Maximum domain of attraction for Ψ_α

We need the following definition

Def. 3.2.8 (Regular variation in zero)
A measurable function $f: (0, A) \longrightarrow (0, \infty)$ is called regularly varying in 0 with index $\alpha \in \mathbb{R}$, if

$$\lim_{x \downarrow 0} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha$$

for all $\lambda > 0$.
The class of such functions is denoted by $RV_\alpha(0)$

Ex.: f given by $f(x) = x^\alpha$ belongs to $RV_\alpha(0)$

Th. 3.2.9 (MDA(Ψ_α), Gnedenko, 1943)
Let F be a distr. and let $x^* := \sup\{x \in \mathbb{R} : F(x) < 1\}$.
Then $F \in MDA(\Psi_\alpha)$, $\alpha > 0$, iff

(i) $x^* < \infty$.

(ii) The function given by $\overline{F}(x^* - x)$, $x > 0$ is in $RV_\alpha(0)$, that is

$$\lim_{x \downarrow 0} \frac{1 - F(x^* - \lambda x)}{1 - F(x^* - x)} = \lambda^\alpha$$

for all $\lambda > 0$.

Rem. 3.2.10: If (i) and (ii) are satisfied, then

$$\frac{M_n - x^*}{a_n} \xrightarrow[n \to \infty]{d} \Psi_\alpha,$$

where $a_n > 0$, $n \geq 1$ is a sequence s.t.

$$\lim_{n \to \infty} n \overline{F}(x^* - a_n) = 1.$$

One can e.g. choose

$$a_n = x^* - F^{\leftarrow}\left(1 - \frac{1}{n}\right).$$

See the proof of Th. 3.2.9

Ex: $F(x) = 1-(x^*-x)^\alpha$ with $x \in (x^*-1, x^*)$, $\alpha>0$
Then $\overline{F}(x^*-\cdot) \in RV_\alpha(0)$ and therefore $F \in MDA(\Psi_\alpha)$.

Proof of Th. 3.2.9:

"$\Leftarrow$": Define the distr. F^* given by

$$\overline{F}^*(x) = \overline{F}\left(x^* - \frac{1}{x}\right)$$

(ii) $\Rightarrow$ $\overline{F}^* \in RV_{-\alpha}$ $\overset{\text{Th. 3.2.5}}{\Longrightarrow}$ $F^* \in MDA(\overline{\Phi}_\alpha)$ (Fréchet)

$\Rightarrow$ For $a_n^* := \gamma_n^* := (F^*)^{\leftarrow}(1-\frac{1}{n})$ and $b_n^* := 0$ we have that

$$\lim_{n\to\infty} \left(F^*(a_n^* x + b_n^*)\right)^n = \lim_{n\to\infty} \left(F^*(a_n^* x)\right)^n$$

$$\overset{\text{def.}}{=} \lim_{n\to\infty} \left(F\left(x^* - \frac{1}{a_n^* x}\right)\right)^n = \overline{\Phi}_\alpha(x) \quad \forall x>0$$

$\Rightarrow$ For $a_n := \frac{1}{a_n^*}$, $b_n := x^*$ we have that

$$\lim_{n\to\infty} \left(F\left(b_n - \frac{a_n}{x}\right)\right)^n = \overline{\Phi}_\alpha(x) \quad \forall x>0$$

$$\Rightarrow \lim_{n\to\infty} \left(F(a_n x + b_n)\right)^n = \overline{\Phi}_\alpha\left(\tfrac{1}{x}\right) = \Psi_\alpha(x) \quad \forall x<0$$

On the other hand,

$$F(a_n x + b_n) = F(\underbrace{a_n}_{>0} x + x^*) \equiv 1 \quad \forall x \geq 0$$

It remains to show that

$$a_n = x^* - \gamma_n, \quad \gamma_n \overset{\text{def.}}{=} F^{\leftarrow}(1-\tfrac{1}{n})$$

We find that

$$\begin{aligned}
a_n^* \overset{\text{def}}{=} \gamma_n^* &\overset{\text{def}}{=} \inf\left\{x>0 : F^*(x) \geq 1-\tfrac{1}{n}\right\} \\
&\overset{\text{def}}{=} \inf\left\{x>0 : F\left(x^* - \tfrac{1}{x}\right) \geq 1-\tfrac{1}{n}\right\} \\
&= \inf\left\{x>0 : u = x^* - \tfrac{1}{x} \text{ and } F(u) \geq 1-\tfrac{1}{n}\right\} \\
&= \inf\left\{x>0 : x = (x^*-u)^{-1} \text{ and } F(u) \geq 1-\tfrac{1}{n}\right\} \\
&= \inf\left\{(x^*-u)^{-1}>0 : F(u) \geq 1-\tfrac{1}{n}\right\} \\
&= \left(\sup\left\{(x^*-u)>0 : F(u) \geq 1-\tfrac{1}{n}\right\}\right)^{-1} \\
&= \left(x^* - \inf\left\{u<x^* : F(u) \geq 1-\tfrac{1}{n}\right\}\right)^{-1} \\
&= (x^* - \gamma_n)^{-1}
\end{aligned}$$

$\overset{\text{def}}{\Rightarrow}$ $a_n = (x^* - \gamma_n)$ $\Rightarrow$ proof of "$\Leftarrow$"

(62) "$\Longrightarrow$" : similar to the case Φ_α

In the sequel we need the following auxiliary result, which plays an important role in extreme value theory:

Prop. 3.2.11 (Poisson approximation)

Let $\tau \in [0,\infty]$, F a distr. and $u_n, n \geq 1$. Then

(i) $n(1-F(u_n)) \xrightarrow[n\to\infty]{} \tau$

$\Longleftrightarrow$

(ii) $P(M_n \leq u_n) = (F(u_n))^n \xrightarrow[n\to\infty]{} e^{-\tau}$

Proof:

1. case: $\tau < \infty$

"$\Longrightarrow$": $(F(u_n))^n = \left(1 - \frac{1}{n}\underbrace{(n(1-F(u_n)))}_{\xrightarrow[n\to\infty]{}\tau}\right)^n \xrightarrow[n\to\infty]{\text{uniform conv.}} e^{-\tau}$

"$\Longleftarrow$": We want to show first that

$$1 - F(u_n) \xrightarrow[n\to\infty]{} 0 \qquad (*)$$

Assume (*) doesn't hold

$\Rightarrow \exists$ subsequence $u_{n_k}, k \geq 1$ and $\varepsilon \in (0,1)$:

$$1 - F(u_{n_k}) \geq \varepsilon$$

$\Rightarrow (F(u_{n_k}))^{n_k} \leq (1-\varepsilon)^{n_k} \xrightarrow[k\to\infty]{} 0 \overset{(i)}{\Longrightarrow} \lightning$

$\Rightarrow$ (*) holds.

$\log(\cdot)$ applied to (ii) gives

$$n \log(1-(1-F(u_n))) = n \log F(u_n) \xrightarrow[n\to\infty]{} -\tau$$

On the other hand,

$$n \log(1-(1-F(u_n))) = n(1-F(u_n)) \underbrace{\frac{\log(1-(1-F(u_n)))}{(1-F(u_n))}}_{\to -1}$$

because of L'Hospital and $1-F(u_n) \xrightarrow[n\to\infty]{} 0$

$$\Rightarrow \lim_{n\to\infty} n(1-F(u_n)) = \tau$$

2. case: $\bar{T} = \infty$

"$\Rightarrow$": Assume that (ii) doesn't hold.

$\Rightarrow \exists\, u_{n_k}, k \geq 1, \varepsilon > 0$ s.t.

$$\left(F(u_{n_k})\right)^{n_k} \geq \varepsilon \quad \forall k \geq 1$$

Further, since $\left(F(u_{n_k})\right)^{n_k}, k \geq 1$ is bounded, there ex. a subseq. of this sequence, which converges. W.l.o.g. assume that

$$\left(F(u_{n_k})\right)^{n_k} \xrightarrow[k\to\infty]{} \mu \geq \varepsilon > 0$$

Proof of 1. case, "$\Leftarrow$"

$\Rightarrow$ $\mu = e^{-\tau}$ for $\tau \in [0, \infty)$

$$\lim_{k\to\infty} n_k \left(1 - F(u_{n_k})\right) = \tau < \infty$$

$\Rightarrow$ $\lightning$ $\Rightarrow$ (ii) holds.

"$\Leftarrow$": Assume (i) doesn't hold.

$\Rightarrow \exists$ subseq. $u_{n_k}, k \geq 1$:

$$\lim_{k\to\infty} n_k \left(1 - F(u_{n_k})\right) = \tau < \infty$$

Proof of 1. case, "$\Rightarrow$"

$$\left(F(u_{n_k})\right)^{n_k} \xrightarrow[k\to\infty]{} e^{-\tau} \neq 0 \Rightarrow \lightning$$

$\Rightarrow$ proof.

Maximum domain of attraction for Λ

Th. 3.2.12 (MDA(Λ), Gnedenko, 1943)
Let F be a distr. and $x^* = \sup\{x \in \mathbb{R} : F(x) < 1\}$.
Then $F \in \mathrm{MDA}(\Lambda)$, iff there ex. a measurable function $g > 0$ s.t.

$$\lim_{x \uparrow x^*} \frac{\bar{F}(x + u g(x))}{\bar{F}(x)} = e^{-u} \qquad (3.11)$$

for all $u \in \mathbb{R}$.

Rem. 3.2.13 (3.11) implies

$$\frac{M_n - b_n}{a_n} \xrightarrow[n\to\infty]{d} \Lambda$$

for sequences $a_n > 0, b_n, n \geq 1$ s.t.

$$\lim_{n\to\infty} n \bar{F}(b_n) = 1, \quad a_n = g(b_n)$$

Here one can e.g. choose

$$b_n = F^{\leftarrow}\left(1-\frac{1}{n}\right)$$

Proof of Th. 3.2.12:

"$\Leftarrow$": One shows that

$$\lim_{y \uparrow 1} F^{\leftarrow}(y) = x^*$$

See exercises.

$$\Rightarrow \quad b_n := F^{\leftarrow}\left(1-\frac{1}{n}\right) \xrightarrow[n\to\infty]{} x^*$$

Using (3.11), one shows similarly to L.3.2.7 that

$$\lim_{n\to\infty} n\,\overline{F}(b_n) = 1$$

$$\Rightarrow \quad n\left(1-F(b_n + x\,g(b_n))\right) =$$

$$= \underbrace{n\,\overline{F}(b_n)}_{\to 1} \; \underbrace{\frac{n\left(1-F(b_n+x\,g(b_n))\right)}{n\,\overline{F}(b_n)}}_{\to e^{-x}} \xrightarrow[n\to\infty]{(3.11)} e^{-x}$$

Prop. 3.2.11 $\Rightarrow$

$$\left(F(b_n + x\,g(b_n))\right)^n \xrightarrow[n\to\infty]{} e^{-e^{-x}} = \Lambda(x)$$

(a_n)

"$\Rightarrow$": omitted.

Convergence rates for normalized extremal events of i.i.d. random variables

We have discussed so far the characterization of the maximum domain of attraction $MDA(G)$ for the three possible extreme value distr. of the type $G = \Lambda, \Phi_\alpha$ and Ψ_α

In order to be able to apply such results in practice, we also need to study convergence rates of normalized extremal events

$$\frac{M_n - b_n}{a_n} \xrightarrow[n\to\infty]{d} G$$

$$\Leftrightarrow \quad \left| F(a_n \cdot x + b_n)^n - G(x) \right| \xrightarrow[n\to\infty]{} 0$$

in the pointwise or uniform sense.

For this purpose, we need the following estimate:

L.3.2.14: For $0 \le \tau < \min(2, \sqrt{2n-1}) =: c_n$ we have that

$$\frac{\tau^2 e^{-\tau}}{2n} \le e^{-\tau} - \left(1 - \frac{\tau}{n}\right)^n \qquad (3.12)$$

Further, for all $0 \le \tau < \sqrt{2n-1}$, $n \ge 2$

$$e^{-\tau} - \left(1 - \frac{\tau}{n}\right)^n \le \frac{\tau^2 e^{-\tau}}{2n-1} \le \frac{0.6}{2n-1}. \qquad (3.13)$$

Proof: Set $h_n(\tau) = e^{\tau}\left(1 - \frac{\tau}{n}\right)^n$ for $0 \le \tau < c_n$

Then (3.12) and (3.13) are equivalent to

$$1 - \frac{\tau^2}{2n} \ge h_n(\tau) \ge 1 - \frac{\tau^2}{2n-1}$$

$$\Leftrightarrow \quad \log\left(1 - \frac{\tau^2}{2n}\right) \ge \tau + n\log\left(1 - \frac{\tau}{n}\right) \ge \log\left(1 - \frac{\tau^2}{2n-1}\right)$$

Define

$$g_n(\tau) = \tau + n\log\left(1 - \frac{\tau}{n}\right) - \log\left(1 - \frac{\tau^2}{2n}\right)$$

for $0 \le \tau < c_n$.

Then
$$g_n'(\tilde{\tau}) = 1 - \frac{1}{1-\frac{\tilde{\tau}}{n}} + \frac{\tilde{\tau}}{n-\frac{\tilde{\tau}^2}{2}} = \frac{\tilde{\tau}-2}{(n-\tilde{\tau})(n-\frac{\tilde{\tau}^2}{2})} \cdot \frac{\tilde{\tau}^2}{2} \leq 0$$

$\Rightarrow$ $g_n \searrow$ on $[0, c_n)$ with $g(0)=0$

$\Rightarrow$ $g_n(\tilde{\tau}) \leq 0$ for all $\tilde{\tau} \in [0, c_n)$

$\Leftrightarrow$ $\log\left(1-\frac{\tilde{\tau}^2}{2n}\right) \geq \tilde{\tau} + n\log\left(1-\frac{\tilde{\tau}}{n}\right)$

for all $\tilde{\tau} \in [0, c_n)$

Similarly, one shows for
$$G_n(\tilde{\tau}) := \tilde{\tau} + n\log\left(1-\frac{\tilde{\tau}}{n}\right) - \log\left(1-\frac{\tilde{\tau}^2}{2n-1}\right), \quad 0 \leq \tilde{\tau} \leq c_n$$
that $G_n'(\tilde{\tau}) \geq 0$, $0 \leq \tilde{\tau} \leq c_n$

$G_n(0)=0$

$\Rightarrow$ $G_n(\tilde{\tau}) \geq 0$, $0 \leq \tilde{\tau} < c_n$

$\Leftrightarrow$ $\tilde{\tau} + n\log\left(1-\frac{\tilde{\tau}}{n}\right) \geq \log\left(1-\frac{\tilde{\tau}^2}{2n-1}\right)$, $0 \leq \tilde{\tau} < c_n$

Since the function
$$f(\tilde{\tau}) := \tilde{\tau}^2 e^{-\tilde{\tau}}, \quad \tilde{\tau} > 0$$
attains its maximum at $\tilde{\tau}=2$ with $f(2) = 4e^{-2} \leq 0.6$, we obtain the upper bound in (3.13).

$\Rightarrow$ proof.

Th. 3.2.15 (Convergence rates for normalized extremal events)

Let $\gamma_n := F^{\leftarrow}(1-\frac{1}{n})$, $x^* := \sup\{y \in \mathbb{R} : F(y) < 1\}$ and g the measurable function in Th. 3.2.12.

Define
$$\tau_n(x) = \begin{cases} n(1-F(\gamma_n + x\,g(\gamma_n))) - e^{-x}, & x \in \mathbb{R}, \text{ if } F \in MDA(\Lambda) \\ n(1-F(\gamma_n x)) - x^{-\alpha}, & x > 0, \text{ if } F \in MDA(\Phi_\alpha) \\ n(1-F(x^* + x(x^*-\gamma_n))) - (-x)^{\alpha}, & x < 0, \text{ if } F \in MDA(\Psi_\alpha) \end{cases}$$

Then for sufficiently large n, we have that

(i) for $F \in MDA(\Lambda)$: (a_n, b_n)
$$|P(M_n \leq g(\gamma_n)x + \gamma_n) - \Lambda(x)| \leq \frac{0.6}{2n-1} + e^{-e^{-x}}\left|e^{-\tau_n(x)} - 1\right| \quad (3.14)$$
$$= \mathcal{O}\left(\max\left(\tfrac{1}{n}, |\tau_n(x)|\right)\right) \quad (x \in \mathbb{R}, n \to \infty)$$

Recall: $c_n = \mathcal{O}(d_n)$, $n \to \infty$ $\Leftrightarrow$ $|c_n/d_n| \leq M < \infty$ for all $n \geq n_0$

(ii) for $F \in MDA(\Phi_\alpha)$:

$$|P(M_n \le x\gamma_n) - \Phi_\alpha(x)| \le \frac{0.6}{2n-1} + e^{-x^{-\alpha}} |e^{-r_n(x)} - 1| \quad (3.15)$$

(γ_n : a_n)

$$= O\left(\max\left(\tfrac{1}{n}, |r_n(x)|\right)\right) \quad (x>0, n\to\infty)$$

(iii) for $F \in MDA(\Psi_\alpha)$:

$$|P(M_n \le (x^* - \gamma_n)x + x^*) - \Psi_\alpha| \le \frac{0.6}{2n-1} + e^{-(-x)^\alpha} |e^{-r_n(x)} - 1| \quad (3.16)$$

($(x^* - \gamma_n)$: a_n, x^* : b_n)

$$= O\left(\max\left(\tfrac{1}{n}, |r_n(x)|\right)\right) \quad (x<0, n\to\infty)$$