small" deviation of $\overline{F}_X$ from the exact power law

(ii) X regularly varying with index $\alpha > 0$

$$\overset{(2.4.9)}{\Longrightarrow} \quad E[X^{\delta}] = \begin{cases} \infty, & \text{if } \delta > \alpha \\ < \infty, & \text{if } \delta < \alpha \end{cases} \qquad (2.4.14)$$

(see exercises)

(iii) Pareto, Burr and Log-Gamma distributed X are regularly varying with an index $\alpha > 0$

In order to introduce a reasonable concept of subexponentiality (or heavy-tailedness), it is natural

(33) to first analyze the influence of regularly varying claim size variables on allocated claims

→ L.2.4.6: Let X_1, X_2 be indep. regularly varying r.v.'s with index $\alpha > 0$, that is

$$\overline{F}_{X_i}(x) = P(X_i > x) = \frac{L_i(x)}{x^\alpha}$$

for slowly varying functions $L_i, i = 1,2$. Then

$$\frac{P(X_1 + X_2 > x)}{x^{-\alpha}(L_1(x) + L_2(x))} \xrightarrow{x \to \infty} 1$$

$$\left(\iff P(X_1 + X_2 > x) = x^{-\alpha}(L_1(x) + L_2(x))(1 + o(1))\right)$$

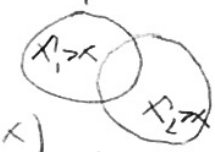
Recall: $f(x) = o(g(x)) \iff \lim_{x \to \infty} \left|\frac{f(x)}{g(x)}\right| = 0$

Proof: We see that

$$\{X_1 + X_2 > x\} \supseteq \{X_1 > x\} \cup \{X_2 > x\} =$$

$$= \{X_1 > x, X_2 \le x\} \dot\cup \{X_2 > x, X_1 \le x\} \dot\cup \{X_1 > x, X_2 > x\}$$

(disjoint, disjoint)



$$\Rightarrow P(X_1 + X_2 > x) \ge P(X_1 > x, X_2 \le x) + P(X_2 > x, X_1 \le x)$$

$$+ P(X_1 > x, X_2 > x) \overset{X_1, X_2 \text{ indep.}}{=} P(X_1 > x) \underbrace{P(X_2 \le x)}_{1 - P(X_2 > x)}$$

$$+ \underbrace{P(X_2 > x) P(X_1 \le x) + P(X_1 > x) P(X_2 > x)}$$

$$= P(X_2 > x)\left(\underbrace{P(X_1 \le x) + P(X_1 > x)}_{=1}\right)$$

$$= P(X_1 > x)(1 - P(X_2 > x)) + P(X_2 > x)$$

$$= \left(P(X_1 > x) + P(X_2 > x)\right)\left(1 - \underbrace{P(X_1 > x)}_{\xrightarrow{x \to \infty} 0} \cdot \underbrace{\frac{P(X_2 > x)}{P(X_1 > x) + P(X_2 > x)}}_{\le 1}\right) \quad (*)$$

On the other hand for $0 < \delta < \frac{1}{2}$, $\xrightarrow{} 0$ for $x \to \infty$

$$\{X_1 + X_2 > x\} \subset \{X_1 > (1-\delta)x\} \cup \{X_2 > (1-\delta)x\} \cup \{X_1 > \delta x, X_2 > \delta x\}$$

σ-additivity

$\Rightarrow$

X_1, X_2 indep.

$$P(X_1 + X_2 > x) \le P(X_1 > (1-\delta)x) + P(X_2 > (1-\delta)x)$$

$$+ P(X_1 > \delta x) P(X_2 > \delta x)$$

(34) $= \left(P(X_1 > (1-\delta)x) + P(X_2 > (1-\delta)x)\right)\left(1 + \frac{P(X_1>\delta x)\,P(X_2>\delta x)}{P(X_1>(1-\delta)x)+P(X_2>(1-\delta)x)}\right)$

$\leq$

$\underbrace{P(X_1>\delta x)}_{\xrightarrow{x\to\infty} 0}\left(\frac{P(X_2>\delta x)}{P(X_2>x)}\right) \xrightarrow[x\to\infty]{(2.4.13)} 0$ (**)

$\overset{(*),(**)}{\underset{(2.4.13)}{\Longrightarrow}}$ $1 \overset{(*)}{\leq} \liminf_{x\to\infty} \frac{P(X_1+X_2>x)}{P(X_1>x)+P(X_2>x)} \overset{\text{def}}{\leq} \limsup_{x\to\infty} \frac{P(X_1+X_2>x)}{P(X_1>x)+P(X_2>x)} \overset{(**)}{\leq} (1-\delta)^{-\alpha}$

L_1+L_2 slowly varying!

$\overset{\delta\searrow 0}{\Longrightarrow}$ $\liminf(\cdot) = \limsup(\cdot) = 1 \Longrightarrow$ proof.

induction on n $\longrightarrow$ important corollary of L. 2.4.6 for general n:

Corollary 2.47: Let $X_1, \dots, X_n$ i.i.d regularly varying r.v.'s with index $\alpha > 0$ and $X_i \sim F$. Then

$$\frac{P(S_n > x)}{n\,\overline{F}(x)} \xrightarrow[x\to\infty]{} 1$$

where $S_n = \sum_{i=1}^n X_i$

Rem. 2.4.8: $M_n \overset{\text{def}}{=} \max(X_1, \dots, X_n) \Longrightarrow$

$$\frac{P(M_n>x)}{n\,P(X_1>x)} = \frac{1}{n}\,\frac{1-(F(x))^n}{1-F(x)} \overset{\text{geom. sum}}{=} \frac{1}{n}\sum_{k=0}^{n-1}(F(x))^k \xrightarrow[x\to\infty]{} \frac{n}{n} = 1 \quad (2.4.15)$$

On the other hand, we know from the previous Lemma for general n that

$$\lim_{x\to\infty} \frac{P(S_n>x)}{\underbrace{P(X_1>x)+\dots+P(X_n>x)}_{=n\overline{F}(x)}} = 1$$

$$\Longrightarrow \lim_{x\to\infty} \frac{P(S_n>x)}{P(M_n>x)} = 1 \quad \forall n \geq 2 \qquad (2.4.16)$$

$\longrightarrow$ for regularly varying X_i the tail-distr. of S_n and M_n are asymptotically the same

$\longrightarrow$ (2.4.16) gives rise to the following definition of subexponentiality (or heavy-tailedness) of distr.:

Def. 2.49 (Subexponential r.v.)
Let X be a positive r.v. with distr. funct. F s.t. $\overline{F}(x) < 1$ for $x \in (0,\infty)$. Further, let X_i, $i \geq 1$ be an i.i.d.-sequence of r.v.'s with $X_i \sim F$.

Then X and F are called subexponential, if

$$\lim_{x\to\infty} \frac{P(S_n>x)}{P(M_n>x)} = 1 \quad \text{for all } n\geq 2$$

The class of such distr. is denoted by S.

Rem. 2.4.10 (equivalent definitions)

(i) One shows that (2.4.16) is equivalent to

$$\lim_{x\to\infty} \frac{P(S_n>x)}{P(M_n>x)} = 1 \quad \text{for } \underline{\text{some}}\ n\geq 2$$

(ii) Since $P(M_n>x) = n\overline{F}(x)(1+o(1))$ (see R 4.15)) cond. (2.4.16) is also equivalent to

$$\lim_{x\to\infty} \frac{P(S_n>x)}{\overline{F}(x)} = n \qquad (2.4.17)$$

L. 2.4.11 (Basic properties of $F\in S$)

(i) Let $F\in S$, then for all $y>0$:

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1 \qquad (2.4.18)$$

(ii) Suppose (2.4.18) holds for all $y>0$. Then for all $\varepsilon>0$

$$e^{\varepsilon x}\overline{F}(x) \xrightarrow[x\to\infty]{} \infty \qquad (2.4.19)$$

(iii) Let $F\in S$. Then for all $\varepsilon>0$ there ex. a $K=K(\varepsilon)<\infty$ s.t. for all $n\geq 2$

$$\frac{P(S_n>x)}{\overline{F}(x)} \leq K(1+\varepsilon)^n, \quad x\geq 0 \qquad (2.4.20)$$

Rem. 2.4.12

(i) Let $F=F_X\in S$. Then

$$E[e^{\varepsilon X}] \geq E[1_{(y,\infty)}(X)\, e^{\varepsilon X}] \geq e^{\varepsilon y}\, \underbrace{E[1_{(y,\infty)}(X)]}_{=P(X>y)=\overline{F}(y)} \xrightarrow{(2.4.19)} \infty$$

$$\Rightarrow E[e^{\varepsilon X}] = \infty$$

$\rightarrow$ (2.4.19) justifies the notion "subexponential", since $\overline{F}(x)$ decays slower to zero than $e^{-\varepsilon x}$ for any $\varepsilon>0$

(36) (ii) If $F = F_X \in S$ it follows that

$$1 \xleftarrow[x\to\infty]{(2.4.18)} \frac{P(X>x+y)}{P(X>x)} = \frac{P(X>x+y,\, X>x)}{P(X>x)} \overset{\text{def}}{=} P(X>x+y \mid X>x)$$

$\longrightarrow$ once X exceeds a high threshold x, it exceeds very likely the even higher value $x+y$

Proof of L.2.4.11:

(iii) For $x \geq y > 0$ we have that

$$\frac{P(S_{n+1}>x)}{P(X_1>x)} = 1 + \frac{P(X_1 \leq x) - P(S_{n+1} \leq x)}{P(X_1>x)} = 1 + \frac{1}{\overline{F}(x)}\left(F(x) - E\left[\mathbb{1}_{\{S_{n+1}-X_1 \leq x-X_1\}}\right]\right)$$

($P(X_1>x) = \overline{F}(x)$, $P(X_1 \leq x) = F(x)$, $P(S_{n+1} \leq x) = P(S_{n+1}-X_1 \leq x-X_1)$ with $S_{n+1}-X_1$ indep. of X_1)

dobbel-forventning: Y_1, Y_2 ind. rand. vectors, f function

$\Rightarrow E[f(Y_1,Y_2)] = E\left[E[f(Y_1,y)]\big|_{y=Y_2}\right]$

Here: $Y_1 = S_{n+1}-X_1$, $Y_2 = X_1$, $f(y_1,y_2) = \mathbb{1}_{\{y_1 \leq x-y_2\}}(y_1,y_2)$

$$= 1 + \frac{1}{\overline{F}(x)}\left(F(x) - E\left[\left(E\left[\mathbb{1}_{\{S_{n+1}-X_1 \leq x-t\}}\right]\right)\Big|_{t=X_1}\right]\right)$$

$$E\left[\mathbb{1}_{\{S_{n+1}-X_1 \leq x-t\}}\right] = P(X_2+\dots+X_{n+1} \leq x-t) = P(S_n \leq x-t)$$

Ex. 2.3.9

$$= 1 + \frac{1}{\overline{F}(x)}\left(\int_0^x 1\, dF(t) - \int_0^x P(S_n \leq x-t)\, dF(t)\right)$$

$$= 1 + \int_0^x \frac{P(S_n>x-t)}{\overline{F}(x)}\, dF(t) \qquad T>0$$

$$\leq 1 + \sup_{0\leq x\leq T}\int_0^x \frac{\overbrace{P(S_n>x-t)}^{\leq 1}}{\overline{F}(x)}\, dF(t)$$

$$+ \sup_{x\geq T}\int_0^x \frac{P(S_n>x-t)}{\overline{F}(x-t)}\,\frac{\overline{F}(x-t)}{\overline{F}(x)}\, dF(t) \leq 1 + \overline{F}(T)^{-1} +$$

$$\sup_{x\geq T}\left\{\left(\sup_{y\geq 0}\frac{P(S_n>y)}{\overline{F}(y)}\right)\left(\frac{1}{\overline{F}(x)}\int_0^x \overline{F}(x-t)\, dF(t)\right)\right\}$$

$$\frac{P(S_2>x)}{\overline{F}(x)} \leq (1+\varepsilon) \quad \forall x \geq T = T(\varepsilon)$$

$$\left(\int_0^x \overline{F}(x-t)\, dF(t) \overset{\text{dobbel-forventning}}{=} F(x) - P(S_2 \leq x)\right)$$

$=: \alpha_n < \infty$

$T = T(\varepsilon)$

$\Rightarrow$ iteration $\Rightarrow$

$$\alpha_{n+1} \leq 1 + \overline{F}(T)^{-1} + \alpha_n(1+\varepsilon)$$

$$\alpha_n \leq \underbrace{\left(1+\overline{F}(T)^{-1}\right)\varepsilon^{-1}}_{=K=K(\varepsilon)}(1+\varepsilon)^n \Longrightarrow \text{(iii)}$$

(i), (ii) similar proof $\Longrightarrow$ proof.

In view of a Cramér ruin bound for large claims, i.e. subexponential claims, we need the following crucial representation of non-ruin prob. $\psi(u)$

Prop. 2.4.13 (Representation of $\psi(u)$ as compound geom. probab.)
Assume the Cramér-Lundberg model with $E[X_1] < \infty$ and NPC. Further, require that F_{X_1} has a density and that $X_{J,j}$, $j \geq 1$ is an i.i.d-sequence of r.v.'s with common distr. given by

$$F_{X_{J,j}}(x) = \frac{1}{E[X_1]} \int_0^x P(X_1 > y)\, dy \quad \text{(see proof of Th. 2.3.1)}$$

(integrated tail distribution)

Then ψ given by

$$\psi(u) = \frac{\delta}{1+\delta}\left[1 + \sum_{n\geq 1} (1+\delta)^{-n} P(X_{J,1} + \dots + X_{J,n} \leq u)\right], \quad u \geq 0 \qquad (2.4.21)$$

solves the integral equation (2.3.1).
Furthermore, ψ defined by (2.4.21) is the unique solution to the fundamental int. eq for non-ruin prob. (2.3.1) in the class of functions

$$\mathcal{G} \overset{\text{def.}}{=} \{G : \mathbb{R} \to [0,\infty) : G \text{ non-decreasing, bounded, right-cont., } G(x) = 0 \text{ for } x < 0\}$$

Proof: Set $q = (1+\delta)^{-1}$ and $p = 1 - q = \delta(1+\delta)^{-1}$

1. Existence: Recall (dobbel-forvent.ing):
X, Y indep. rand. variables, f function
$$\Rightarrow E[f(X,Y)] = E\big[\underbrace{E[f(X,Y)]_{Y=y}}_{\text{Ex. 2.3.9}}\big] = \int_{-\infty}^{\infty} E[f(X,y)]\, dF_Y(y)$$
$$\overset{\text{indep.}}{=}$$

(choose $f(x,y) = \mathbb{1}_{\{y+x \leq u\}}$, $Y = X_{J,1}$, $X = X_{J,2} + \dots + X_{J,n}$)

$E[\mathbb{1}_A] = P(A)$ $\Rightarrow$

$$P(X_{3,1}+\dots+X_{3,n}\le u) = E[f(X,Y)]$$
$$= \int_0^\infty P(y+X_{3,2}+\dots+X_{3,n}\le u)\, dF_{X_1,3}(y)$$
$$= \int_0^u P(y+X_{3,2}+\dots+X_{3,n}\le u)\, dF_{X_1,3}(y)$$

(2.4.21)
$$\Rightarrow \quad g(u) = p + p\sum_{n\ge 1} q^n P(X_{3,1}+\dots+X_{3,n}\le u)$$
$$= p + qp\, E\, F_{X_1,3}(u) + \sum_{n\ge 2} q^{n-1} P(X_{3,1}+\dots+X_{3,n}\le u)$$
$$= p + qp\, E\, F_{X_1,3}(u) + \sum_{n\ge 2} q^{n-1}\int_0^u \underbrace{P(y+X_{3,2}+\dots+X_{3,n}\le u)}_{= P(X_{3,2}+\dots+X_{3,n}\le u-y)\ \sim\ X_{3,1}+\dots+X_{3,n-1}} dF_{X_1,3}(y)$$
$$= p + qp\, E\, F_{X_1,3}(u) + \sum_{n\ge 2} q^{n-1}\int_0^u P(X_{3,1}+\dots+X_{3,n-1}\le u-y)\, dF_{X_1,3}(y)$$
$$= p + q\int_0^u g(u-y)\, dF_{X_1,3}(y) \quad \Rightarrow \text{ existence.}$$

(more)

(ii) Uniqueness: See Embrechts $\Rightarrow$ proof.

Rem. 2.4.14: M r.v. geometrically distributed $\iff$
$p_n \overset{\text{def}}{=} P(M=n) = p\cdot q^n$, $n\ge 0$ for some $p=1-q\in(0,1)$
Let $X_i, i\ge 1$ be i.i.d. r.v.'s being indep. of M
$\longrightarrow$ r.v. of the form
$$S_M \overset{\text{def}}{=} \sum_{i=1}^{M} X_i$$
is called a compound geometrically distributed r.v.

$$\longrightarrow F_{S_M}(x) = P(S_M\le x) = E\Big[1_{\{\sum_{i=1}^{M} X_i\le x\}}\Big]$$
$$= E\Big[1_{\{\sum_{i=1}^{M} X_i\le x\}}\Big(\sum_{n\ge 0} 1_{\{M=n\}}\Big)\Big] =$$
$$= \sum_{n\ge 0} E\Big[1_{\{\sum_{i=1}^{M} X_i\le x\}} 1_{\{M=n\}}\Big] = \sum_{n\ge 0} E\Big[1_{\{\sum_{i=1}^{n} X_i\le x\}} 1_{\{M=n\}}\Big]$$

($M=n$ on $\{\sum_{i=1}^{M} X_i\le x\}$; $P(X_1+\dots+X_n\le x)^{n\ge 0}$; indep. (2.4.22))

$E[X\cdot Y] = E[X]\cdot E[Y]$, if X, Y indep.

$$\sum_{n\ge 0} \underbrace{E[1_{\{M=n\}}]}_{=p_n=p\cdot q^n}\cdot E\Big[1_{\{\sum_{i=1}^{n} X_i\le x\}}\Big]$$

(2.4.21) $\longrightarrow$ interpretation:

$g(u)$ distr. of a compound geometrically distr. r.v. with M for $q=(1+s)^{-1}$ and $X_i' = X_{I,i}$

$\longrightarrow$

Th. 2.4.15 (Cramér's ruin bound for subexponential $F_{X_{1,I}}$)

Assume the Cramér-Lundberg model for $S(t)$. Require that $E[X_1] < \infty$ and that the NPC holds. Further, suppose that F_{X_1} has a prob. density and that $F_{X_{1,I}}$ is subexponential. Then

$$\lim_{u\to\infty} \frac{\psi(u)}{g^{-1}\overline{F}_{X_{1,I}}(u)} = 1 \qquad (2.4.23)$$

Rem. 2.4.16:

(i) (2.4.23) $\longrightarrow$ $\psi(u)$ is asymptotically of the same order as $\overline{F}_{X_{1,I}}(u)$, which is subexponential

$\longrightarrow$ dangerous compared to the light-tailed case (Th. 2.3.1):

$$\psi(u) \sim C \cdot e^{-Ru} \quad \text{for } u\to\infty$$

(like the exponential distr.)

Spontaneous ruin through single large claims is characteristic for heavy-tailed (i.e. subexponential) distr.

(ii) If X_1 is regularly varying with index $\alpha > 1$ (i.e. $\overline{F}_{X_1}(x) = x^{-\alpha} L(x)$ ← slowly varying) then $F_{X_{1,I}}$ is subexponential (see exercises)

$\longrightarrow$ example: $\overline{F}_{X_1}(x) = \left(\frac{\lambda}{\lambda+x}\right)^{\alpha}$, $\alpha > 1$ ($\overset{(2.4.14)}{\Rightarrow} E[X_1] < \infty$)

(2.4.16) $\Rightarrow$ Pareto distr. $\longrightarrow$ $\overline{F}_{X_{1,I}}$ regularly varying with index $\alpha - 1$

$\Rightarrow$ $F_{X_{1,I}}$ subexponential

(iii) Let $q(x) = f(x)/\overline{F}_{X_1}(x)$ be the hazard rate (f density of X_1)

If $\lim_{x\to\infty} x\, q(x) < \infty$ then $F_{X_{1,I}}$ subexponential (i.e. $F_{X_{1,I}} \in S$). See Embrechts.

(iv) $F_{X_{1,I}} \in S \iff g = 1-\psi \in S$. See Embrechts.

Proof of Th. 2.4.15: Idea: Application of Prop. 2.4.13

$$\frac{\psi(u)}{\overline{F}_{X_{1,J}}(u)} \overset{\text{Prop. 2.4.13}}{=} \frac{\delta}{1+\delta} \sum_{n\geq 1} (1+\delta)^{-n} \frac{P(X'_{J,1}+\dots+X'_{J,n}>u)}{\overline{F}_{X_{1,J}}(u)} \quad (*)$$

On the other hand, we know that

$$\lim_{u\to\infty} \frac{P(X'_{J,1}+\dots+X'_{J,n}>u)}{\overline{F}_{X_{1,J}}(u)} = n,$$

since $F_{X_{1,J}}$ is subexponential.

Further, (iii) in L. 2.4.11 yields for all $\varepsilon>0$

$$\frac{P(X_{J,1}+\dots+X_{J,n}>u)}{\overline{F}_{X_{1,J}}(u)} \leq K_\varepsilon (1+\varepsilon)^n \quad (K_\varepsilon \text{ constant})$$

(choose $\varepsilon<\delta$. Then

$$\sum_{n\geq 1} \underbrace{(1+\delta)^{-n} K_\varepsilon (1+\varepsilon)^n}_{K_\varepsilon \left(\frac{1+\varepsilon}{1+\delta}\right)^n} < \infty$$

$\left(\frac{1+\varepsilon}{1+\delta}<1\right)$

$\overset{\text{dom. convergence}}{\Longrightarrow}$

$$\lim_{u\to\infty} \frac{\psi(u)}{\overline{F}_{X_{1,J}}(u)} = \frac{\delta}{(1+\delta)} \sum_{n\geq 1} (1+\delta)^{-n} \overbrace{\lim_{u\to\infty} \frac{P(X'_{J,1}+\dots+X'_{J,n}>u)}{\overline{F}_{X_{1,J}}(u)}}^{(*)\ =n} \quad (**)$$

Recall: $f(q) := \sum_{k\geq 0} q^k$, $|q|<1 \Rightarrow f'(q) = \sum_{k\geq 1} k q^{k-1}$

$f(q) = \frac{1}{1-q} \Rightarrow f'(q) = \frac{1}{(1-q)^2}$ on $(-1,1)$

$q=(1+\delta)^{-1}$ $\Rightarrow$ $(**)$

$$\lim_{u\to\infty} \frac{\psi(u)}{\overline{F}_{X_{1,J}}(u)} = \frac{\delta}{(1+\delta)} (1+\delta)^{-1} \cdot \frac{1}{\left(1-\frac{1}{(1+\delta)}\right)^2} = \frac{\delta}{(1+\delta)^2} \frac{(1+\delta)^2}{\delta^2} = \frac{1}{\delta}$$

$\Rightarrow$ proof.

same proof $\rightarrow$ **Th 2.4.17** (General $S(t)$ for subexponential claims)

Let $S(t) = \sum_{i=1}^{N(t)} X_i$ for a general claim number proc. $N(t)$, which is indep. of the i.i.d. X_i, $i\geq 1$.

Set $P_t(n) = P(N(t)=n)$. If $\overline{F}_{X_1} \in S$ and

$$\sum_{n\geq 0} (1+\varepsilon)^n P_t(n) < \infty$$

for some $\varepsilon>0$ Then $\overline{F}_{S(t)} \in S$ and

$$\lim_{x\to\infty} \overline{F}_{S(t)}(x) \Big/ E[N(t)] \cdot \overline{F}_{X_1}(x) = 1$$

3. Extreme value distributions

In extreme value theory one is typically interested in the study of extremal values of samples e.g. of the form

$$M_n = \max(X_1, \dots, X_n)$$

or

$$m_n = \min(X_1, \dots, X_n)$$

The latter, however, requires e.g. some knowledge on the distributional behaviour of extremal values of samples

For i.i.d.-samples $X_1, \dots, X_n$ we directly saw that e.g.

$$F_{M_n}(x) = P(M_n \le x) = (F_{X_1}(x))^n$$

In practice, however, the distribution F_{X_1} is unknown.

$\longleftrightarrow$ What could be a possible solution to this problem?

To get an idea let us recall the classical central limit theorem (CLT):

If $X_i, i \ge 1$ is i.i.d with mean μ and variance $\sigma^2 > 0$ then for $S_n \stackrel{\text{def}}{=} X_1 + \dots + X_n$

$$\frac{\frac{1}{n} S_n - \mu}{\sigma/\sqrt{n}} \xrightarrow[n\to\infty]{d} \zeta \sim N(0,1) \qquad (3.1)$$

($N(0,1)$: standard normal distr. function)

$\Longleftrightarrow$

$$P\left(\frac{\frac{1}{n} S_n - \mu}{\sigma/\sqrt{n}} \le x\right) \xrightarrow[n\to\infty]{} P(\zeta \le x) = \Phi(x)$$

for all x

Recall: $Y_n \xrightarrow[n\to\infty]{d} Y \Longleftrightarrow P(Y_n \le x) \longrightarrow P(Y \le x)$ for all continuity points of $F_Y(x) = P(Y \le x)$

So, using the CLT (3.1), we can get an approximation of the distribution F_{S_n} of S_n by means of the standard normal distribution Φ for large n

→ Natural question:
Is there an analogon of a CLT (3.1) for M_n instead of S_n, which would allow for an approximation of F_{M_n} by means of a distribution G (instead of Φ) for large n?

More precisely: Can we find constants $a_n > 0$, $b_n \in \mathbb{R}$, $n \geq 1$ and a distr. function G such that

$$\frac{M_n - b_n}{a_n} \xrightarrow[n\to\infty]{d} G \qquad (3.2)$$

?

$$(3.2) \iff P\left(\frac{M_n - b_n}{a_n} \leq x\right) \xrightarrow[n\to\infty]{} G(x)$$

for all continuity points x of $G(x)$

$$\iff$$

$$P(M_n \leq a_n x + b_n) = \left(F_{X_1}(a_n x + b_n)\right)^n \xrightarrow[n\to\infty]{} G(x) \qquad (3.3)$$

for all such points x

→ answere: Yes

→ Def. 3.1 (Maximum domain of attraction)
The maximum domain of attraction of a non-degenerate distribution function G (i.e. $\nexists c$ s.t. $G(t) = 0$, if $t < c$ and 1 else), denoted by $MDA(G)$ consists of all distr. functions F, for which there ex. $a_n > 0$, $b_n \in \mathbb{R}$, $n \geq 1$ s.t.

$$\frac{M_n - b_n}{a_n} \xrightarrow[n\to\infty]{d} G$$

for $M_n = \max(X_1, \dots, X_n)$, where X_i, $i \geq 1$ i.i.d with $X_i \sim F$.

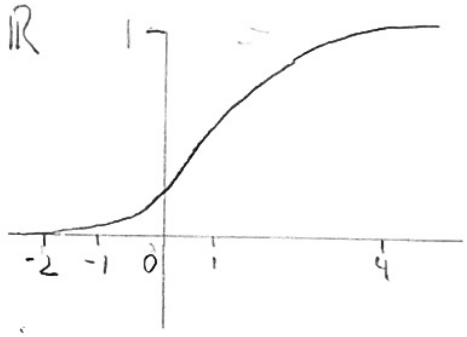
Def. 3.2 (Extreme value distr.)
A non-degenerate distr. G is called extreme value distribution, if $MDA(G) \neq \emptyset$

We will see that there are only few extreme value distr. G

→ Examples: Gumbel, Fréchet and Weibull distr.:

Ex 3.3 (Gumbel distr. Λ)
Recall: A r.v. has a Gumbel distr., if its distr. is given by

$$\Lambda(x) = e^{-e^{-x}}, \quad x \in \mathbb{R}$$



Let $X_i, i \geq 1$ i.i.d with $X_1 \sim Exp(1)$.
Then

$$P(M_n - \log n \leq x) = P(M_n \leq x + \log n)$$
$$F = F_{X_1} = (F(x+\log n))^n \overset{def}{=} (1 - e^{-(x+\log n)})^n$$
$$= \left(1 - \frac{e^{-x}}{n}\right)^n \xrightarrow[n\to\infty]{} e^{-e^{-x}}$$

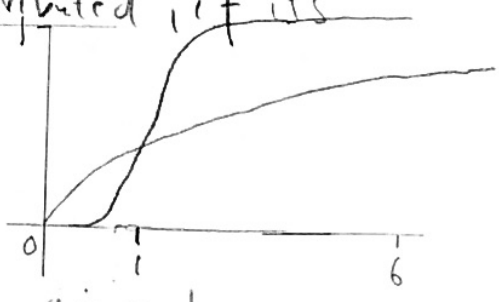
$$\Rightarrow \quad \frac{M_n - b_n}{a_n} \xrightarrow[n\to\infty]{d} G = \Lambda$$

for $a_n \equiv 1$ and $b_n = \log n$, $n \geq 1$
$\Rightarrow$ Λ extreme value distr. and $Exp(1) \in MDA(\Lambda)$

Ex. 3.4 (Fréchet distr. Φ_α)
Recall: A r.v. is said to be Fréchet distributed, if its distr. is given by

$$\Phi_\alpha(x) = \begin{cases} e^{-x^{-\alpha}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

for $\alpha > 0$.



Let $X_i, i \geq 1$ i.i.d. with common law, given by the following Pareto distr.

$$F(x) = \begin{cases} 1 - x^{-\alpha}, & x \geq 1 \\ 0, & x \leq 1 \end{cases}$$

for $\alpha > 0$.

Then $P\left(\frac{M_n}{n^{1/\alpha}} \le x\right) = P(M_n \le x n^{1/\alpha}) \overset{F=F_X}{=} \left(F(x \cdot n^{1/\alpha})\right)^n$

$= \left(1 - \frac{1}{(x n^{1/\alpha})^\alpha}\right)^n = \left(1 - \frac{x^{-\alpha}}{n}\right)^n \xrightarrow[n\to\infty]{} e^{-x^{-\alpha}}$

$\Rightarrow$ for $a_n = n^{1/\alpha}$, $b_n \equiv 0$, $n \ge 1$ we get that

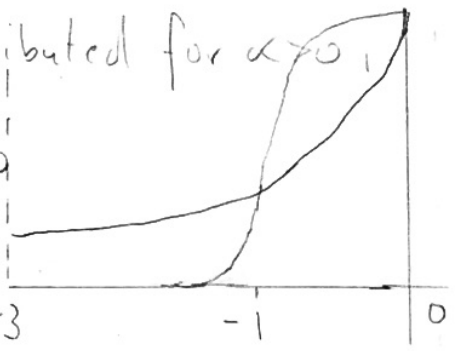
$$\frac{M_n - b_n}{a_n} \xrightarrow[n\to\infty]{d} \Phi_\alpha$$

$\Rightarrow$ Pareto distr. in $MDA(\Phi_\alpha)$ $\Rightarrow$ Φ_α extreme value distr.

Ex. 3.5 (Weibull distr. Ψ_α)

Recall: A r.v. is called Weibull distributed for $\alpha > 0$, if its distr is given by

$$\Psi_\alpha(x) = \begin{cases} e^{-(-x)^\alpha}, & x \le 0 \\ 1, & x \ge 0 \end{cases}$$



Let X_i, $i \ge 1$ i.i.d. with common law

$$F(x) = \begin{cases} 0, & x \le -1 \\ 1 - (-x)^\alpha, & x \in [-1, 0] \\ 1, & x \ge 0 \end{cases}$$

for $\alpha > 0$.

Then for $x \le 0$, we get

$$P(n^{1/\alpha} M_n \le x) = \left(F(x n^{-1/\alpha})\right)^n \overset{x n^{-1/\alpha} \in [-1,0] \text{ for large } n}{=} \left(1 - (-x n^{-1/\alpha})^\alpha\right)^n = \left(1 - \frac{(-x)^\alpha}{n}\right)^n \xrightarrow[n\to\infty]{} e^{-(-x)^\alpha}.$$

We also see that

$$P(n^{1/\alpha} M_n \le x) = 1$$

for $x \ge 0$, since $M_n \le 0$.

$\Rightarrow$ for $a_n = n^{-1/\alpha}$, $b_n \equiv 0$, $n \ge 1$ (3.2) holds

$\Rightarrow$ $F \in MDA(\Psi_\alpha)$ $\Rightarrow$ Ψ_α extreme value distr.

Def. 3.6 (r.v.'s of the same type)

The r.v.'s Y_1, Y_2 and distr. F_1 and F_2 are said to be of the same type, if

$$Y_1 \overset{d}{=} c_1 Y_2 + d_1 \quad \text{(notation: } Y_1 \rhd\lhd Y_2\text{)}$$

and

$$F_1(x) = F_2\left(\frac{x - d_2}{c_2}\right) \text{ for all } x \quad \text{(notation: } F_1 \rhd\lhd F_2\text{)}$$

for some $c_1, c_2 > 0$, $d_1, d_2 \in \mathbb{R}$, respectively.

(45) Ex.: $F_1 \sim N(\mu, \sigma^2)$, $F_2 \sim N(0,1) \Rightarrow F_1 \not\asymp F_2$

$\longrightarrow$ other examples:

Prop. 3.7: If G is an extreme value distr., then $G\left(\frac{x-d}{c}\right)$ is an extreme value distr. for all $c>0$, $d \in \mathbb{R}$.

Proof: exerc.

Rem: $MDA(G(x)) = MDA\left(G\left(\frac{x-d}{c}\right)\right)$

It follows from Prop. 3.7 that the following distr. are extreme value distr.:

Gumbel type: $\Lambda\left(\frac{x-\mu}{\sigma}\right) = \exp\left(-e^{\frac{x-\mu}{\sigma}}\right)$, $x \in \mathbb{R}$ (3.4)

Fréchet type: $\Phi_\alpha\left(\frac{x-\mu}{\sigma}\right) = \begin{cases} \exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^{-\alpha}\right\}, & x > \mu \\ 0 \text{ else} \end{cases}$ (3.5)

Weibull type: for $\alpha > 0$.
$\Psi_\alpha\left(\frac{x-\mu}{\sigma}\right) = \begin{cases} \exp\left\{-\left(-\frac{x-\mu}{\sigma}\right)^{\alpha}\right\}, & x < \mu \\ 1 \text{ else} \end{cases}$ (3.6)

$\longrightarrow$ central result of extreme value theory:

Th. 3.8 (Fisher-Tippett (1928), Gnedenko (1943))
If G is a extreme value distribution, then G belongs to one of the families of distr. (3.4), (3.5) or (3.6).

Rem. 3.9 (Jenkinson-von Mises-representation)
The following family of distr. contains (3.4), (3.5) and (3.6) as special cases:

$$G_\gamma(x) = \begin{cases} \exp\left\{-(1+\gamma x)^{-1/\gamma}\right\}, & 1+\gamma x > 0 \\ 0, & \gamma > 0 \text{ and } x \le -\frac{1}{\gamma} \\ 1, & \gamma < 0 \text{ and } x \ge -\frac{1}{\gamma} \end{cases} \quad (3.7)$$

for $\gamma \in \mathbb{R}$.

(i) For $\gamma > 0$, G_γ is of the same type as $\Phi_{1/\gamma}(x) = e^{-x^{-1/\gamma}}$, $x > 0$

(ii) For $\gamma < 0$, G_γ is of the same type as $\Psi_{-1/\gamma}(x) = e^{-(-x)^{-1/\gamma}}$, $x < 0$

(iii) For $\gamma = 0$, $(1+\gamma x)^{-1/\gamma}$ is defined as
$$\lim_{\gamma \searrow 0} (1+\gamma x)^{-1/\gamma} = e^{-x}$$
So $G_0(x) = e^{-e^{-x}} = \Lambda(x)$

Th. 3.8 $\longrightarrow$ every extreme value distr. is of the form $G_\gamma(cx+d)$ for $\gamma \in \mathbb{R}$, $c>0$, $d \in \mathbb{R}$ (then extreme value distr. (GED))

In order to prove Th. 3.8, we need some notions and auxiliary results.

Consider r.v.'s $Z_1, Z_2, \dots$ and $a_n > 0$, $b_n \in \mathbb{R}$, $n \geq 1$ s.t.

$$\frac{Z_n - b_n}{a_n} \xrightarrow[n\to\infty]{d} Z \qquad (3.8)$$

→ Can we choose other sequences $\widehat{a}_n, \widehat{b}_n, n \geq 0$, without "destroying" the convergence in (3.8) ?

→ **Prop. 3.10** : Assume (3.8). Let $\widehat{a}_n > 0$, $\widehat{b}_n \in \mathbb{R}$, $n \geq 1$.

(i) If

$$\lim_{n\to\infty} \frac{\widehat{a}_n}{a_n} = 1 \text{ and } \lim_{n\to\infty} \frac{\widehat{b}_n - b_n}{a_n} = 0,$$

then

$$\frac{Z_n - \widehat{b}_n}{\widehat{a}_n} \xrightarrow[n\to\infty]{d} Z.$$

(ii) If

$$\lim_{n\to\infty} \frac{\widehat{a}_n}{a_n} = a \in (0,\infty) \text{ and } \lim_{n\to\infty} \frac{\widehat{b}_n - b_n}{a_n} = b \in \mathbb{R},$$

then

$$\frac{Z_n - \widehat{b}_n}{\widehat{a}_n} \xrightarrow[n\to\infty]{d} \frac{Z-b}{a}.$$

Lemma 3.11 (Slutsky)
Suppose that

$$X'_n \xrightarrow[n\to\infty]{d} X'$$

and

$$Y_n \xrightarrow[n\to\infty]{} c \qquad (c \text{ constant})$$

in probability.
Then

(i) $X'_n + Y_n \xrightarrow[n\to\infty]{d} X' + c$

(ii) $X'_n \cdot Y_n \xrightarrow[n\to\infty]{d} c \cdot X'$

(iii) $X'_n / Y_n \xrightarrow[n\to\infty]{d} X'/c \quad (c \neq 0)$

Proof of Prop. 3.10 : Follows from the L. 3.11 applied to

$$\frac{Z_n - \widehat{b}_n}{\widehat{a}_n} = \frac{a_n}{\widehat{a}_n} \cdot \frac{Z_n - \widehat{b}_n}{a_n} = \frac{a_n}{\widehat{a}_n} \left(\frac{Z_n - b_n}{a_n} - \frac{\widehat{b}_n - b_n}{a_n} \right).$$

$\Rightarrow$ proof.

What can we say about the uniqueness of $a_n, b_n, n \geq 1$ in (3.8)?

→ Th. 3.12 (Khintschin)
Let $Z_1, Z_2 \dots$ be r.v.'s and $a_n, \widehat{a_n} > 0$, $b_n, \widehat{b_n} \in \mathbb{R}$, $n \geq 1$ s.t.

$$\frac{Z_n - b_n}{a_n} \xrightarrow[n\to\infty]{d} Z \quad \text{and} \quad \frac{Z_n - \widehat{b_n}}{\widehat{a_n}} \xrightarrow[n\to\infty]{d} \widehat{Z}$$

for non-degenerate r.v.'s Z and $\widehat{Z}$.
Then

$$a \overset{\text{def.}}{=} \lim_{n\to\infty} \frac{\widehat{a_n}}{a_n} \in (0,\infty), \quad b \overset{\text{def.}}{=} \lim_{n\to\infty} \frac{\widehat{b_n} - b_n}{a_n} \in \mathbb{R}$$

and

$$\widehat{Z} \overset{d}{=} (Z-b)/a$$

Proof: See e.g. Billingsley, Probability and Measure (2012).

Lemma 3.13: Let Z be a non-degenerate r.v. and $c > 0$, $d \in \mathbb{R}$ s.t.

$$c \cdot Z + d \overset{d}{=} Z.$$

Then $c = 1$ and $d = 0$.

Proof: See e.g. Billingsley.

In order to characterize all extreme value distr. G_γ, we also need the following notion:

Def 3.14 (Max-stable distribution)
A non-degenerate distr. G is called max-stable, if there exist $c_n > 0$, $d_n \in \mathbb{R}$, $n \geq 1$ s.t.

$$(G(c_n \cdot x + d_n))^n = G(x) \qquad (3.9)$$

(i.e. G^n is of the same type as G)

Ex 3.15: $\Lambda(x) = e^{-e^{-x}}$ Gumbel distr. is max-stable, since:

$$(\Lambda(x + \log n))^n = \exp(-n e^{-(x+\log n)}) = \Lambda(x)$$

Other examples are Φ_α and Ψ_α

→ Th. 3.16 A distr. G is max-stable, iff G is an extreme value distr.

Proof:

"$\Rightarrow$": $\exists\ c_n > 0,\ d_n \in \mathbb{R}$ s.t.

$$(G(c_n x + d_n))^n = G(x)$$

$$\Rightarrow \lim_{n\to\infty} (G(c_n \cdot x + d_n))^n = G(x)$$

$\Rightarrow$ G extreme value distr.

"$\Leftarrow$": $\exists$ distr. $F \in MDA(G)$ and $a_n > 0,\ b_n \in \mathbb{R},\ n \geq 1$ s.t.

$$\lim_{n\to\infty} (F(a_n x + b_n))^n = G(x) \quad (*)$$

for all continuity points x.

$n \Leftrightarrow nk$ $\Rightarrow$ $\lim_{n\to\infty} (F(a_{n\cdot k} \cdot x + b_{n\cdot k}))^{n\cdot k} = G(x)$ ← distr.

$(\cdot)^{1/k}$ $\Rightarrow$ $\lim_{n\to\infty} (F(a_{nk} x + b_{nk}))^n = (G(x))^{1/k}$ $(**)$

Th. 3.12 (Khintchin) applied to $(*), (**)$ $\Rightarrow$ G and $G^{1/k}$ are of the same type

$\Rightarrow$ $\exists\ c_k > 0,\ d_k \in \mathbb{R},\ k \geq 1$ s.t.

$$(G(x))^{1/k} = G(c_k \cdot x + d_k)$$

$$\Leftrightarrow G(x) = (G(c_k \cdot x + d_k))^k$$

$\Rightarrow$ G max-stable. $\Rightarrow$ proof.

Rem.: Proof of $(*)$ implies that $G \in MDA(G)$

→ Generalization of (3.7) to the case of non-integers n:

Prop. 3.17: Let G be max-stable. Then there exist measurable functions $c : (0,\infty) \to (0,\infty)$ and $d : (0,\infty) \to \mathbb{R}$ such that for all real numbers $s > 0$:

$$(G(c(s)x + d(s)))^s = G(x) \qquad (3.10)$$

for all x.

Proof: Let $[x]$ be the Gauss bracket of x, that is

$$[x] \stackrel{\text{def}}{=} \max\{n \in \mathbb{Z} : n \leq x\}.$$

We have that

$$(G(c_n \cdot x + d_n))^n = G(x)$$

$\Rightarrow$

(49) $\forall s > 0$:

$$\left(G\left(c_{[n\cdot s]}\cdot x + d_{[n\cdot s]}\right)\right)^{\overbrace{[n\cdot s]}^{\in \mathbb{N}_0}} = G(x)$$

$$\Rightarrow \quad \left(G\left(c_{[n\cdot s]}\cdot x + d_{[n\cdot s]}\right)\right)^{n} = \left(G\left(c_{[n\cdot s]}\cdot x + d_{[n\cdot s]}\right)\right)^{\frac{n}{[n\cdot s]}}$$

$$= \left(G(x)\right)^{\frac{n}{[n\cdot s]}} \xrightarrow[n\to\infty]{} \left(G(x)\right)^{1/s}$$

Recall: $\frac{[n\cdot x]}{n} \xrightarrow[n\to\infty]{} x$

On the other hand,

$$\left(G(c_n\cdot x + d_n)\right)^n \xrightarrow[n\to\infty]{} G(x)$$

$\longrightarrow$ Khintchin for $a_n = c_n$, $b_n = d_n$, $\hat{a}_n = c_{[n\cdot s]}$, $\hat{b}_n = d_{[ns]}$ gives

$$c(s) \overset{\text{def.}}{=} \lim_{n\to\infty} \frac{c_{[n\cdot s]}}{c_n} \in (0,\infty), \quad d(s) \overset{\text{def.}}{=} \lim_{n\to\infty} \frac{d_{[n\cdot s]} - d_n}{c_n} \in \mathbb{R}$$

and

$$\left(G(x)\right)^{1/s} = G(c(s)\cdot x + d(s))$$

$$\Rightarrow \quad G(x) = \left(G(c(s)x + d(s))\right)^s \quad \Rightarrow \text{ proof.}$$

Using the functional equation (3.10) we can prove the following characterization of max-stable distr.:

<u>Th. 3.18</u> (Characterization of max-stable G)

Let G be max-stable. Then G is of the same type as one of the distr.: Λ (Gumbel), Φ_α (Fréchet) or Ψ_α (Weibull)

In order to prove Th. 3.18 we want to solve the equation (3.10) by using the following deep result

<u>Lemma 3.19</u> (<u>Cauchy functional equation</u>; Ostrowski (1929))

Let $f: \mathbb{R} \to \mathbb{R}$ be (Lebesgue-) measurable s.t.

$$f(x+y) = f(x) + f(y), \quad x, y \in \mathbb{R}$$

Then there exists a $C \in \mathbb{R}$ s.t.

$$f(x) = C\cdot x$$

for all x.

Corollary 3.20: let $g:(0,\infty)\to(0,\infty)$ be meas. with

$$g(x\cdot y)=g(x)\cdot g(y)\ ,\ x,y\in(0,\infty)$$

Then

$$g(x)=x^c\ ,\ x>0$$

for a $c\in\mathbb{R}$.

Proof: $f(x):=\log(g(e^x))$ in L.3.19

Proof of Th. 3.18: We know from (3.10) that

$$\left(G\left(c(s)\cdot x+d(s)\right)\right)^s=G(x) \qquad (*)$$

First we want to find the functions c and d

1. step: Choose $s=s_1\cdot s_2$, $s_1,s_2>0$ in $(*)$

$$\Rightarrow \left(G\left(c(s_1\cdot s_2)x+d(s_1\cdot s_2)\right)\right)^{s_1\cdot s_2}=G(x)$$

$$\overset{(*)}{=} \left(G\left(c(s_1)x+d(s_1)\right)\right)^{s_1} \overset{(*)}{=} \left(G\left(c(s_2)\underbrace{[c(s_1)\cdot x+d(s_1)]}_{x=c(s_1)\cdot x+d(s_1)}+d(s_2)\right)\right)^{s_1 s_2}$$

$\overset{\text{L.3.13}}{\Rightarrow}$ $(\cdot)^{1/s_1 s_2}$

$$\begin{cases} c(s_1\cdot s_2)=c(s_1)\cdot c(s_2) \\ d(s_1\cdot s_2)=c(s_2)\,d(s_1)+d(s_2) \end{cases} \qquad (**)$$

$\overset{\text{Cor. 3.20}}{\Rightarrow}$ applied to c

$$c(s)=s^\rho\ ,\ s>0 \quad \text{for some } \rho\in\mathbb{R}.$$

2. step: $\rho=0 \Rightarrow G \bowtie \Lambda$ (exercise)

3. step: $\rho\neq 0 \overset{(**),\ c(s)=s^\rho}{\Rightarrow} d(s_1\cdot s_2)=s_2^\rho\, d(s_1)+d(s_2)$

and $d(s_2\cdot s_1)=s_1^\rho\, d(s_2)+d(s_1)$

$$\Rightarrow \quad s_2^\rho\, d(s_1)+d(s_2)=s_1^\rho\, d(s_2)+d(s_1)$$

Set $s_2=2$.

$$\Rightarrow \quad d(s_1)=\frac{d(2)}{\underbrace{2^\rho-1}_{\neq 0 \text{ since } \rho\neq 0}}\left(s_1^\rho-1\right)=\left(s_1^\rho-1\right)\mu$$

with $\mu \overset{\text{def.}}{=} \frac{d(2)}{2^\rho-1}$

$$\Rightarrow \quad c(s)=s^\rho \text{ and } d(s)=\left(s^\rho-1\right)\mu$$

$$\overset{(*)}{\Rightarrow} \left(G\left(s^\rho\cdot x+(s^\rho-1)\mu\right)\right)^s=G(x) \qquad (+)$$

(51) Define the distr.

$$H(u) = G(u-\mu), \quad u \in \mathbb{R}$$

(+) for $x'' = x - \mu$ $\Rightarrow$ $x \geq 0$

$$(H(s^{\vartheta} x))^s = H(x) \qquad (++)$$

$\Rightarrow (H(0))^s = H(0)$ for all $s > 0$ $\Rightarrow H(0) = 0$ or $H(0) = 1$

1. case. $H(0) = 0 \overset{H \uparrow 1}{\Rightarrow} H(y) = 0$ for all $y \leq 0$

Let $y > 0$. Since G is non-degenerate we can find a x^* s.t. $H(x^*) \in (0,1)$. W.l.o.g. $x^* = 1$.

Set $s = y^{1/\vartheta}$

$\overset{(++)}{\Rightarrow} H(y) = (H(1))^{1/s} = (H(1))^{y^{-1/\vartheta}} = \exp\{\log(H(1)) \cdot y^{-1/\vartheta}\}, \; y > 0$

$\Rightarrow$ H and therefore G are of the same type as Φ_α (Fréchet) for $\alpha = 1/\vartheta$ ($\vartheta > 0$ since $H(y) \xrightarrow[y \to \infty]{} 1$)

2. case: $H(0) = 1 \overset{H \uparrow 1}{\Rightarrow} H(y) = 1$ for all $y \geq 0$

Let $y < 0$. Assume $H(-1) \in (0,1)$.

Set $s = (-y)^{1/\vartheta} > 0$

$\overset{(H)}{\Rightarrow} H(y) = (H(-1))^{1/s} = (H(-1))^{(-y)^{-1/\vartheta}} = \exp\{\log(H(-1)) \cdot (-y)^{-1/\vartheta}\}$,

$y < 0$.

$\Rightarrow$ H and therefore G are of the same type as Ψ_α (Weibull) for $\alpha = -1/\vartheta$ ($\vartheta < 0$ since $H(y) \xrightarrow[y \to -\infty]{} 0$)

$\Rightarrow$ proof.

$\Rightarrow$ Th. 3.8