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Discussion of "Copulas: Tales and facts", by Thomas Mikosch

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The copula epidemic described by Dr. Mikosch is a recent phenomenon. We have witnessed some of the abuses he alludes to, particularly in the financial sector, where analysts are under high pressure to account for the impact of dependence and extreme values, e.g., in risk management and the pricing of derivatives.

While copulas are useful for understanding and modeling association in various settings, neophytes sometimes cut corners short and out of ignorance or inexperience, end up overextending methodology or reinventing the wheel. Combined with the current tendency to circulate working papers widely and to post technical reports on the web, a little carelessness can go a long way towards eroding the knowledge base, here as in many other areas of science. Alas, the literature abounds with overly enthusiastic, naive or improper uses of tools that may be perfectly fit for an end, but not for all ends.

We are thus generally sympathetic to the primary objective pursued by Dr. Mikosch, which is to caution optimism about what copulas can and cannot achieve as a dependence modeling tool. Given his reputation, we expected him to present a scholarly, balanced view of things. We hoped that although

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critical of misuses, he would recognize that copulas are a natural concept with a long tradition and demonstrated value in a variety of contexts, including multivariate and survival analysis, nonparametric statistics, and more recently actuarial science, finance, and hydrology.

However we find that in conveying his message, Dr. Mikosch presents a partial and almost completely antagonistic view of copula theory. Regretfully, he has chosen to produce a pamphlet which, written as it is in a lively but distinctly unscientific style, clearly does a disservice to the community by "throwing the baby out with the bathwater." While we respect his right to "ask some naive questions," we can hardly accept that either through ignorance or malice, he depicts copula theory as an unsubstantiated fad that leads to "a biased view of stochastic dependence." But a strong response is most definitely called for when unrealistic standards, inappropriate comparisons, dubious logic, as well as deceptive and tendentious statements are used to slander a perfectly respectable body of work.

After giving an overview of the emergence of copula theory in Section 1, we expose some of the fallacies hidden in Dr. Mikosch's discourse in Section 2. Our main source of disagreement with his views on dependence is touched on in Section 3, where we address some of his questions briefly, in addition to providing guidelines to the literature wherever he may have left the impression than little or none existed.

1 A Historical Perspective

Although copula theory has only recently emerged as a distinct field of investigation, its roots go back at least to the 1940s, with the seminal work of Hoeffding on margin-free measures of association, which he expressed in terms of "standardized distributions" with uniform margins on $[-1/2, 1/2]$.

The notion must have been rather natural. For, it kept creeping up under various guises in the work of some of many reputable mathematicians of the past century. Examples include Kantorovitch's efforts to define a "distance" between probability measures; Fréchet's work on distributions with fixed margins; Menger's theory of probabilistic metric spaces; Rényi's study of measures of dependence; Lehmann's introduction of concepts of association and their applications; Kimeldorf and Sampson's exploration of multivariate models via their "uniform representation," and Sibuya's use of "dependence functions" in extreme-value analysis.

At least four decades passed before the notion of copula as we know it today emerged from these investigations and crystallized in a form that is now widely considered as useful in statistical circles. It was possibly Deheuvels who, in a series of papers published around 1980, revealed the full potential of the fecund link between multivariate analysis and rank-based statistical techniques. The connection stems from the invariance of the unique underlying copula of a continuous random vector through increasing transformations of its components.

However, the generalized use of copulas for model building (and Archimedean copulas in particular) seems to have been largely fuelled at the end of the 1980s by the publication of significant papers by Marshall and Olkin (1988) and by Oakes (1989) in the influential *Journal of the American Statistical Association*. The first article showed how Archimedean copulas derived from mixture models, while the second established a connection between this class of copulas and a bivariate extension of Cox's proportional hazards model through frailties.

Important advances in multivariate and survival analysis ensued, and the relevant rank-based statistical theory began to develop. Early examples of nonparametric inference for copulas may be found, e.g., in Genest and Rivest (1993) and Shih and Louis (1995). The work of Pickands (1981) and Deheuvels (1982) also led several authors to adhere to the copula point of view in studying multivariate extremes, though exponential margins are often preferred in that context.

Finally, the books by Joe (1997) and Nelsen (1999) were instrumental in consolidating and popularizing this methodology, and the techniques began to spread to actuarial science, finance and hydrology, thanks in part to the review papers by Frees and Valdez (1998), Embrechts et al. (2002), and Favre et al. (2004), respectively.

2 Recipes for Copula Bashing

Instead of presenting straight facts in a critical light, Dr. Mikosch chooses to tell fairy tales and resorts to cunning tricks to persuade his audience that copulas are "something like the emperor's new clothes."

2.1 Judging by Unrealistic Standards

At various points in his paper, Dr. Mikosch ascribes unrealistic objectives to copula theory, and then he throws discredit on the methodology for failing to meet them. For example, his introduction states (the emphasis is ours, here and in all subsequent quotes) that "this very simple concept... *promises to solve all problems of stochastic dependence* but it falls short in achieving the goal." He further suggests that some researchers and practitioners think that "*all the world's problems* related to stochastic dependence and multivariate distributions can be solved via copulas."

We have a fairly good command of the copula literature and have never encountered anything remotely approaching such claims. Who would be so naive as to hold such beliefs anyway? Are there scientists who seriously think that there isn't always a benefit in looking at problems different ways?

As amply illustrated, e.g., in recent special issues of *Insurance: Mathematics and Economics* (IME) and *The Canadian Journal of Statistics* (CJS), copulas are helpful in studying and modeling both serial and non-serial dependence. However, in no way do they make obsolete other methods of tackling these

issues. Would anybody in their right mind ever claim that distribution functions and characteristic functions are redundant equivalents of probability measures and that as such, they "only create theoretical confusion" and that "one does not gain new information" from them?

Other unrealistic standards by which Dr. Mikosch chooses to condemn copula methodology include its failure to evade *all* problems associated with (a) simulation from an arbitrary multivariate distribution; (b) curse of dimensionality; and (c) the classification and interpretation of *all* possible types of dependence. Does he know of any technique that addresses *all* these issues successfully? With respect to the third point, how would he account for the fact—obvious, e.g., from the review of Scarsini and Shaked (1996)—that virtually all modern concepts, measures and orderings of dependence are copula-based? Would he go so far as to deny the value of their many statistical and financial implications, e.g., in proving the unbiasedness of various tests and the monotonicity of their power functions, or in classifying the riskiness of portfolios?

2.2 Invoking Dogmas and Using Dubious Logic

Many assertions in Dr. Mikosch's pamphlet are based on fallacious arguments and dogmatic views. Consider, e.g., his statement that "A multivariate non-Gaussian distribution with Gaussian marginal distributions is *usually considered pathological* and not of much practical use. But such a distribution corresponds to a copula. Would this copula not be 'pathological' as well?" When first exposed to calculus, most people get the false impression that continuous, non-differentiable functions are pathological. Since with probability one, Brownian paths are continuous but nowhere differentiable, should one conclude that Brownian motion is pathological?

Among dogmatic views expressed by Dr. Mikosch is the notion that "EVDs should be fitted to data *only* if the latter are generated by an extremal mechanism." We disagree. With this kind of logic, how many analyses would have to be discarded because they relied on a continuous distribution to model data that were effectively discrete, e.g., due to limited precision? Simplicity, interpretability, convenience and "fitness for purpose" are also important considerations in selecting a model, whether it be copula-based or not.

A careful analysis of Dr. Mikosch's pamphlet also reveals constant wavering in the discourse. For instance, his conclusion states that "since copulas generate any distribution the class is *too big* to be understood and to be useful." Yet, when authors concentrate on specific copula models, he complains that they are textbook, "toy examples" that are "mostly chosen because they are mathematically convenient." The latter is said in particular of Archimedean copulas; yet their rationale is no more "murky" than the popular mixture and frailty models that they stem from. Does Dr. Mikosch think that diffusions form too large a class of stochastic processes to be useful? Or does he regard Engle's ARCH models as a "toy example," given that they are a simple approximation of a tiny subset of solutions to stochastic differential equations?

2.3 Making Deceptive or Tendentious Statements

Several deceptive or tendentious statements can be detected in Dr. Mikosch's exposition. An example of a deceptive assertion is the claim that "Copulas *completely* fail in describing complex space-time dependence structures." It can be objected, e.g., that the stochastic behavior of a stationary Markov chain (X_i) is characterized by its stationary distribution (taken here to be continuous) and a bivariate copula associated with the law of the pair (X_i, X_{i+1}). A few papers have exploited this idea. This is not to say that copulas are always the most appropriate tool for describing complex space-time dependence structures, of course.

As an example of a tendentious statement, consider the claim that the copula of a vector of dependent risks "does not help one to evaluate VaR." While it is true that the marginal distributions are indispensable to the *evaluation* of Value-at-Risk, information about the dependence structure is just as essential. Besides, useful bounds on the VaR can be derived from copula theory even in the absence of exact knowledge about the nature of association. See, e.g., the previously cited issue of the IME for contributions along these lines.

More damaging, however, are those of Dr. Mikosch's claims that suggest either his lack of knowledge or his lack of appreciation of the literature about copulas and dependence. Speaking of estimation methods for copula models, he writes authoritatively: "There exists *very little statistical theory* about fitting multivariate data by using copulas and about their goodness-of-fit; most papers focus on some *particular 2-dimensional copulas*..." Is he aware that the consistency of many of these estimators has been established in *general dimension* and for *broad classes of copulas* that meet *minimal regularity conditions*? See, e.g., Genest et al. (1995), Shih and Louis (1995), Joe (2005) and papers in the previously mentioned issue of the CJS. In fact, even questions related to efficiency have already been considered, e.g., by Klaassen and Wellner (1997) or Genest and Werker (2002). The search for an asymptotically semiparametrically efficient estimator for dependence parameters in copula models recently culminated in the work of Chen et al. (2006).

As for copula goodness-of-fit techniques, it is true that they are only beginning to emerge. Nevertheless, see, e.g., Wang and Wells (2000), Fermanian (2005) or Genest et al. (2006). We also grant that owing in part to the pressing demand for appropriate tests in this context, some theoretically unsupported proposals (not listed here) have recently been made.

3 Setting the Record Straight

From a scientific perspective, our main point of disagreement with Dr. Mikosch is in our conception of a *dependence structure*. From his article, it seems that he regards as biased and incomplete any representation of association that fails to take into account the margins. If in his view anything short of a full distribution distorts dependence, then no wonder he thinks so poorly of copulas. Is this

also why he raises doubts subtly about the usefulness of such time-honored concepts as Kendall's tau and Spearman's rho by referring to them as "*some kind* of dependence measure?"

Certainly, Dr. Mikosch would agree that independence is a purely margin-free concept. This justifies the use of rank-based procedures based on the empirical copula for testing independence; see, e.g., Genest and Rémillard (2004) and references therein. To us, association is also a margin-free notion, and we are not alone to hold this view. In fact, quite apart from copula theory, many traditional notions of dependence are defined in terms of "distances" between two σ-algebras. As the latter are invariant by strictly increasing transformations of the margins, copula-based concepts emerge. The $\alpha-$mixing coefficient and Gebelein's monotone correlation coefficient provide but two illustrations.

Besides his objectionable point of view on the meaning of dependence, Dr. Mikosch asks questions that are sometimes surprising, naive, broad or even (purposely?) ambiguous about copulas and their usefulness. In the rest of this section, we address some of them briefly.

3.1 Is there any Reason for Considering Uniform Marginals?

Dr. Mikosch asserts that the choice of uniform margins in the definition of copula is arbitrary and that he could not find any *scientific* or *practical* reason for it. Beyond the fact that there is always a certain degree of arbitrariness in mathematical definitions (e.g., that probabilities take values in $[0, 1]$), his question is surprising, considering his knowledge of stochastic processes.

Paraphrasing Dr. Mikosch, one might as well ask why a Brownian bridge β is arbitrarily defined on $[0, 1]$; is it because it can be visualized on a computer screen? A more cogent explanation would be that the limit of any empirical process $\sqrt{n}(F_n - F)$ can be written in the form $\beta \circ F$, so that β enjoys an "invariance property" with respect to the continuous distribution F that makes it of intrinsic interest. It is the very same invariance argument that led to Sklar's representation theorem and to copulas. That sounds fairly compelling to us theoretically.

Nevertheless, there are also many *practical* reasons why uniform margins are often more convenient than, say, exponential ones. Since Dr. Mikosch apparently sees little virtue in them, let us leave aside the facts that the standard probability integral transformation is natural and that many algorithms for generating distributions are based on the uniform. But consider this: *if* a copula were defined as a distribution function with unit exponential margins, its determination would require the transformation of the margins of the original distribution into uniforms as an *intermediate* calculation step. Furthermore, the invariance argument mentioned earlier would still lead to base inference about the copula on normalized ranks. Only, the latter would then have to be processed via the transformation $T : u \mapsto -\log(1 - u)$ in order to achieve exponentiality! Why impose needless contortions?

3.2 What is the Interest in Transforming the Marginals to Uniform?

In a data analytic context where the marginal distributions are typically unknown, transforming the margins to uniforms involves using the marginal empirical distribution functions and hence amounts to working with the normalized vectors of ranks. Dr. Mikosch is concerned that this transformation is arbitrary and affects to be naive in wondering whether the resulting graph might not provide a "biased view of stochastic dependence."

As a picture is worth a thousand words, consider the four graphs in Fig. 1. Shown on the left are scatter plots of pairs of normalized ranks deduced from 1,000 observations from bivariate distributions with uniform margins; shown on the right are the graphs resulting from their coordinatewise transformation by T. The top panels correspond to independence, while the bottom panels arise from a Gumbel copula with Kendall's tau equal to 3/4. All graphs are virtually the same as we would get from the raw data (i.e., uniform on the left and unit exponential on the right).

Most people would agree that the graphs on the right give the misleading impression that the pairs of points are dependent, whereas this is only true in the lower panel. In addition, the lower right panel suggests that the association is strongest in the small values of X and Y, when in fact upper tail dependence

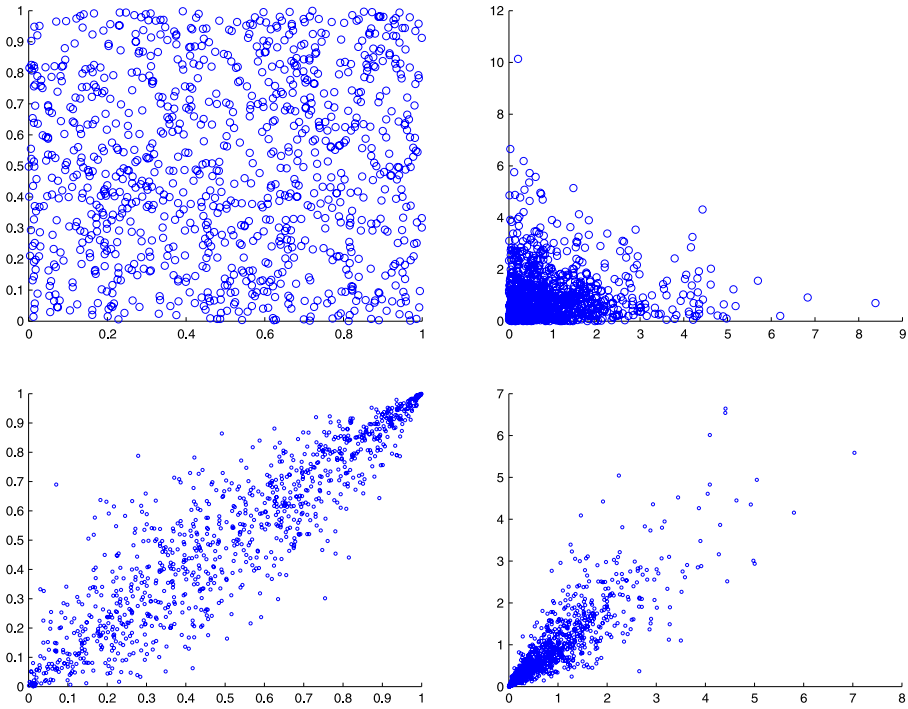


Fig. 1 Four scatter plots; see text for details

is the dominant feature. On the contrary, the appropriate conclusions are obvious from the left panels.

Frankly, we don't see why Dr. Mikosch finds it so "*hard to imagine* what dependence of two uniformly distributed random variables means." The homogeneous dispersion of points on the unit square is not only characteristic of independence between two uniform variables; it is easily detected using pairs of normalized ranks, and with minimum training, looking at the support of the empirical copula is also suggestive of the type of association present.

3.3 How Does One Choose a Copula?

Model selection is a broad question for which a completely satisfying answer does not yet exist, even in the univariate case. The same strategies can be used here as in any other modeling exercise, i.e., choices can be guided by model properties and characterizations, diagnostic tools, cross-validation, predictive accuracy, etc. Given that copula modeling is still in a relatively early stage of development, we concur that much remains to be done in this regard. For a state-of-the-art illustration of the methodology currently available, see Genest and Favre (2007).

3.4 Can the Estimation of the Dependence Structure Only be a Matter of Relative Ranks?

This is a loaded question, whose answer depends on the interpretation of "dependence structure," and in Dr. Mikosch's paper, we find that the meaning of the term sometimes changes with the context, to suit the author's purpose.

If "dependence structure" means the joint distribution, as Dr. Mikosch often implies, then clearly estimation is not just a matter of ranks. For, while multiplying the data by $10^{\pm 50}$ does not change their ranks, it does affect the margins. However if, as we already advocated, association is a margin-free concept, the invariance of the copula by increasing transformations entirely justifies basing inference on maximal invariant statistics, i.e., the ranks. The marginal distributions may then be regarded as (infinite-dimensional) nuisance parameters. Such invariance arguments are common in statistical inference.

Incidentally, there are concrete settings in which the margins are not only nuisance parameters but completely irrelevant, or even impossible to estimate given the data at hand. Testing for (serial or non-serial) independence is one instance where the margins are extraneous. A prime example of situation where the copula and the margins must be estimated separately arises in pricing multi-name credit risk derivatives. In that context, the copula of default times of different companies can be estimated using joint prices for financial instruments (e.g., credit default swaps), but not the margins. To estimate the latter, different data need to be used, e.g., credit ratings as in Berrada et al. (2006).

3.5 Given Good Fits for the Marginals and the Copula, How Good is the Overall Fit of the Model?

In his conclusion, Dr. Mikosch raises doubts about "the main selling point of the copula technology—separation of the copula... from the marginal distributions," in particular because "it is unclear whether a good fit of the copula of the data yields a good fit to the distribution of the data." Again, we disagree.

Indeed, basic facts from copula theory imply that a good fit of the joint distribution function necessarily results from good fits of the copula and the margins taken separately, especially in low dimension. To see this, let $\hat{C}$ be an estimate of C and let $\hat{F}_1, \ldots, \hat{F}_d$ be estimations of the margins $F_1, \ldots, F_d$. Since C satisfies the Lipschitz condition

$$|C(u_1, \ldots, u_d) - C(v_1, \ldots, v_d)| \leq |u_1 - v_1| + \cdots + |u_d - v_d|,$$

it follows that if $\hat{H}(x_1, \ldots, x_d) = \hat{C}\{\hat{F}_1(x_1), \ldots, \hat{F}_d(x_d)\}$, then

$$\begin{aligned}
\|\hat{H} - H\|_\infty &= \sup_{x_1,\ldots,x_d \in \mathbb{R}} |\hat{C}\{\hat{F}_1(x_1), \ldots, \hat{F}_d(x_d)\} - C\{F_1(x_1), \ldots, F_d(x_d)\}| \\
&\leq \|\hat{C}(\hat{F}_1, \ldots, \hat{F}_d) - C(\hat{F}_1, \ldots, \hat{F}_d)\|_\infty \\
&\quad + \|C(\hat{F}_1, \ldots, \hat{F}_d) - C(F_1, \ldots, F_d)\|_\infty \\
&\leq \|\hat{C} - C\|_\infty + \sum_{i=1}^{d} \|\hat{F}_i - F_i\|_\infty.
\end{aligned}$$

We would be interested to hear what leads Dr. Mikosch to assert that the use of $\hat{H}$ "implies a much higher statistical uncertainty than" any other estimator $\breve{H}$ of H. Certainly, if the interest lies in the estimation of C, the (rank-based) empirical copula provides a convergent estimator, while things "can go terribly wrong" for a copula deduced from $\breve{H}$ if an inappropriate choice of margins is made!

4 Conclusion

This pamphlet provides a few fine specimens of the kind of non-sense that is likely to be produced by someone who believes that "one needs less than 10 minutes to understand the fundamentals" of a topic. It sounds to us as though Dr. Mikosch has pushed editorial commentary to new *Extremes*. Nevertheless, we thank him for bringing attention to copulas and dependence with his scathing review of the subject, and we are grateful to the Editor for the opportunity to respond.

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