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Copulas: tales and facts

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Table of contents

1 **Overview**

2 **What is a copula?**

3 **Copulas: tales and facts**
- Introduction
- Some naïve questions
- Conclusion

4 **The aftermath**
- Discussion by Genest and Rémillard
- Copulas: tales and facts - rejoinder

5 **Concluding remarks**

6 **References**

What is a copula?

Definition

A copula is a distribution function on $[0,1]^d$ with uniform marginal distribution functions.

Theorem (Sklar's theorem)

Let F be the joint distribution function of $X_1, \ldots, X_p$ and F_j be the marginal distribution of X_j. Then there exists a copula, C, such that

$$F(x_1, \ldots, x_p) = C\left[F_1(x_1), \ldots, F_p(x_p)\right].$$

Literature:

(Introduction): [Nelson (1999)]

(Estimation): [Genest, Ghoudi, Rivest (1995)] and [Joe (2005)]

(Vines): [Aas et al. (2009)]

Copulas: tales and facts

Published in *Extremes* in 2006

- Extreme value theory

Thomas Mikosch

- Professor at University of Copenhagen
- Editor of *Extremes*
- Research interests:
  - Time series
  - Stochastic processes
  - Extreme value theory
  - Insurance and financial statistics

Three goals:

(1) Disprove that copulas solve all problems related to stochastic dependence

(2) Comment on statistical problems

(3) Show that copulas do not fit into theory of stochastic processes

Some naïve questions

Why does one transform the marginals to a uniform distribution?

- No *scientific* answer
- Easy to visualize on a computer screen
- Does not help with modeling

Why does one transform the marginals to a standard distribution?

- Value at risk
- Needs simulation and Monte Carlo
- Expensive method
- We can only simulate from copulas corresponding to already known distributions

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How does one fit the marginal distributions and the copula?

- Two staged procedure
- Fit of copula is affected by fit of marginals
- Few ways to check goodness of fit

How does one choose a copula?

- Parametric families
- Copula families are made to be mathematical convenient
  - Archimedean copulas
  - Copulas of Elliptical Distributions
  - Extreme value copulas

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Upper tail dependence and multivariate extremes

- Choose a copula based on the upper tail dependence
- Upper tail dependence parameter
  - Two dimensional
  - Direction and magnitude of extremes
- λ_B approximates probability of rare events
  - Hard to estimate in high dimensions

How do copulas fit into the models of time series analysis and the theory of stochastic processes?

- Spatial dependence vs dependence through time

Do copulas avoid the curse of dimensionality?

- Reformulation of the problem
- Number of parameters usually increases
  - Needs to fit marginal distributions

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The copula fashion - the emperor's new clothes?

Summary:

- There is no particular reason to use copulas when dealing with multivariate distributions.
- Marginals and the copula is linked.
- Models are chosen because of mathematical convenience rather than reasoning.
- The class of copulas is too big to be useful.
- Little statistical theory
- Copulas do not fit into the existing framework of stochastic processes and time series analysis

The aftermath

Discussion of “Copulas: tales and facts”, by Thomas Mikosch

Christian Genest

- Professor at McGill University (Canada)
- Worked a lot with copulas

Bruno Rémillard

- Professor at Université du Québec à Trois-Rivières (Canada)

> *While we respect his right to “ask some naive questions,” we can hardly accept that either through ignorance or malice, he depicts copula theory as an unsubstantiated fad that leads to “a biased view of stochastic dependence.” But a strong response is most definitely called for when unrealistic standards, inappropriate comparisons, dubious logic, as well as deceptive and tendentious statements are used to slander a perfectly respectable body of work.*

A historical perspective

- Copulas are more than a fad

Recipes for Copula bashing

- Judging by unrealistic standards
- Invoking dogmas and using dubious logic
- Making deceptive or tendentious statements

Instead of presenting straight facts in a critical light, Dr. Mikosch chooses to tell fairy tales and resorts to cunning tricks to persuade his audience that copulas are "something like the emperor's new clothes."

Setting the record straight

Is there a reason for considering uniform marginals?

- All definitions are somewhat arbitrary
- To transform marginals to other distributions one has to go via uniform
- Easier to visually detect dependence

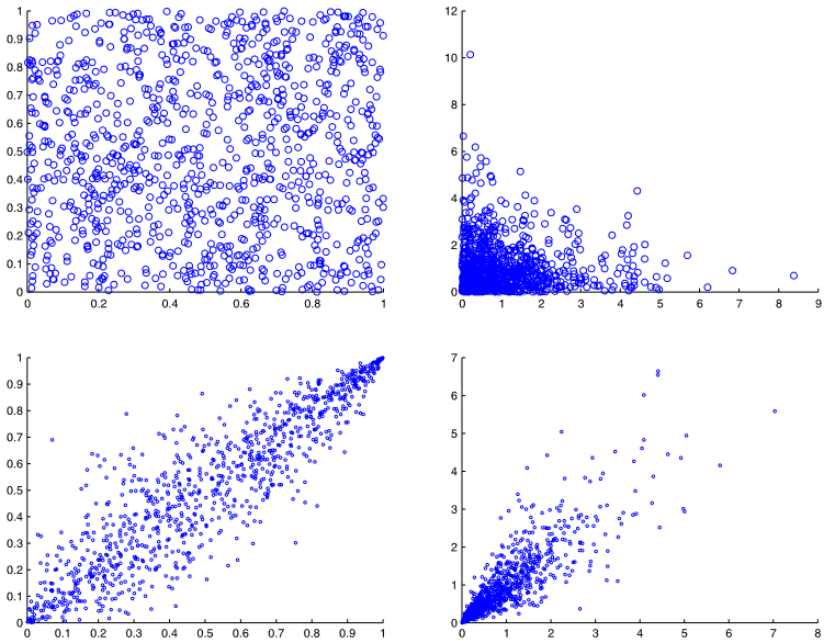


Figure: From [Genest & Rémillard (1995)]

How does one choose a copula?

- Standard model diagnostic tools

Given good fits for the marginals and the copula, how good is the overall fit of the model?

- Let $F(x_1, \ldots, x_p) = C[F_1(x_1), \ldots, F_p(x_p)]$ and $\widehat{F}$, $\widehat{F}_j$ and $\widehat{C}$ be their estimators. Then

$$||F - \widehat{F}||_\infty \leq ||C - \widehat{C}||_\infty + \sum_{j=1}^{p} ||F_j - \widehat{F}_j||_\infty$$

- The empirical copula estimates C consistently.

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Copulas: tales and facts - rejoinder

I do not want to react to all the statements (accusations?) presented by Professors Genest and Rémillard, but I want to make clear where we differ. This contribution is typical for people who are partly blind since they are used to look through copula glasses.

Two main points:

- Why should we use copulas in finance and risk analysis?
- Copula theory does not fit into theory of time series and stochastic processes

Concluding remarks

Mikosch has some good points

- How do you choose a copula?
- How do you check if a copula fits well?

A lot of the critique is too harsh

- Copulas are criticized for not dealing perfectly with all statistical problems at once.
- Copulas are compared to direct and correct modeling of the full distribution.
- Bad fits for the marginals result in a bad fit for the copula.

If you want to convince someone, don't insult them.

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