for functions h in

$$C_0^+(E) := \{ f : E \to [0,\infty) : f \text{ continuous and } f \text{ vanishes outside a compact set} \}$$

Here the functions $h_i \in C_0^+(E)$ in (5.6) have the following properties:

(i) If $\mu, \mu_n \in M_+(E)$, $n \geq 1$ then $\mu_n(h_i) \xrightarrow[n\to\infty]{} \mu(h_i)$ for all $i \geq 1$ iff

$$\int_E f(y)\, \mu_n(dy) \xrightarrow[n\to\infty]{} \underbrace{\int_E f(y)\, \mu(dy)}_{= \mu(f)} \qquad (5.7)$$

for all $f \in C_0^+(E)$.

(ii) If $\mu, \mu' \in M_+(E)$ with

$$\mu(h_i) = \mu'(h_i) \text{ for all } i \geq 1$$

then $\mu = \mu'$.

See S. I. Resnick, Springer 1987

$$\xrightarrow{(5.6)} \quad d(\mu_n, \mu) \xrightarrow[n\to\infty]{} 0 \iff \mu_n(h_i) \xrightarrow[n\to\infty]{} \mu(h_i) \text{ for all } i$$

$$\xLeftrightarrow{(5.7)} \quad \mu_n(f) \xrightarrow[n\to\infty]{} \mu(f) \text{ for all } f \in C_0^+(E)$$

$\longrightarrow$ Def. 5.2 (weak convergence in $M_+(E)$)

If $d(\mu_n, \mu) \xrightarrow[n\to\infty]{} 0$ $\left(\iff \mu_n(f) \xrightarrow[n\to\infty]{} \mu(f) \text{ for all } f \in C_0^+(E)\right)$

we say μ_n converges to μ weakly (or vaguely) and write

$$\mu_n \xrightarrow[n\to\infty]{v} \mu$$

Rem: (5.5) $\iff m_n \xrightarrow[n\to\infty]{v} m$ (see later).

Th. 5.3: $(M_+(E), d)$ is a complete separable metric space, that is

(i) $d(\mu_n, \mu_m) \xrightarrow[n,m\to\infty]{} 0 \iff \exists \mu \in M_+(E) : d(\mu_n, \mu) \xrightarrow[n\to\infty]{} 0$

(ii) There exists a countable set $\mathcal{U} \subseteq M_+(E)$ such that

for all $\mu \in M_+(E)$ there exists $\mu_n \in \mathcal{U}$, $n \geq 1$ such that

$$d(\mu_n, \mu) \xrightarrow[n\to\infty]{} 0$$

So $\mathcal{U}$ is dense in $M_+(E)$

Prop. 5.4: For $\mathcal{M}_+(E) = \mathcal{B}(M_+(E))$ we have that

$$\mathcal{M}_+(E) = \text{smallest } \sigma\text{-algebra containing all sets of the form } \{\mu \in M_+(E) : \mu(f) \in B\},\ f \in C_0^+(E),\ B \in \mathcal{B}([0,\infty]) \quad (5.8)$$

$$= \text{smallest } \sigma\text{-algebra containing all sets of the form } \{\mu : \mu(G) \in B\},\ G \in \mathcal{B}(E),\ B \in \mathcal{B}([0,\infty]) \quad (5.9)$$

Proof:

(5.8): Because of (5.7) we see that for $\mu_n \xrightarrow[n\to\infty]{v} \mu$

$\mu_n(f) \xrightarrow[n\to\infty]{} \mu(f)$ for all $f \in C_0^+(E)$.

$\Rightarrow$ the function $T : M_+(E) \to \mathbb{R}$, $T(\mu) := \mu(f)$ is continuous in μ

$\Rightarrow$ T measurable, that is

$$\{\mu : \underbrace{T(\mu)}_{=\mu(f)} \in B\} \in \mathcal{M}_+(E) = \mathcal{B}(M_+(E))$$

for all $B \in \mathcal{B}([0,\infty])$

$\Rightarrow$ right hand side of (5.8) is contained in $\mathcal{M}_+(E)$

The other inclusion is left as an excercise

(5.9): see later.

In the sequel we need the following useful notions and results w.r.t. $M_+(E)$:

Th. 5.5 (Portmanteau)

Let $\mu, \mu_n \in M_+(E)$, $n \geq 1$. Then

(i) $\mu_n \xrightarrow[n\to\infty]{v} \mu$

$\Longleftrightarrow$

(ii) $\mu_n(B) \xrightarrow[n\to\infty]{} \mu(B)$ for all B with $\overline{B}$ compact

and $\mu(\partial B) = 0$.

Here $\overline{B}$ = closure of B, $\partial B = \overline{B} \cap \overline{B^c}$ boundary of B



$\Leftrightarrow$

(iii) $\overline{\lim_{n\to\infty}} \mu_n(K) \leq \mu(K)$ and $\underline{\lim_{n\to\infty}} \mu_n(G) \geq \mu(G)$

for compact K and open G with $\overline{G}$ compact

Let (S, d) be a complete separable metric space.

Recall a set K in in S is compact iff for all sequences $x_n, n \geq 1$ in K there exists a subsequence $x_{n_k}, k \geq 1$ and a $x \in K$ such that

$$x_{n_k} \xrightarrow[k\to\infty]{} x \quad (\Leftrightarrow d(x_{n_k}, x) \xrightarrow[k\to\infty]{} 0)$$

The next result gives a characterization of compact sets in $S = M_+(E)$!

closure of M ↓ $\overline{M}$

Th. 5.6 (Compactness of sets in $M_+(E)$)

A subset M of $M_+(E)$ is relatively compact (i.e. $\overline{M}$ is compact) iff $\sup_{\mu \in M} \mu(f) < \infty$

for all $f \in C_0^+(E)$.

See e.g. S. I. Resnick, Springer, 1987

Def. 5.7 (tightness of probability measures)

Let (S, d) be a complete separable metric space. A family Π of probability measures on $\mathcal{B}(S)$ (Borel-σ-algebra on S) is called tight, if for all $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq S$ such that for all $P \in \Pi$

$$P(K_\varepsilon) > 1 - \varepsilon$$

Def. 5.8 (Weak convergence of probability measures on S)
Let $P_n, n \geq 1, P$ be probability measures on $\mathcal{B}(S)$.
We shall say P_n converges weakly (or vaguely) to P and write $P_n \xrightarrow{v} P$ for $n \to \infty$, if

$$P_n(f) \xrightarrow[n\to\infty]{} P(f) = \int_S f(y) P(dy)$$

for all $f \in C_b(S)$ (class of bounded continuous functions).

Rem. 5.9: Denote by $\mathcal{Z}$ the class of probability measures on $\mathcal{B}(S)$.
One shows (as in the case of $M_+(E)$) that there exists a metric d on $\mathcal{Z}$ such that
(i) $(\mathcal{Z}, d)$ is a complete separable metric space
and (ii) $d(P_n, P) \xrightarrow[n\to\infty]{} 0 \iff P_n \xrightarrow[n\to\infty]{v} P$
See e.g. P. Billingsley, Wiley 1968.

Th. 5.10: $\Pi \subset \mathcal{Z}$ is tight iff $\overline{\Pi}$ is compact ($\overline{\Pi}$: closure of Π)
See e.g. Billingsley, Wiley 1968 for a proof.

We are now coming to the definition of a random measure:

Def. 5.11 (random measure)
Let $X : \Omega \longrightarrow M_+(E)$ be a r.v., that is

$$\{\omega : X(\omega) \in B\} \in \mathcal{F}$$ ($\mathcal{F}$: σ-algebra on Ω)

for all $B \in \mathcal{B}(M_+(E))$ (Borel-σ-algebra of $M_+(E)$)
Then X is called a <u>random measure</u>.

In order to verify that a given map $X : \Omega \longrightarrow M_+(E)$ is a random measure, we need a more practical criterion than that in Def. 5.11.
For this purpose recall the following result from

probability theory

<u>Lemma 5.12</u> : Let $(\Omega_1, \mathcal{A}_1), (\Omega_2, \mathcal{A}_2)$ be measurable spaces ($\mathcal{A}$-algebras) and $\mathcal{B}_2$ a family of sets in $\mathcal{A}_2$ such that $\mathcal{A}_2 = \sigma(\mathcal{B}_2) :=$ smallest σ-algebra which contains all sets from $\mathcal{B}_2$.

Consider a function $f: \Omega_1 \to \Omega_2$ such that

$$\{\omega : f(\omega) \in B\} \in \mathcal{A}_1$$

for all $B \in \mathcal{B}_2$.

Then f is $\mathcal{A}_1, \mathcal{A}_2$-measurable, that is

$$\{\omega : f(\omega) \in B\} \in \mathcal{A}_1$$

for all $B \in \mathcal{A}_2$

→ <u>Prop. 5.13</u> : X is a random measure iff $X(A)$ is a r.v. for all $A \in \mathcal{B}(E)$

<u>Proof</u> : We can choose in L. 5.12 $f = X$, $\Omega_1 = \Omega$, $\Omega_2 = M_+(E)$, $\mathcal{A}_1 = \mathcal{F}$, $\mathcal{A}_2 = \mathcal{B}(M_+(E)) \overset{\text{Def}}{=} \mathcal{M}_+(E)$ and

$$\mathcal{B}_2 = \{\{\mu \in M_+(E) : \mu(G) \in B\} : G \in \mathcal{B}(E), B \in \mathcal{B}([0,\infty])\}$$

We know from Prop. 5.4, (5.9) that $\mathcal{A}_2 = \sigma(\mathcal{B}_2)$

$\overset{\text{L.5.12}}{\Longrightarrow}$ X is a random measure iff

$$\{\omega : \underbrace{X(A)}_{= X(\omega, A)} \in B\} = \{\omega : X(\omega) \in \{\mu \in M_+(E) : \mu(A) \in B\}\}$$

$\in \mathcal{A}_1 = \mathcal{F}$ for all $B \in \mathcal{B}([0,\infty])$

$\overset{\text{Def.}}{\Longleftrightarrow}$ $X(A)$ is a (real valued) r.v. $\Rightarrow$ proof.