(ii) for $F \in MDA(\Phi_\alpha)$:

$$|P(M_n \le x\gamma_n) - \Phi_\alpha(x)| \le \frac{0.6}{2n-1} + e^{-x^{-\alpha}} |e^{-\tau_n(x)} - 1| \quad (3.15)$$

($\gamma_n = a_n$)

$$= \mathcal{O}\left(\max\left(\tfrac{1}{n}, |\tau_n(x)|\right)\right) \quad (x>0, n\to\infty)$$

(iii) for $F \in MDA(\Psi_\alpha)$:

$$|P(M_n \le (x^* - \gamma_n)x + x^*) - \Psi_\alpha| \le \frac{0.6}{2n-1} + e^{-(-x)^\alpha} |e^{-\tau_n(x)} - 1| \quad (3.16)$$

($\gamma_n = a_n$, $x^* = b_n$)

$$= \mathcal{O}\left(\max\left(\tfrac{1}{n}, |\tau_n(x)|\right)\right) \quad (x<0, n\to\infty)$$

Proof: Define

$$u_n(x) = \begin{cases} \gamma_n + x\, g(\gamma_n) & , F \in MDA(\Lambda) \\ \gamma_n x & , F \in MDA(\Phi_\alpha) \\ (x^* - \gamma_n)x + x^* & , F \in MDA(\Psi_\alpha) \end{cases}$$

So we can write $u_n(x)$ as

$$u_n(x) = a_n x + b_n$$

for sequences $a_n, b_n, n \ge 1$ as in L. 3.2.7 (i.e. Φ_α), Rem. 3.2.10 (i.e. Ψ_α) and Rem. 3.2.13 (i.e. Λ)

$$\Rightarrow \quad \tau_n(x) \overset{\text{Def}}{=} n(1 - F(u_n(x))) + \log G(x)$$

for $G = \Lambda, \Phi_\alpha$ or Ψ_α

Define $\tilde{\tau}_n(x) = n(1 - F(u_n(x))) = \tau_n(x) - \log(G(x))$

From Prop. 3.2.11 (Poisson approximation) we obtain that

$$\tau_n(x) \xrightarrow[n\to\infty]{} 0. \quad (\Leftrightarrow \tilde{\tau}_n(x) \xrightarrow[n\to\infty]{} -\underbrace{\log(G(x))}_{>0})$$

$\tilde{\tau}_n(x) \ge 0$ for all $n \ge 1$

L. 3.2.14 $\Rightarrow$

$$0 \le e^{-\tilde{\tau}_n(x)} - (F(u_n(x)))^n = e^{-\tilde{\tau}_n(x)} - \left(1 - \frac{\overbrace{n(1 - F(u_n(x)))}^{=\tilde{\tau}_n(x)}}{n}\right)^n$$

$$\le \frac{0.6}{2n-1} \quad \text{for sufficiently large } n$$

$$\Rightarrow |(F(u_n(x)))^n - G(x)| \le \underbrace{|e^{-\tilde{\tau}_n(x)} - G(x)|}_{= G(x)|e^{-\tau_n(x)} - 1|} + |e^{-\tilde{\tau}_n(x)} - (F(u_n(x)))^n|$$

$$\le \frac{0.6}{2n-1} + G(x)|e^{-\tau_n(x)} - 1| \quad \Rightarrow \text{proof.}$$

Ex. 3.2.16 ($F \sim Exp(\lambda)$)

$F(x) = 1 - e^{-\lambda x}$, $x \geq 0$, $\lambda > 0$

We know that $F \in MDA(\Lambda)$.

In fact, we can choose in Th. 3.2.12

$g(x) \equiv \frac{1}{\lambda}$

$\Rightarrow r_n(x) = n(1 - F(\gamma_n + \frac{x}{\lambda})) - e^{-x}$

$= n e^{-(\lambda \gamma_n + x)} - e^{-x} = 0$, $x > -\log n$

since $\gamma_n = F^{\leftarrow}(1 - \frac{1}{n}) = F^{-1}(1 - \frac{1}{n}) = -\frac{1}{\lambda}\log(\frac{1}{n})$

Th. 3.2.15 $\Rightarrow$ $|P(M_n \leq \frac{x}{\lambda} + \frac{\log(n)}{\lambda}) - e^{-e^{-x}}| \leq \frac{0.6}{2n-1}$

for $x > -\log n$

On the other hand, it follows from (3.12) for $\lambda = e^{-x}$, $x > 0$ that

$$\frac{e^{-2x} e^{-e^{-x}}}{2n} \leq |P(M_n \leq \frac{x}{\lambda} + \frac{\log(n)}{n}) - e^{-e^{-x}}|$$

$\Rightarrow$ $O(\frac{1}{n})$ exact order of convergence

Ex 3.2.17 ($F \sim N(0,1)$)

We know from Exerc. 6 that $F \in MDA(\Lambda)$

One can choose

$$\gamma_n^2 = 2\log(n) - \log(\log(n)) - \log(4\pi)$$

and gets that

$$|r_n(x)| = O\left(\frac{\log(\log n)}{\log(n)}\right)$$

Th. 3.2.15 $\Rightarrow$ $|P(M_n \leq \frac{x}{\gamma_n} + \gamma_n) - \Lambda(x)| = O\left(\frac{\log(\log(n))}{\log(n)}\right)$

Exercise.

4. Statistics of extreme value distributions

→ objective:
Estimation of the parameters of extreme value distributions from empirical data

→ two possible statistical methods:

1. Block maxima method based on generalized extreme value distributions (GEV)
2. Peaks over threshold method based on generalized Pareto distributions (GPD)

1. Statistics of block maxima: GEV-distribution

We have seen in Section 3 that extreme value distributions are given by GEV-distr., that is

$$G_{\gamma,\mu,\beta}(z) = \exp\left\{-\left(1+\gamma\frac{z-\mu}{\beta}\right)^{-\frac{1}{\gamma}}\right\}, \text{ if } 1+\gamma\frac{z-\mu}{\beta}>0, \tag{4.1}$$

where $\gamma \in \mathbb{R}$ is the shape parameter, μ the location parameter and β the scale parameter

We have the following cases for γ:

$\gamma > 0$: G is of the type Φ_α for $\alpha = \frac{1}{\gamma}$, that is

$$G(z) = \begin{cases} \exp\left(-\left(1+\gamma\frac{z-\mu}{\beta}\right)^{-\frac{1}{\gamma}}\right), \text{ if } \frac{z-\mu}{\beta} > -\frac{1}{\gamma} \\ 0 \text{ else} \end{cases} \tag{4.2}$$

$\gamma < 0$: G is of the type Ψ_α for $\alpha = -\frac{1}{\gamma}$, that is

$$G(z) = \begin{cases} \exp\left(-\left(1+\frac{\gamma(z-\mu)}{\beta}\right)^{-\frac{1}{\gamma}}\right), \text{ if } \frac{z-\mu}{\beta} < -\frac{1}{\gamma} \\ 1 \text{ else} \end{cases} \tag{4.3}$$

$\gamma = 0$: G is of the type Λ, that is

$$G(z) = \exp\left(-\exp\left(\frac{z-\mu}{\delta}\right)\right), \quad z \in \mathbb{R} \qquad (4.4)$$

In order to illustrate the block maxima method consider for day $j \in \{1, \dots, 365\}$ in the year $i \in \{1, \dots, n\}$ the water level

$$x_{ij}$$

at a dike.

→ block maxima:

$$x_i := \max_{j=1}^{365} x_{ij}$$

→ problem: How can we estimate the dike height z_p from the data $x_i, i = 1, \dots, n$, which prevents the dike from being flooded with probability p?

In the sequel we want to distinguish between a stationary and non-stationary model with respect to $x_1, \dots, x_n$ as realizations of r.v.'s $X_1, \dots, X_n$:

1. stationary model:

Assumption: $X_1, \dots, X_n$ i.i.d. r.v.'s

→ in particular: $X_1, \dots, X_n$ identically distr.

→ this model can only be applied to stationary data x_i, that is to data x_i which don't show a trend (e.g. the trend that the maximal water levels x_i increase (or decrease) over time)

In addition

$$X_i = \max_{j=1}^{365} X_{ij}$$

(71)

where

$X_{i,j}, j = 1, \dots, 365$

are i.i.d. r.v.'s for all $i = 1, \dots, n$

Since a distr. belongs to the maximum domain of attraction of an extreme value distr. under very general conditions and since $m = 365$ is a large sample size it is also reasonable to assume that

$$F_{X_1} = G_{\gamma,\mu,\beta}$$

for the parameter vector $\theta = (\gamma, \mu, \beta) \in \mathbb{R}^2 \times (0, \infty)$

Here the unknown normalizing constants $a_m > 0, b_m \in \mathbb{R}$ can be assumed to already be included in the parameters γ, μ, β.

→ it remains to estimate θ from $x_1, \dots, x_n$!

→ possible method : maximum likelihood estimation (MLE)

→ differentiation of $G_{\gamma,\mu,\beta}$ yields the prob. density $f_{\gamma,\mu,\beta}$:

$\gamma \neq 0$:

$$f_\theta(z) = f_{\gamma,\mu,\beta}(z) = \begin{cases} \frac{1}{\beta}\left(1+\gamma\frac{z-\mu}{\beta}\right)^{-\frac{1}{\gamma}-1} \exp\left(-\left(1+\gamma\frac{z-\mu}{\beta}\right)^{-\frac{1}{\gamma}}\right), & 1+\gamma\frac{z-\mu}{\beta} > 0 \\ 0 \text{ else} \end{cases} \tag{4.5}$$

$\gamma = 0$:

$$f_\theta(z) = f_{0,\mu,\beta}(z) = \frac{1}{\beta} e^{-\frac{z-\mu}{\beta}} \exp\left(-e^{-\frac{z-\mu}{\beta}}\right), \quad z \in \mathbb{R} \tag{4.6}$$

→ log-likelihood function

$$\ell(\theta) := \sum_{i=1}^{n} \log\left(f_\theta(x_i)\right)$$

$\gamma \neq 0$:

$$\ell(\theta) = n \log \sigma - \left(\frac{1}{\gamma}+1\right) \sum_{i=1}^{n} \log\left(1+\gamma \frac{x_i-\mu}{\sigma}\right) - \sum_{i=1}^{n} \left(1+\gamma \frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\gamma}}$$

if $1+\gamma \frac{x_i-\mu}{\sigma} > 0$ for all $i = 1, \dots, n$

and

$\ell(\theta) = -\infty$ else

$\gamma = 0$:

$$\ell(\theta) = -n \log \sigma - \sum_{i=1}^{n} \frac{x_i-\mu}{\sigma} - \sum_{i=1}^{n} e^{-\frac{x_i-\mu}{\sigma}}$$

→ maximization of $\ell(\theta)$ w.r.t. θ (in general numerically) gives the maximum likelihood estimator

$$\hat{\theta} = (\hat{\gamma}, \hat{\mu}, \hat{\sigma}) = \operatorname{argmax} \ell(\gamma, \mu, \sigma)$$

→ the latter estimation enables us to find the dike height z_p as a $(1-p)$-quantile of the annual maxima, that is as the (smallest) solution $\hat{z}_p$ to the equation

$$G_{\hat{\gamma},\hat{\mu},\hat{\sigma}}(\hat{z}_p) = 1-p$$

(4.2), (4.3) →

$$\hat{z}_p = \begin{cases} \hat{\mu} - \frac{\hat{\sigma}}{\hat{\gamma}} \left(1 - (-\log(1-p))^{-\hat{\gamma}}\right), & \hat{\gamma} \neq 0 \\ \hat{\mu} - \hat{\sigma} \log(-\log(1-p)), & \hat{\gamma} = 0 \end{cases} \qquad (4.7)$$