If

$$\frac{\max(X_1,\dots,X_n)-b_n}{a_n} \xrightarrow[n\to\infty]{d} G \qquad (3.11)$$

for $a_n>0$, $b_n\in\mathbb{R}$, $n\geq 1$ and a non-degenerate distr. G, then we know from the Fisher-Tippett-Gnedenko Theorem (Th. 3.8) that G must be of the same type as one of the distr.: Λ (Gumbel), Φ_α (Fréchet) or Ψ_α (Weibull)

→ natural question:

Can we characterize the distribution of X, for which (3.11) holds with respect to the different extreme value distr. $G=\Lambda, \Phi_\alpha$ and Ψ_α ?

→ our next aim:

Characterization of the maximum domain of attraction $MDA(G)$ for $G=\Lambda, \Phi_\alpha$ and Ψ_α

For this purpose, we need some notions and auxiliary results:

Recall

Def. 3.21 (Quantile function)
Let F be a distr. funct.. Then the function $F^{\leftarrow}$ defined by

$$F^{\leftarrow}(t)=\inf\{x\in\mathbb{R}: F(x)\geq t\},\quad t\in(0,1)$$

is called quantile function of F.

Lemma 3.22 (Properties of F)
(i) $F^{\leftarrow}$ left-continuous, non-decreasing
(ii) $F^{\leftarrow}=F^{-1}$ if F is strictly increasing and continuous
(iii) $F^{\leftarrow}(F(x))\leq x$ for all $x\in\mathbb{R}$ with $F(x)\in(0,1)$.
(iv) $F(F^{\leftarrow}(y))\geq y$ for all $y\in(0,1)$.

Proof: Exercise

Lemma 3.23 (Potter, 1942)
Let L be slowly varying. Then for all $C > 1$, $\delta > 0$ there exists $K = K(C, \delta)$ s.t.

$$\frac{L(y)}{L(x)} \leq C \max\left\{\left(\frac{x}{y}\right)^{\delta}, \left(\frac{y}{x}\right)^{\delta}\right\}$$

for all $x, y > K$

Notation: RV_α set of regularly varying functions, i.e. of functions f of the form $f(x) = x^\alpha L(x)$ for $\alpha \in \mathbb{R}$.

Lemma 3.24: Let $f \in RV_\alpha$ for $\alpha > 0$. Then

$$\lim_{x \to \infty} f(x) = \infty.$$

Proof: Exercise

Maximum domain of attraction of Φ_α

Th 3.2.5 (MDA(Φ_α), Gnedenko, 1943)
Let F be a distr. and let $x^* = \sup\{x \in \mathbb{R} : F(x) < 1\}$. Then $F \in MDA(\Phi_\alpha)$, $\alpha > 0$, iff

(i) $x^* = \infty$

(ii) $\overline{F} \in RV_{-\alpha}$, that is

$$\lim_{x \to \infty} \frac{1 - F(\lambda x)}{1 - F(x)} = \lambda^{-\alpha} \quad \text{for all } \lambda > 0.$$

Ex.: F Pareto distr., i.e. $\overline{F}(x) = x^{-\alpha}$, $x \geq 1$
$\Rightarrow x^* = \infty$ and $F \in MDA(\Phi_\alpha)$

We first prove the direction "$\Leftarrow$" in Th. 3.2.5:

Th. 3.2.6: Let X_i, $i \geq 1$ i.i.d. with $X_1 \sim F$. Assume that $x^* = \infty$, $\overline{F} \in RV_{-\alpha}$ and

$$\lim_{n \to \infty} n \cdot \overline{F}(a_n) = 1$$

for a sequence $a_n, n \geq 1$.

54 Then $\frac{M_n}{a_n} \xrightarrow[n\to\infty]{d} \Phi_\alpha$.

Proof: We know that $x^* = \infty$ and $\overline{F} \in RV_{-\alpha}$.
We want to show that

$$\lim_{n\to\infty} \left(F(a_n x)\right)^n = \Phi_\alpha(x)$$

1. step: We want to show that

$$\lim_{n\to\infty} a_n = \infty$$

Assume this is not true. Then there ex. a bounded subseq. a_{n_k}

$$\Rightarrow \overline{F}(a_{n_k}) \geq \varepsilon > 0 \text{ for all } k \text{ for some } \varepsilon > 0$$

(because of $x^* = \infty$)

$$\Rightarrow \lim_{k\to\infty} n_k \cdot \overline{F}(a_{n_k}) = \infty \quad \Rightarrow \lightning \text{ (contradiction)}$$

$$\Rightarrow \lim_{n\to\infty} a_n = \infty$$