(54) Then $\frac{M_n}{a_n} \xrightarrow[n\to\infty]{d} \Phi_\alpha$.

Proof: We know that $x^* = \infty$ and $\overline{F} \in RV_{-\alpha}$.
We want to show that

$$\lim_{n\to\infty} \left(F(a_n x)\right)^n = \Phi_\alpha(x)$$

1. step: We want to show that

$$\lim_{n\to\infty} a_n = \infty$$

Assume this is not true. Then there ex. a bounded subseq. a_{n_k}

$\Rightarrow$ $\overline{F}(a_{n_k}) \geq \varepsilon > 0$ for all n for some $\varepsilon > 0$
(because of $x^* = \infty$)

$\Rightarrow$ $\lim_{k\to\infty} n_k \cdot \overline{F}(a_{n_k}) = \infty$ $\Rightarrow$ ↯ (contradiction)

$\Rightarrow$ $\lim_{n\to\infty} a_n = \infty$

2. step: Let $x > 0$.

$\overline{F} \in RV_{-\alpha}$ $\Rightarrow$ $n\overline{F}(a_n x) = n\overline{F}(a_n) \cdot \frac{\overline{F}(a_n x)}{\overline{F}(a_n)} \xrightarrow[n\to\infty]{} 1 \cdot x^{-\alpha} = x^{-\alpha}$

$\Rightarrow$ $\left(F(a_n \cdot x)\right)^n = \left(1 - \overline{F}(a_n x)\right)^n$

$= \left(1 - \frac{n\overline{F}(a_n \cdot x)}{n}\right)^n \xrightarrow[n\to\infty]{\text{unif. conv.}} e^{-x^{-\alpha}} = \Phi_\alpha(x)$

3. step: Let $x \leq 0$.

$a_n > 0$ $\forall n \geq n_0$ (because of 1. step)

$\Rightarrow \left(F(\underbrace{a_n \cdot x}_{\leq 0})\right)^n \overset{F\uparrow}{\leq} \left(\underbrace{F(0)}_{<1 \text{ since } x^* = \infty}\right)^n \xrightarrow[n\to\infty]{} 0 = \Phi_\alpha(x)$

$\Rightarrow$ proof.

$\longrightarrow$ <u>possible choice of a_n</u> in Th. 3.2.6

<u>L. 3.2.7</u>: Let $x^* = \infty$ and $\overline{F} \in RV_{-\alpha}$. Define

$$a_n = F^{\leftarrow}\left(1 - \frac{1}{n}\right).$$

Then $$\lim_{n\to\infty} n\,\overline{F}(a_n) = 1.$$

<u>Proof</u>:

<u>1. step</u>: $\overline{F}(a_n) = 1 - \underbrace{F\left(F^{\leftarrow}\left(1-\frac{1}{n}\right)\right)}_{\geq 1-\frac{1}{n} \text{ (L. 3.2.2 (iv))}}$

$\Rightarrow \overline{F}(a_n) \leq 1 - \left(1-\frac{1}{n}\right) = \frac{1}{n}$

$\Rightarrow n\,\overline{F}(a_n) \leq 1 \Rightarrow \overline{\lim_{n\to\infty}}\, n\,\overline{F}(a_n) \leq 1$

<u>2. step</u>: It remains to show that

$$\underline{\lim_{n\to\infty}}\, n\,\overline{F}(a_n) \geq 1$$

Recall: $\underline{\lim}\, b_n \leq \overline{\lim}\, b_n$.

If $\underline{\lim} b_n = \overline{\lim} b_n$ then $\lim b_n = \underline{\lim} b_n = \overline{\lim} b_n$

Let $x \in (0,1)$. Then $x \cdot a_n \to 0$ for all $n \geq n_0$ (since $a_n \xrightarrow[n\to\infty]{} \infty$).

On the other hand,

$F(x\,a_n) < 1 - \frac{1}{n}$ (because of the def. of a_n)

$$\Rightarrow n\,\overline{F}(x\,a_n) = n\left(1 - F(x\,a_n)\right) > n\left(1-\left(1-\tfrac{1}{n}\right)\right)$$
$$= n \cdot \frac{1}{n} = 1$$

$$\Rightarrow n\,\overline{F}(a_n) = n\,\overline{F}(x\,a_n) \cdot \frac{\overline{F}(a_n)}{\overline{F}(x\,a_n)} > \frac{\overline{F}(a_n)}{\overline{F}(x\,a_n)} \xrightarrow[n\to\infty]{\overline{F}\in RV_{-\alpha}} x^{\alpha}$$

$$\Rightarrow \underline{\lim_{n\to\infty}}\, n\,\overline{F}(a_n) \geq x^{\alpha} \quad \text{for all } x \in (0,1).$$

$$\xrightarrow{x\to 1} \underline{\lim_{n\to\infty}}\, n\,\overline{F}(a_n) \geq 1$$

$$\Rightarrow \lim_{n\to\infty} n\,\overline{F}(a_n) = 1 \quad \Longrightarrow \text{proof.}$$

Proof of Th. 3.2.5: "$\Longrightarrow$"

By assumption there ex. a distr. F and $a_n > 0$, $b_n \in \mathbb{R}$ s.t

$$\lim_{n\to\infty} \left(F(a_n x + b_n)\right)^n = \Phi_\alpha(x)$$

We have to show that $x^* = \infty$ and $F \in RV_{-\alpha}$.

1. step: For $s \geq 0$ define

$$a_s = a_{[s]}, \quad b_s = b_{[s]} \qquad ([\cdot] \text{ Gauss bracket})$$

Then

$$\lim_{s\to\infty} \frac{a_{\lambda s}}{a_s} = \lambda^{1/\alpha}, \quad \lim_{s\to\infty} \frac{b_{\lambda\cdot s} - b_s}{a_s} = 0 \qquad (*)$$

for all $\lambda > 0$ (see Exercises 6).

$\Longrightarrow$ the function a_s belongs to $RV_{1/\alpha}$

2. step: We want to show

$$\lim_{s\to\infty} \frac{b_s}{a_s} = 0 \qquad (**)$$

by using $(*)$

We know that $a_x = x^{1/\alpha} L(x)$, L slowly varying function

$$\Longrightarrow \frac{a_{\frac{x}{\lambda}}}{a_x} = \frac{\left(\frac{x}{\lambda}\right)^{1/\alpha} L\left(\frac{x}{\lambda}\right)}{x^{1/\alpha} L(x)} = \lambda^{-1/\alpha} \frac{L\left(\frac{x}{\lambda}\right)}{L(x)}$$

$$\overset{\text{L.3.2.3 (Potter)}}{\underset{\delta = \frac{1}{2\alpha},\, C=2}{\leq}} \lambda^{-1/\alpha}\, 2 \max\left\{\underbrace{\left(\frac{x}{\lambda}\right)^{\frac{1}{2\alpha}}}_{= \left(\frac{1}{\lambda}\right)^{\frac{1}{2\alpha}}}, \underbrace{\left(\frac{x}{\frac{x}{\lambda}}\right)^{\frac{1}{2\alpha}}}_{= (\lambda)^{\frac{1}{2\alpha}}}\right\}$$

$$\overset{\lambda > 1}{\leq} \lambda^{-1/\alpha}\, 2\, (\lambda)^{\frac{1}{2\alpha}} = 2\, \lambda^{-1/2\alpha} \quad \forall\, \frac{x}{\lambda} \geq A \quad (\lambda > 1)$$

$$\Longrightarrow \frac{a_{\frac{x}{\lambda}}}{a_x} \leq 2\, \lambda^{-1/2\alpha} \quad \forall\, \frac{x}{\lambda} \geq A \quad (\lambda > 1) \qquad (+)$$

(57) It also follows from (*) that for $\varepsilon > 0$

$$\left|\frac{b_{2x} - b_x}{a_x}\right| < \varepsilon \quad \forall x \geq \widehat{A} \qquad (++)$$

W.l.o.g. $A = \widehat{A}$

Let $s > A$. Define

$$n = n(s) := \sup\left\{ j \in \mathbb{N}_0 : A < \frac{s}{2^j} \right\}$$

$\Rightarrow$ $s/2^{n+1} \leq A$ and $A < s/2^j$ for all $j = 0, \dots, n$

$$\Rightarrow \left|\frac{b_s}{a_s}\right| = \left| \frac{\left(\sum_{\kappa=1}^{n} b_{s/2^{\kappa-1}} - b_{s/2^\kappa}\right) + b_{s/2^n}}{a_s} \right|$$

$$\leq \sum_{\kappa=1}^{n} \left| \frac{b_{s/2^{\kappa-1}} - b_{s/2^\kappa}}{a_{s/2^\kappa}} \right| \frac{a_{s/2^\kappa}}{a_s} + \frac{|b_{s/2^n}|}{a_s}$$

$\overset{(+),(++)}{\leq}$ $\lambda = 2^\kappa$ $\sum_{\kappa=1}^{n} \varepsilon \cdot 2\, 2^{-\kappa/2\kappa} + \frac{|b_{s/2^n}|}{a_s}$

$s/2^{n+1} \leq A \Rightarrow s/2^n \leq 2A \Rightarrow s/2^n \in [A, 2A]$

$b_s \overset{\text{def}}{=} b[s]$ $\Longrightarrow$ $|b_{s/2^n}| \leq C_1 = C_1(\varepsilon, \kappa)$

$$\Rightarrow \left|\frac{b_s}{a_s}\right| \leq 2\varepsilon \left(\sum_{\kappa \geq 1} 2^{-\kappa/2\kappa} \right) + \frac{C_1}{a_s}$$

$a_s \xrightarrow[s\to\infty]{} \infty$ because of L.3.2.4

$$\Rightarrow \overline{\lim_{s\to\infty}} \left|\frac{b_s}{a_s}\right| \leq 2\varepsilon \underbrace{\left(\sum_{\kappa \geq 1} 2^{-\kappa/2\kappa} \right)}_{< \infty}$$

$\varepsilon \searrow 0$ $\Longrightarrow$ $\overline{\lim_{s\to\infty}} \left|\frac{b_s}{a_s}\right| = 0 \Rightarrow \lim_{s\to\infty} \frac{b_s}{a_s} = 0$

We know that

$$\frac{M_n - b_n}{a_n} \xrightarrow[n\to\infty]{d} \Phi_\kappa \quad \nearrow^0$$

L.3.11 (Slutsky) $\Longrightarrow$

$$\frac{M_n}{a_n} = \frac{M_n - b_n}{a_n} + \frac{b_n}{a_n} \longrightarrow \Phi_\kappa$$

$\Longleftrightarrow \quad (F(a_n x))^n \xrightarrow[n\to\infty]{} \Phi_\alpha(x) \qquad (+++)$

3. step: We want to show that $x^* = \infty$.

Assume that $x^* < \infty$

$\Rightarrow \quad a_n > x^* \ \forall n \geq n_0 \xRightarrow{\text{def. of } x^*} F(a_n) = 1 \ \forall n \geq n_0$

$\Rightarrow \quad (F(a_n))^n \xrightarrow[n\to\infty]{} 1 \neq \Phi_\alpha(1) \Rightarrow \lightning$ (contradiction)

$\Rightarrow \quad x^* = \infty$.

4. step: It follows from (+++) that $e^{-x^{-\alpha}}$

$\log\left((F(a_n x))^n\right) \xrightarrow[n\to\infty]{} \log\left(\Phi_\alpha(x)\right)$

$\Longleftrightarrow \quad n \log(F(a_n x)) \xrightarrow[n\to\infty]{} -x^{-\alpha}, \ x > 0$

We know that

$$\lim_{x\to 0} \frac{\log(1-x)}{x} = -1$$

$\overline{F}(a_n x) \xrightarrow[n\to\infty]{} 0$ (since $a_n \xrightarrow[n\to\infty]{} \infty$) $\qquad F(a_n x)$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{\log\left(1-\overline{F}(a_n x)\right)}{\overline{F}(a_n x)} = -1 \qquad \overline{F}(a_n x) \xrightarrow{x\to\infty} -1$$

$$\Rightarrow \quad n\,\overline{F}(a_n x) = n \log(F(a_n x)) \cdot \frac{\overline{F}(a_n x)}{\log\left(1-\overline{F}(a_n x)\right)}$$

$\log(F(a_n x)) \xrightarrow{x\to\infty} \log(\Phi_\alpha(x))$

$$\xrightarrow[n\to\infty]{} -\log\left(\Phi_\alpha(x)\right) = x^{-\alpha} \qquad (\star\star\star)$$

5. step: Using $(\star\star\star)$ we want to show that

$$\overline{F} \in RV_{-\alpha}$$

Let $x > 0$ and define

$$n(x) = \inf\{ m \in \mathbb{N} : a_{m+1} > x \} \qquad (a_m \xrightarrow[m\to\infty]{} \infty)$$

$\Rightarrow \quad a_{n(x)} \leq x < a_{n(x)+1}$

Further, $\lim_{x\to\infty} n(x) = \infty$, otherwise one can find $l_k \xrightarrow[k\to\infty]{} \infty$ s.t. $n(l_k), k \geq 1$ is bounded

However,

$$a_{\underbrace{n(l_k)+1}_{\text{finitely many numbers}}} > l_k \xrightarrow[k\to\infty]{} \infty \Rightarrow \lightning$$

(59) $\Rightarrow n(x) \xrightarrow[x\to\infty]{} \infty$

$\overline{F}\downarrow$

$$\Rightarrow \quad \frac{\overline{F}(\lambda x)}{\overline{F}(x)} \leq \frac{\overline{F}(\lambda a_{n(x)})}{\overline{F}(a_{n(x)+1})} = \frac{\overline{F}(\lambda a_{n(x)})}{\overline{F}(a_{n(x)+1})} \cdot \frac{n(x)}{(n(x)+1)} \cdot \frac{n(x)+1}{n(x)}$$

$(\lambda \cdot 1.)^{-\alpha}$; $(\ast\ast\ast) \to \infty$; $\frac{1}{1} \to \infty$; $(1)^{-\alpha}$

$$\xrightarrow[x\to\infty]{} \lambda^{-\alpha}$$

$$\Rightarrow \quad \overline{\lim_{x\to\infty}} \frac{\overline{F}(\lambda x)}{\overline{F}(x)} \leq \lambda^{-\alpha}$$

Similarly, one shows that

$$\underline{\lim_{x\to\infty}} \frac{\overline{F}(\lambda x)}{\overline{F}(x)} \geq \lambda^{-\alpha} \quad \Rightarrow \lim_{x\to\infty} \frac{\overline{F}(\lambda x)}{\overline{F}(x)} = \lambda^{-\alpha}$$

$\Rightarrow$ proof.