Chapter 2

Fourier analysis for periodic functions: Fourier series

In Chapter 1 we identified audio signals with functions and discussed informally the idea of decomposing a sound into basis sounds to make its frequency content available. In this chapter we will make this kind of decomposition precise by discussing how a given function can be expressed in terms of the basic trigonometric functions. This is similar to Taylor series where functions are approximated by combinations of polynomials. But it is also different from Taylor series because polynomials are different from polynomials, and the approximations are computed in a very different way. The theory of approximation of functions with trigonometric functions is generally referred to as *Fourier analysis*. This is a central tool in practical fields like image and signal processing, but it also an important field of research within pure mathematics. We will only discuss Fourier analysis for functions defined on a finite interval and for finite sequences (vectors), but Fourier analysis may also be applied to functions defined on the whole real line and to infinite sequences.

Perhaps a bit surprising, linear algebra is a very useful tool in Fourier analysis. This is because the sets of functions involved are vector spaces, both of finite and infinite dimension. Therefore many of the tools from your linear algebra course will be useful, in a situation that at first may seem far from matrices and vectors.

2.1 Basic concepts

The basic idea of Fourier series is to approximate a given function by a combination of simple cos and sin functions. This means that we have to address at least three questions:

1. How general do we allow the given function to be?

2. What exactly are the combinations of cos and sin that we use for the approximations?
3. How do we determine the approximation?

Each of these questions will be answered in this section.

We have already indicated that the functions we consider are defined on an interval, and without much loss of generality we assume this interval to be $[0, T]$, where T is some positive number. Note that any function f defined on $[0, T]$ gives rise to a related function defined on the whole real line, by simply gluing together copies of f. The result is a periodic function with period T that agrees with f on $[0, T]$.

We have to make some more restrictions. Mostly we will assume that f is continuous, but the theory can also be extended to functions which are only Riemann-integrable, more precisely, that the square of the function is integrable.

Definition 2.1 (Continuous and square-integrable functions)**.** The set of continuous, real functions defined on an interval $[0, T]$ is denoted $C[0, T]$.

A real function f defined on $[0, T]$ is said to be square integrable if f^2 is Riemann-integrable, i.e., if the Riemann integral of f^2 on $[0, T]$ exists,

$$\int_0^T f(t)^2 \, dt < \infty.$$

The set of all square integrable functions on $[0, T]$ is denoted $L^2[0, T]$.

The sets of continuous and square-integrable functions can be equippped with an inner-product, a generalisation of the so-called dot-product for vectors.

Theorem 2.2. Both $L^2[0, T]$ and $C[0, T]$ are vector spaces. Moreover, if the two functions f and g lie in $L^2[0, T]$ (or in $C[0, T]$), then the product fg is also in $L^2[0, T]$ (or in $C[0, T]$). Moreover, both spaces are inner product spaces[1], with inner product[2] defined by

$$\langle f, g \rangle = \frac{1}{T} \int_0^T f(t) g(t) \, dt, \tag{2.1}$$

and associated norm

$$\|f\| = \sqrt{\frac{1}{T} \int_0^T f(t)^2 dt}. \tag{2.2}$$

The mysterious factor $1/T$ is included so that the constant function $f(t) = 1$ has norm 1, i.e., its role is as a normalizing factor.

Definition 2.1 and Theorem 2.2 answer the first question above, namely how general do we allow our functions to be. Theorem 2.2 also gives an indication of how we are going to determine approximations—we are going to use inner products. We recall from linear algebra that the projection of a function f onto a subspace W with respect to an inner product $\langle \cdot, \cdot \rangle$ is the function $g \in W$ which minimizes $\|f-g\|$, which we recognise as the error[3]. This projection is therefore also called a best approximation of f from W and is characterised by the fact that the error should be orthogonal to the subspace W, i.e., we should have

$$\langle f, g \rangle = 0, \quad \text{for all } g \in W.$$

More precisely, if $\boldsymbol{\phi} = \{\phi_i\}_{i=1}^m$ is an orthogonal basis for W, then the best approximation g is given by

$$g = \sum_{i=1}^{m} \frac{\langle f, \phi_i \rangle}{\langle \phi_i, \phi_i \rangle} \phi_i. \tag{2.3}$$

The error $\|f - g\|$ in the approximation is often referred to as the *least square error*.

We have now answered the second of our primary questions. What is left is a description of the subspace W of trigonometric functions. This space is spanned by the pure tones we discussed in Chapter 1.

Definition 2.3 (Fourier series)**.** Let $V_{N,T}$ be the subspace of $C[0, T]$ spanned by the set of functions given by

$$\begin{aligned}\mathcal{D}_{N,T} = \{&1, \cos(2\pi t/T), \cos(2\pi 2t/T), \cdots, \cos(2\pi Nt/T), \\ &\sin(2\pi t/T), \sin(2\pi 2t/T), \cdots, \sin(2\pi Nt/T)\}.\end{aligned} \tag{2.4}$$

The space $V_{N,T}$ is called the N'th order Fourier space. The Nth-order Fourier series approximation of f, denoted f_N, is defined as the best approximation of f from $V_{N,T}$ with respect to the inner product defined by (2.1).

The space $V_{N,T}$ can be thought of as the space spanned by the pure tones of frequencies $1/T$, $2/T$, ..., N/T, and the Fourier series can be thought of as linear combination of all these pure tones. From our discussion in Chapter 1, we see that if N is sufficiently large, we get a space which can be used to approximate most sounds in real life. The approximation f_N of a sound f from a space $V_{N,T}$ can also serve as a compressed version if many of the coefficients can be set to 0 without the error becomingg too big.

Note that all the functions in the set $\mathcal{D}_{N,T}$ are periodic with period T, but most have an even shorter period. More precisely, $\cos(2\pi nt/T)$ has period T/n, and frequency n/T. In general, the term *fundamental frequency* is used to denote the lowest frequency of a given periodic function.

Definition 2.3 characterises the Fourier series. The next lemma gives precise expressions for the coefficients.

[3]See Section 6.3 in [7] for a review of projections and least squares approximations.

Theorem 2.4. The set $\mathcal{D}_{N,T}$ is an orthogonal basis for $V_{N,T}$. In particular, the dimension of $V_{N,T}$ is $2N+1$, and if f is a function in $L^2[0,T]$, we denote by $a_0,\ldots,a_N$ and $b_1,\ldots,b_N$ the coordinates of f_N in the basis $\mathcal{D}_{N,T}$, i.e.

$$f_N(t) = a_0 + \sum_{n=1}^{N} \left(a_n \cos(2\pi nt/T) + b_n \sin(2\pi nt/T)\right). \qquad (2.5)$$

The $a_0,\ldots,a_N$ and $b_1,\ldots,b_N$ are called the (real) Fourier coefficients of f, and they are given by

$$a_0 = \langle f, 1\rangle = \frac{1}{T}\int_0^T f(t)\,dt, \qquad (2.6)$$

$$a_n = 2\big\langle f, \cos(2\pi nt/T)\big\rangle = \frac{2}{T}\int_0^T f(t)\cos(2\pi nt/T)\,dt \quad \text{for } n \geq 1, \qquad (2.7)$$

$$b_n = 2\langle f, \sin(2\pi nt/T)\rangle = \frac{2}{T}\int_0^T f(t)\sin(2\pi nt/T)\,dt \quad \text{for } n \geq 1. \qquad (2.8)$$

Proof. To prove orthogonality, assume first that $m \neq n$. We compute the inner product

$$\begin{aligned}
&\langle \cos(2\pi mt/T), \cos(2\pi nt/T)\rangle \\
&= \frac{1}{T}\int_0^T \cos(2\pi mt/T)\cos(2\pi nt/T)dt \\
&= \frac{1}{2T}\int_0^T (\cos(2\pi mt/T + 2\pi nt/T) + \cos(2\pi mt/T - 2\pi nt/T)) \\
&= \frac{1}{2T}\left[\frac{T}{2\pi(m+n)}\sin(2\pi(m+n)t/T) + \frac{T}{2\pi(m-n)}\sin(2\pi(m-n)t/T)\right]_0^T \\
&= 0.
\end{aligned}$$

Here we have added the two identities $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$ together to obtain an expression for $\cos(2\pi mt/T)\cos(2\pi nt/T)dt$ in terms of $\cos(2\pi mt/T + 2\pi nt/T)$ and $\cos(2\pi mt/T - 2\pi nt/T)$. By testing all other combinations of sin and cos also, we obtain the orthogonality of all functions in $\mathcal{D}_{N,T}$ in the same way.

We find the expressions for the Fourier coefficients from the general formula (2.3). We first need to compute the following inner products of the basis functions,

$$\begin{aligned}
\langle \cos(2\pi mt/T), \cos(2\pi mt/T)\rangle &= \frac{1}{2} \\
\langle \sin(2\pi mt/T), \sin(2\pi mt/T)\rangle &= \frac{1}{2} \\
\langle 1, 1\rangle &= 1,
\end{aligned}$$

which are easily derived in the same way as above. The orthogonal decomposition theorem (2.3) now gives

$$
\begin{aligned}
f_N(t) &= \frac{\langle f, 1\rangle}{\langle 1, 1\rangle}1 + + \sum_{n=1}^{N} \frac{\langle f, \cos(2\pi nt/T)\rangle}{\langle \cos(2\pi nt/T), \cos(2\pi nt/T)\rangle} \cos(2\pi nt/T) \\
&+ \sum_{n=1}^{N} \frac{\langle f, \sin(2\pi nt/T)\rangle}{\langle \sin(2\pi nt/T), \sin(2\pi nt/T)\rangle} \sin(2\pi nt/T) \\
&= \frac{\frac{1}{T}\int_0^T f(t)dt}{1} + \sum_{n=1}^{N} \frac{\frac{1}{T}\int_0^T f(t)\cos(2\pi nt/T)dt}{\frac{1}{2}} \cos(2\pi nt/T) \\
&+ \sum_{n=1}^{N} \frac{\frac{1}{T}\int_0^T f(t)\sin(2\pi nt/T)dt}{\frac{1}{2}} \sin(2\pi nt/T) \\
&= \frac{1}{T}\int_0^T f(t)dt + \sum_{n=1}^{N} \left(\frac{2}{T}\int_0^T f(t)\cos(2\pi nt/T)dt\right) \cos(2\pi nt/T) \\
&+ \sum_{n=1}^{N} \left(\frac{2}{T}\int_0^T f(t)\sin(2\pi nt/T)dt\right) \sin(2\pi nt/T).
\end{aligned}
$$

The relations (2.6)- (2.8) now follow by comparison with (2.5). □

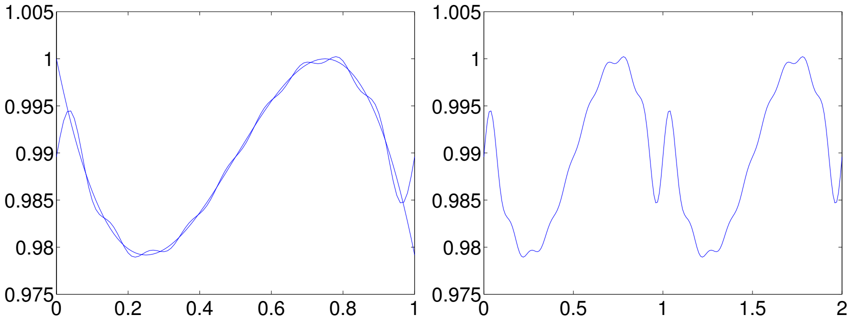
Since f is a function in time, and the a_n, b_n represent contributions from different frequencies, the Fourier series can be thought of as a change of coordinates, from what we vaguely can call the *time domain*, to what we can call the *frequency domain* (or *Fourier domain*). We will call the basis $\mathcal{D}_{N,T}$ the *N'th order Fourier basis* for $V_{N,T}$. We note that $\mathcal{D}_{N,T}$ is not an orthonormal basis; it is only orthogonal.

In the signal processing literature, Equation (2.5) is known as *the synthesis equation*, since the original function f is synthesized as a sum of trigonometric functions. Similarly, equations (2.6)- (2.8) are called *analysis equations*.

A major topic in harmonic analysis is to state conditions on f which guarantees the convergence of its Fourier series. We will not discuss this in detail here, since it turns out that, by choosing N large enough, any reasonable periodic function can be approximated arbitrarily well by its Nth-order Fourier series approximation. More precisely, we have the following result for the convergence of the Fourier series, stated without proof.

Theorem 2.5 (Convergence of Fourier series)**.** Suppose that f is periodic with period T, and that

1. f has a finite set of discontinuities in each period.
2. f contains a finite set of maxima and minima in each period.
3. $\int_0^T |f(t)|dt < \infty$.



(a) The function and its Fourier series (b) The Fourier series on a larger interval

Figure 2.1: The cubic polynomial $f(x) = -\frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{3}{16}x + 1$ on the interval $[0, 1]$, together with its Fourier series approximation from $V_{9,1}$.

Then we have that $\lim_{N\to\infty} f_N(t) = f(t)$ for all t, except at those points t where f is not continuous.

The conditions in Theorem 2.5 are called the Dirichlet conditions for the convergence of the Fourier series. They are just one example of conditions that ensure the convergence of the Fourier series. There also exist much more general conditions that secure convergence — these can require deep mathematical theory, depending on the generality.

An illustration of Theorem 2.5 is shown in Figure 2.1 where the cubic polynomial $f(x) = -\frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{3}{16}x + 1$ is approximated by a 9th order Fourier series. The trigonometric approximation is periodic with period 1 so the approximation becomes poor at the ends of the interval since the cubic polynomial is not periodic. The approximation is plotted on a larger interval in Figure 2.1(b), where its periodicity is clearly visible.

Example 2.6. Let us compute the Fourier coefficients of the square wave, as defined by (1.1) in Example 1.11. If we first use (2.6) we obtain

$$a_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{T}\int_0^{T/2} dt - \frac{1}{T}\int_{T/2}^T dt = 0.$$

Using (2.7) we get

$$
\begin{aligned}
a_n &= \frac{2}{T}\int_0^T f(t)\cos(2\pi nt/T)dt \\
&= \frac{2}{T}\int_0^{T/2} \cos(2\pi nt/T)dt - \frac{2}{T}\int_{T/2}^{T} \cos(2\pi nt/T)dt \\
&= \frac{2}{T}\left[\frac{T}{2\pi n}\sin(2\pi nt/T)\right]_0^{T/2} - \frac{2}{T}\left[\frac{T}{2\pi n}\sin(2\pi nt/T)\right]_{T/2}^{T} \\
&= \frac{2}{T}\frac{T}{2\pi n}((\sin(n\pi) - \sin 0) - (\sin(2n\pi) - \sin(n\pi)) = 0.
\end{aligned}
$$

Finally, using (2.8) we obtain

$$
\begin{aligned}
b_n &= \frac{2}{T}\int_0^T f(t)\sin(2\pi nt/T)dt \\
&= \frac{2}{T}\int_0^{T/2} \sin(2\pi nt/T)dt - \frac{2}{T}\int_{T/2}^{T} \sin(2\pi nt/T)dt \\
&= \frac{2}{T}\left[-\frac{T}{2\pi n}\cos(2\pi nt/T)\right]_0^{T/2} + \frac{2}{T}\left[\frac{T}{2\pi n}\cos(2\pi nt/T)\right]_{T/2}^{T} \\
&= \frac{2}{T}\frac{T}{2\pi n}((-\cos(n\pi) + \cos 0) + (\cos(2n\pi) - \cos(n\pi))) \\
&= \frac{2(1 - \cos(n\pi)}{n\pi} \\
&= \begin{cases} 0, & \text{if } n \text{ is even;} \\ 4/(n\pi), & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}
$$

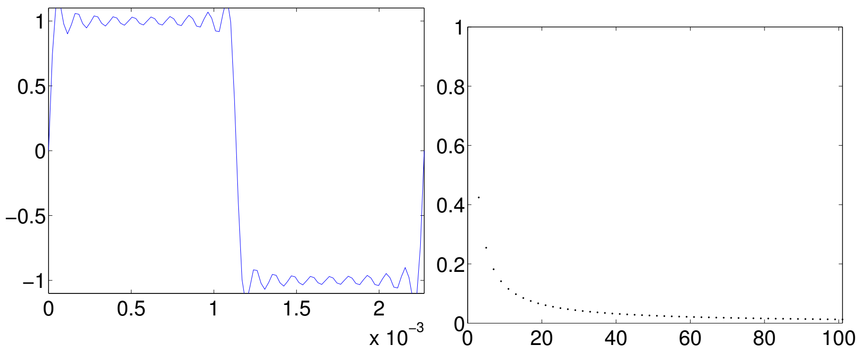
In other words, only the b_n-coefficients with n odd in the Fourier series are nonzero. From this it is clear that the Fourier series is

$$
\frac{4}{\pi}\sin(2\pi t/T) + \frac{4}{3\pi}\sin(2\pi 3t/T) + \frac{4}{5\pi}\sin(2\pi 5t/T) + \frac{4}{7\pi}\sin(2\pi 7t/T) + \cdots .
$$

With $N = 20$, there are 10 trigonometric terms in this sum. The corresponding Fourier series can be plotted on the same interval with the following code.

```
t=0:(1/fs):3;
y=zeros(1,length(t));
for n=1:2:19
  y = y + (4/(n*pi))*sin(2*pi*n*t/T);
end
plot(t,y)
```

In Figure 2.2(a) we have plotted the Fourier series of the square wave when $T = 1/440$, and when $N = 20$. In Figure 2.2(b) we have also plotted the values



(a) The Fourier series for $N = 20$

(b) Values of the first 100 Fourier coefficients b_n

Figure 2.2: The Fourier series of the square wave of Example 2.6

of the first 100 Fourier coefficients b_n, to see that they actually converge to zero. This is clearly necessary in order for the Fourier series to converge.

Even though f oscillates regularly between -1 and 1 with period T, the discontinuities mean that it is far from the simple $\sin(2\pi t/T)$ which corresponds to a pure tone of frequency $1/T$. From Figure 2.2(b) we see that the dominant coefficient in the Fourier series is b_1, which tells us how much there is of the pure tone $\sin(2\pi t/T)$ in the square wave. This is not surprising since the square wave oscillates T times every second as well, but the additional nonzero coefficients pollute the pure sound. As we include more and more of these coefficients, we gradually approach the square wave, as shown for $N = 20$.

There is a connection between how fast the Fourier coefficients go to zero, and how we percieve the sound. A pure sine sound has only one nonzero coefficient, while the square wave Fourier coeffcients decrease as $1/n$, making the sound less pleasant. This explains what we heard when we listened to the sound in Example 1.11. Also, it explains why we heard the same pitch as the pure tone, since the first frequency in the Fourier series has the same frequency as the pure tone we listened to, and since this had the highest value.

The Fourier series approximations of the square wave can be played with the `play` function, just as the square wave itself. For $N = 1$ and with $T = 1/440$ as above, it sounds like this. This sounds exactly like the pure sound with frequency 440Hz, as noted above. For $N = 5$ the Fourier series approximation sounds like this, and for $N = 9$ it sounds like this. Indeed these sounds are more like the square wave itself, and as we increase N we can hear how introduction of more frequencies gradually pollutes the sound more and more. In Exercise 7 you will be asked to write a program which verifies this.

Example 2.7. Let us also compute the Fourier coefficients of the triangle wave, as defined by (1.2) in Example 1.12. We now have

$$a_0 = \frac{1}{T}\int_0^{T/2} \frac{4}{T}\left(t - \frac{T}{4}\right)dt + \frac{1}{T}\int_{T/2}^{T} \frac{4}{T}\left(\frac{3T}{4} - t\right)dt.$$

Instead of computing this directly, it is quicker to see geometrically that the graph of f has as much area above as below the x-axis, so that this integral must be zero. Similarly, since f is symmetric about the midpoint $T/2$, and $\sin(2\pi nt/T)$ is antisymmetric about $T/2$, we have that $f(t)\sin(2\pi nt/T)$ also is antisymmetric about $T/2$, so that

$$\int_0^{T/2} f(t)\sin(2\pi nt/T)dt = -\int_{T/2}^{T} f(t)\sin(2\pi nt/T)dt.$$

This means that, for $n \geq 1$,

$$b_n = \frac{2}{T}\int_0^{T/2} f(t)\sin(2\pi nt/T)dt + \frac{2}{T}\int_{T/2}^{T} f(t)\sin(2\pi nt/T)dt = 0.$$

For the final coefficients, since both f and $\cos(2\pi nt/T)$ are symmetric about $T/2$, we get for $n \geq 1$,

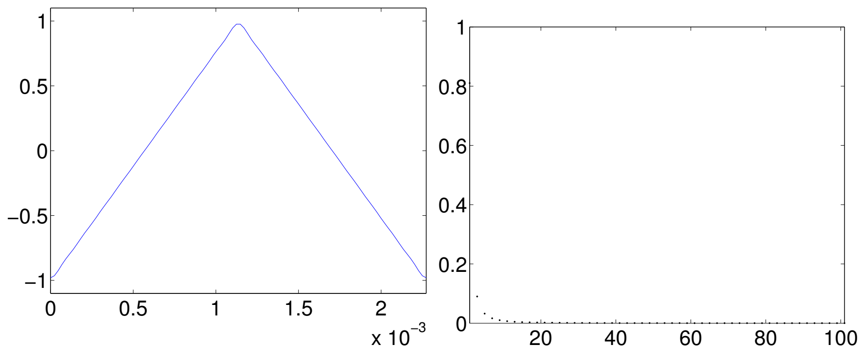
$$\begin{aligned}
a_n &= \frac{2}{T}\int_0^{T/2} f(t)\cos(2\pi nt/T)dt + \frac{2}{T}\int_{T/2}^{T} f(t)\cos(2\pi nt/T)dt \\
&= \frac{4}{T}\int_0^{T/2} f(t)\cos(2\pi nt/T)dt = \frac{4}{T}\int_0^{T/2} \frac{4}{T}\left(t - \frac{T}{4}\right)\cos(2\pi nt/T)dt \\
&= \frac{16}{T^2}\int_0^{T/2} t\cos(2\pi nt/T)dt - \frac{4}{T}\int_0^{T/2} \cos(2\pi nt/T)dt \\
&= \frac{4}{n^2\pi^2}(\cos(n\pi) - 1) \\
&= \begin{cases} 0, & \text{if } n \text{ is even;} \\ -8/(n^2\pi^2), & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

where we have dropped the final tedious calculations (use integration by parts). From this it is clear that the Fourier series of the triangle wave is

$$-\frac{8}{\pi^2}\cos(2\pi t/T) - \frac{8}{3^2\pi^2}\cos(2\pi 3t/T) - \frac{8}{5^2\pi^2}\cos(2\pi 5t/T) - \frac{8}{7^2\pi^2}\cos(2\pi 7t/T) + \cdots.$$

In Figure 2.3 we have repeated the plots used for the square wave, for the triangle wave. As before, we have used $T = 1/440$. The figure clearly shows that the Fourier series coefficients decay much faster.

We can play different Fourier series approximations of the triangle wave, just as those for the square wave. For $N = 1$ and with $T = 1/440$ as above, it sounds like this. Again, this sounds exactly like the pure sound with frequency 440Hz. For $N = 5$ the Fourier series approximation sounds like this, and for $N = 9$ it



(a) The Fourier series for $N = 20$

(b) Values of the first 100 Fourier coefficients a_n

Figure 2.3: The Fourier series of the triangle wave of Example 2.7

sounds like this. Again these sounds are more like the triangle wave itself, and as we increase N we can hear that introduction of more frequencies pollutes the sound. However, since the triangle wave Fourier coefficients decrease as $1/n^2$ instead of $1/n$ as for the square wave, the sound is, although unpleasant due to pollution by many frequencies, not as unpleasant as the square wave. Also, it converges faster to the triangle wave itself, as also can be heard. In Exercise 7 you will be asked to write a program which verifies this.

From the previous examples we understand how we can use the Fourier coefficients to analyse or improve the sound: Noise in a sound often corresponds to the presence of some high frequencies with large coefficients, and by removing these, we remove the noise. For example, we could set all the coefficients except the first one to zero. This would change the unpleasant square wave to the pure tone $\sin 2\pi 440t$, which we started our experiments with.

2.1.1 Fourier series for symmetric and antisymmetric functions

In Example 2.6 we saw that the Fourier coefficients b_n vanished, resulting in a sine-series for the Fourier series. Similarly, in Example 2.7 we saw that a_n vanished, resulting in a cosine-series. This is not a coincident, and is captured by the following result, since the square wave was defined so that it was antisymmetric about 0, and the triangle wave so that it was symmetric about 0.

Theorem 2.8 (Symmetry and antisymmetry)**.** If f is antisymmetric about 0 (that is, if $f(-t) = -f(t)$ for all t), then $a_n = 0$, so the Fourier series is actually a sine-series. If f is symmetric about 0 (which means that $f(-t) = f(t)$ for all t), then $b_n = 0$, so the Fourier series is actually a cosine-series.

Proof. Note first that we can write

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(2\pi nt/T)dt \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(2\pi nt/T)dt,$$

i.e. we can change the integration bounds from $[0,T]$ to $[-T/2,T/2]$. This follows from the fact that all $f(t)$, $\cos(2\pi nt/T)$ and $\sin(2\pi nt/T)$ are periodic with period T.

Suppose first that f is symmetric. We obtain

$$\begin{aligned}
b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(2\pi nt/T)dt \\
&= \frac{2}{T}\int_{-T/2}^{0} f(t)\sin(2\pi nt/T)dt + \frac{2}{T}\int_{0}^{T/2} f(t)\sin(2\pi nt/T)dt \\
&= \frac{2}{T}\int_{-T/2}^{0} f(t)\sin(2\pi nt/T)dt - \frac{2}{T}\int_{0}^{-T/2} f(-t)\sin(-2\pi nt/T)dt \\
&= \frac{2}{T}\int_{-T/2}^{0} f(t)\sin(2\pi nt/T)dt - \frac{2}{T}\int_{-T/2}^{0} f(t)\sin(2\pi nt/T)dt = 0.
\end{aligned}$$

where we have made the substitution $u = -t$, and used that sin is antisymmetric. The case when f is antisymmetric can be proved in the same way, and is left as an exercise. □

In fact, the connection between symmetric and antisymmetric functions, and sine- and cosine series can be made even stronger by observing the following:

1. Any cosine series $a_0 + \sum_{n=1}^{N} a_n \cos(2\pi nt/T)$ is a symmetric function.
2. Any sine series $\sum_{n=1}^{N} b_n \sin(2\pi nt/T)$ is an antisymmetric function.
3. Any periodic function can be written as a sum of a symmetric and antisymmetric function by writing

$$f(t) = \frac{f(t)+f(-t)}{2} + \frac{f(t)-f(-t)}{2}. \tag{2.9}$$

4. If $f_N(t) = a_0 + \sum_{n=1}^{N}(a_n\cos(2\pi nt/T) + b_n\sin(2\pi nt/T))$, then

$$\begin{aligned}
\frac{f_N(t)+f_N(-t)}{2} &= a_0 + \sum_{n=1}^{N} a_n\cos(2\pi nt/T) \\
\frac{f_N(t)-f_N(-t)}{2} &= \sum_{n=1}^{N} b_n\sin(2\pi nt/T).
\end{aligned}$$

Exercises for Section 2.1

Ex. 1 — Find a function f which is Riemann-integrable on $[0, T]$, and so that $\int_0^T f(t)^2 dt$ is infinite.

Ex. 2 — Given the two Fourier spaces V_{N_1,T_1}, V_{N_2,T_2}. Find necessary and sufficient conditions in order for $V_{N_1,T_1} \subset V_{N_2,T_2}$.

Ex. 3 — Prove the second part of Theorem 2.8, i.e. show that if f is anti-symmetric about 0 (i.e. $f(-t) = -f(t)$ for all t), then $a_n = 0$, i.e. the Fourier series is actually a sine-series.

Ex. 4 — Find the Fourier series coefficients of the periodic functions with period T defined by being $f(t) = t$, $f(t) = t^2$, and $f(t) = t^3$, on $[0, T]$.

Ex. 5 — Write down difference equations for finding the Fourier coefficients of $f(t) = t^{k+1}$ from those of $f(t) = t^k$, and write a program which uses this recursion. Use the program to verify what you computed in Exercise 4.

Ex. 6 — Use the previous exercise to find the Fourier series for $f(x) = -\frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{3}{16}x + 1$ on the interval $[0, 1]$. Plot the 9th order Fourier series for this function. You should obtain the plots from Figure 2.1.

Ex. 7 — Let us write programs so that we can listen to the Fourier approximations of the square wave and the triangle wave.

a. Write functions

```
function playsquaretrunk(T,N)
function playtriangletrunk(T,N)
```

which plays the order N Fourier approximation of the square wave and the triangle wave, respectively, for three seconds. Verify that you can generate the sounds you played in examples 2.6 and 2.7.

b. For these Fourier approximations, how high must you choose N for them to be indistuingishable from the square/triangle waves themselves? Also describe how the characteristics of the sound changes when n increases.

2.2 Complex Fourier series

In Section 2.1 we saw how a function can be expanded in a series of sines and cosines. These functions are related to the complex exponential function via Eulers formula

$$e^{ix} = \cos x + i \sin x$$

where i is the imaginary unit with the property that $i^2 = -1$. Because the algebraic properties of the exponential function are much simpler than those of the cos and sin, it is often an advantage to work with complex numbers, even though the given setting is real numbers. This is definitely the case in Fourier analysis. More precisely, we would like to make the substitutions

$$\cos(2\pi nt/T) = \frac{1}{2}\left(e^{2\pi int/T} + e^{-2\pi int/T}\right) \tag{2.10}$$

$$\sin(2\pi nt/T) = \frac{1}{2i}\left(e^{2\pi int/T} - e^{-2\pi int/T}\right) \tag{2.11}$$

in Definition 2.3. From these identities it is clear that the set of complex exponential functions $e^{2\pi int/T}$ also is a basis of periodic functions (with the same period) for $V_{N,T}$. We may therefore reformulate Definition 2.3 as follows:

Definition 2.9 (Complex Fourier basis)**.** We define the set of functions

$$\mathcal{F}_{N,T} = \{e^{-2\pi int/T}, e^{-2\pi i(n-1)t/T}, \cdots, e^{-2\pi it/T}, \tag{2.12}$$

$$1, e^{2\pi it/T}, \cdots, e^{2\pi i(n-1)t/T}, e^{2\pi int/T}\}, \tag{2.13}$$

and call this the order N complex Fourier basis for $V_{N,T}$.

The function $e^{2\pi int/T}$ is also called a pure tone with frequency n/T, just as for sines and cosines. We would like to show that these functions also are orthogonal. To show this, we need to say more on the inner product we have defined by (2.1). A weakness with this definition is that we have assumed real functions f and g, so that this can not be used for the complex exponential functions $e^{2\pi int/T}$. For general complex functions we will extend the definition of the inner product as follows:

$$\langle f, g\rangle = \frac{1}{T}\int_0^T f\bar{g}\,dt. \tag{2.14}$$

The associated norm now becomes

$$\|f\| = \sqrt{\frac{1}{T}\int_0^T |f(t)|^2 dt}. \tag{2.15}$$

The motivation behind Equation 2.14, where we have conjugated the second function, lies in the definition of an *inner product for vector spaces over complex* *numbers.* From before we are used to vector spaces over real numbers, but vector spaces over complex numbers are defined through the same set of axioms as for real vector spaces, only replacing real numbers with complex numbers. For complex vector spaces, the axioms defining an inner product are the same as for real vector spaces, except for that the axiom

$$\langle f, g\rangle = \langle g, f\rangle \tag{2.16}$$

is replaced with the axiom

$$\langle f, g\rangle = \overline{\langle g, f\rangle}, \tag{2.17}$$

i.e. a conjugation occurs when we switch the order of the functions. This new axiom can be used to prove the property $\langle f, cg\rangle = \bar{c}\langle f, g\rangle$, which is a somewhat different property from what we know for real inner product spaces. This property follows by writing

$$\langle f, cg\rangle = \overline{\langle cg, f\rangle} = \overline{c\langle g, f\rangle} = \bar{c}\overline{\langle g, f\rangle} = \bar{c}\langle f, g\rangle.$$

Clearly the inner product 2.14 satisfies Axiom 2.17. With this definition it is quite easy to see that the functions $e^{2\pi int/T}$ are orthonormal. Using the orthogonal decomposition theorem we can therefore write

$$\begin{aligned} f_N(t) &= \sum_{n=-N}^{N} \frac{\langle f, e^{2\pi int/T}\rangle}{\langle e^{2\pi int/T}, e^{2\pi int/T}\rangle} e^{2\pi int/T} = \sum_{n=-N}^{N} \langle f, e^{2\pi int/T}\rangle e^{2\pi int/T} \\ &= \sum_{n=-N}^{N} \left(\frac{1}{T}\int_0^T f(t)e^{-2\pi int/T}dt \right) e^{2\pi int/T}. \end{aligned}$$

We summarize this in the following theorem, which is a version of Theorem 2.4 which uses the complex Fourier basis:

Theorem 2.10. We denote by $y_{-N}, \ldots, y_0, \ldots, y_N$ the coordinates of f_N in the basis $\mathcal{F}_{N,T}$, i.e.

$$f_N(t) = \sum_{n=-N}^{N} y_n e^{2\pi int/T}. \tag{2.18}$$

The y_n are called the complex Fourier coefficients of f, and they are given by.

$$y_n = \langle f, e^{2\pi int/T}\rangle = \frac{1}{T}\int_0^T f(t)e^{-2\pi int/T}dt. \tag{2.19}$$

If we reorder the real and complex Fourier bases so that the two functions $\{\cos(2\pi nt/T), \sin(2\pi nt/T)\}$ and $\{e^{2\pi int/T}, e^{-2\pi int/T}\}$ have the same index in the bases, equations (2.10)-(2.11) give us that the change of basis matrix[4] from

[4]See Section 4.7 in [7], to review the mathematics behind change of basis.

$\mathcal{D}_{N,T}$ to $\mathcal{F}_{N,T}$, denoted $P_{\mathcal{F}_{N,T}\leftarrow\mathcal{D}_{N,T}}$, is represented by repeating the matrix

$$\frac{1}{2}\begin{pmatrix} 1 & 1/i \\ 1 & -1/i \end{pmatrix}$$

along the diagonal (with an additional 1 for the constant function 1). In other words, since a_n, b_n are coefficients relative to the real basis and y_n, y_{-n} the corresponding coefficients relative to the complex basis, we have for $n > 0$,

$$\begin{pmatrix} y_n \\ y_{-n} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1/i \\ 1 & -1/i \end{pmatrix}\begin{pmatrix} a_n \\ b_n \end{pmatrix}.$$

This can be summarized by the following theorem:

Theorem 2.11 (Change of coefficients between real and complex Fourier bases)**.** The complex Fourier coefficients y_n and the real Fourier coefficients a_n, b_n of a function f are related by

$$\begin{aligned} y_0 &= a_0, \\ y_n &= \frac{1}{2}(a_n - ib_n), \\ y_{-n} &= \frac{1}{2}(a_n + ib_n), \end{aligned}$$

for $n = 1, \ldots, N$.

Combining with Theorem 2.8, Theorem 2.11 can help us state properties of complex Fourier coefficients for symmetric- and antisymmetric functions. We look into this in Exercise 8.

Due to the somewhat nicer formulas for the complex Fourier coefficients when compraed to the real Fourier coefficients, we will write most Fourier series in complex form in the following.

Exercises for Section 2.2

Ex. 1 — Show that the complex functions $e^{2\pi int/T}$ are orthonormal.

Ex. 2 — Repeat Exercise 2.1.4, computing the complex Fourier series instead of the real Fourier series.

Ex. 3 — Show that both $\cos^n t$ and $\sin^n t$ are in $V_{N,T}$, and find an expression for their complex Fourier coefficients.

Ex. 4 — Consider a sum of two complex exponentials. When is their sum also periodic? What is the fundamental period of the sum if the sum also is periodic?

Ex. 5 — Compute the complex Fourier coefficients of the square wave using Equation 2.19, i.e. repeat the calculations from Example 2.6 for the complex case. Use Theorem 2.11 to verify your result.

Ex. 6 — Repeat Exercise 5 for the triangle wave.

Ex. 7 — Use Equation 2.19 to compute the complex Fourier coefficients of the periodic functions with period T defined by, respectively, $f(t) = t$, $f(t) = t^2$, and $f(t) = t^3$, on $[0, T]$. Use Theorem 2.11 to verify you calculations from Exercise 4.

Ex. 8 — In this exercise we will prove a version of Theorem 2.8 for complex Fourier coefficients.

a. If f is symmetric about 0, show that y_n is real, and that $y_{-n} = y_n$.
b. If f is antisymmetric about 0, show that the y_n are purely imaginary, $y_0 = 0$, and that $y_{-n} = -y_n$.
c. Show that $\sum_{n=-N}^{N} y_n e^{2\pi int/T}$ is symmetric when $y_{-n} = y_n$ for all n, and rewrite it as a cosine-series.
d. Show that $\sum_{n=-N}^{N} y_n e^{2\pi int/T}$ is antisymmetric when $y_0 = 0$ and $y_{-n} = -y_n$ for all n, and rewrite it as a sine-series.

2.3 Rate of convergence for Fourier series

We have earlier mentioned criteria which guarantee that the Fourier series converges. Another important topic is the rate of convergence of the Fourier series, given that it converges. If the series converges quickly, we may only need a few terms in the Fourier series to obtain a reasonable approximation, meaning that good Fourier series approximations can be computed quickly. We have already seen examples which illustrate convergence rates that appear to be different: The square wave seemed to have very slow convergence rate near the discontinuities, while the triangle wave did not seem to have the same problem.

Before discussing results concerning convergence rates we consider a simple lemma which will turn out to be useful.

Lemma 2.12. If the complex Fourier coefficients of f are y_n and f is differentiable, then the Fourier coefficients of $f'(t)$ are $\frac{2\pi in}{T} y_n$.

Proof. The Fourier coefficients of $f'(t)$ are

$$\begin{aligned}\frac{1}{T}\int_0^T f'(t)e^{-2\pi int/T}dt &= \frac{1}{T}\left(\left[f(t)e^{-2\pi int/T}\right]_0^T + \frac{2\pi in}{T}\int_0^T f(t)e^{-2\pi int/T}dt\right)\\ &= \frac{2\pi in}{T}y_n.\end{aligned}$$

where the second equation was obtained from integration by parts. □

If we turn this around, we note that the Fourier coefficients of $f(t)$ are $T/(2\pi in)$ times those of $f'(t)$. If f is s times differentiable, we can repeat this argument to show that the Fourier coefficients of $f(t)$ are $\big(T/(2\pi in)\big)^s$ times those of $f^{(s)}(t)$. In other words, the Fourier coefficients of a function which is many times differentiable decay to zero very fast.

Observation 2.13. The Fourier series converges quickly when the function is many times differentiable.

An illustration is found in examples 2.6 and 2.7, where we saw that the Fourier series coefficients for the triangle wave converged more quickly to zero than those of the square wave. This is explained by the fact that the square wave is discontinuous, while triangle wave is continuous with a discontinuous first derivative.

Very often, the slow convergence of a Fourier series is due to some discontinuity of (a derivative of) the function at a given point. In this case a strategy to speed up the convergence of the Fourier series could be to create an extension of the function which is continuous, if possible, and use the Fourier series of this new function instead. With the help of the following definition, we will show that this strategy works, at least in cases where there is only one single point of discontinuity (for simplicity we have assumed that the discontinuity is at 0).

Definition 2.14 (Symmetric extension of a function)**.** Let f be a function defined on $[0,T]$. The symmetric extension of f denotes the function $\breve{f}$ defined on $[0,2T]$ by

$$\breve{f}(t) = \begin{cases} f(t), & \text{if } 0 \le t \le T;\\ f(2T-t), & \text{if } T < t \le 2T.\end{cases}$$

Clearly $\breve{f}(0) = \breve{f}(2T)$, so when f is continuous, it can be periodically extended to a continuous function with period $2T$, contrary to the function f we started with. Also, $\breve{f}$ keeps the characteristics of f, since they are equal on $[0,T]$. Also, $\breve{f}$ is clearly a symmetric function, so that it can be expressed as a cosine-series. The Fourier coefficients of the two functions are related.

Theorem 2.15. The complex Fourier coefficients y_n of f, and the cosine-coefficients a_n of $\breve{f}$ are related by $a_{2n} = y_n + y_{-n}$.

Proof. The $2n$th complex Fourier coefficient of $\breve{f}$ is

$$\begin{aligned}
&\frac{1}{2T}\int_0^{2T} \breve{f}(t)e^{-2\pi i 2nt/(2T)}dt \\
&= \frac{1}{2T}\int_0^{T} f(t)e^{-2\pi int/T}dt + \frac{1}{2T}\int_T^{2T} f(2T-t)e^{-2\pi int/T}dt.
\end{aligned}$$

Substituting $u = 2T - t$ in the second integral we see that this is

$$\begin{aligned}
&= \frac{1}{2T}\int_0^{T} f(t)e^{-2\pi int/T}dt - \frac{1}{2T}\int_T^{0} f(u)e^{2\pi inu/T}du \\
&= \frac{1}{2T}\int_0^{T} f(t)e^{-2\pi int/T}dt + \frac{1}{2T}\int_0^{T} f(t)e^{2\pi int/T}dt \\
&= \frac{1}{2}y_n + \frac{1}{2}y_{-n}.
\end{aligned}$$

Therefore we have $a_{2n} = y_n - y_{-n}$. □

This result is not enough to obtain the entire Fourier series of $\breve{f}$, but at least it gives us half of it.

Example 2.16. Let f be the function with period T defined by $f(t) = 2t/T - 1$ for $0 \leq t < T$. In each period the function increases linearly from 0 to 1. Because f is discontinuous at the boundaries between the periods, we would except the Fourier series to converge slowly. Since the function is antisymmetric, the coefficients a_n are zero, and we compute b_n as

$$\begin{aligned}
b_n &= \frac{2}{T}\int_0^T \frac{2}{T}\left(t - \frac{T}{2}\right)\sin(2\pi nt/T)dt = \frac{4}{T^2}\int_0^T \left(t - \frac{T}{2}\right)\sin(2\pi nt/T)dt \\
&= \frac{4}{T^2}\int_0^T t\sin(2\pi nt/T)dt - \frac{2}{T}\int_0^T \sin(2\pi nt/T)dt \\
&= -\frac{2}{\pi n},
\end{aligned}$$

so that the Fourier series is

$$-\frac{2}{\pi}\sin(2\pi t/T) - \frac{2}{2\pi}\sin(2\pi 2t/T) - \frac{2}{3\pi}\sin(2\pi 3t/T) - \frac{2}{4\pi}\sin(2\pi 4t/T) - \cdots,$$

which indeed converges slowly to 0. Let us now instead consider the symmetrization of f. Clearly this is the triangle wave with period $2T$, and the Fourier series of this is

$$\begin{aligned}
&-\frac{8}{\pi^2}\cos(2\pi t/(2T)) - \frac{8}{3^2\pi^2}\cos(2\pi 3t/(2T)) - \frac{8}{5^2\pi^2}\cos(2\pi 5t/(2T)) \\
&-\frac{8}{7^2\pi^2}\cos(2\pi 7t/(2T)) + \cdots.
\end{aligned}$$

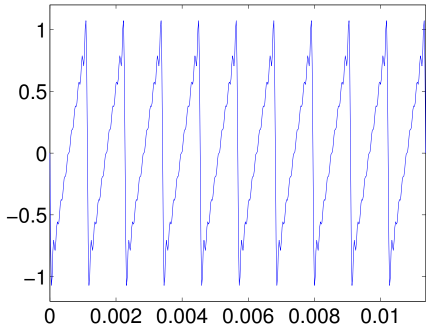


Figure 2.4: The Fourier series for $N = 10$ for the function in Example 2.16

Comparing the two series, we see that the coefficient at frequency n/T in the first series has value $-2(n\pi)$, while in the second series it has value

$$-\frac{8}{(2n)^2\pi^2} = -\frac{2}{n^2\pi^2}.$$

The second series clearly converges faster than the first. Also, we could have obtained half of the second set of coefficients from the first, by using Theorem 2.15.

If we use $T = 1/880$, the symmetrization will be the square wave of Example 2.7. Its Fourier series for $N = 10$ is shown in Figure 2.3(b) and the Fourier series for for f $N = 20$ is shown in Figure 2.4. The value $N = 10$ is used since this corresponds to the same frequencies as the previous figure for $N = 20$. It is clear from the plot that the Fourier series of f is not a very good approximation. However, we cannot differentiate between the Fourier series and the function itself for the triangle wave.

2.4 Some properties of Fourier series

We will end this section by establishing some important properties of the Fourier series, in particular the Fourier coefficients for some important functions. In these lists, we will use the notation $f \rightarrow y_n$ to indicate that y_n is the n'th Fourier coefficient of $f(t)$.

Theorem 2.17 (Fourier series pairs)**.** The functions 1, $e^{2\pi int/T}$, and $\chi_{-a,a}$

have the Fourier coefficients

$$1 \to \boldsymbol{e}_0 = (1, 0, 0, 0 \ldots,)$$
$$e^{2\pi int/T} \to \boldsymbol{e}_k = (0, 0, \ldots, 1, 0, 0, \ldots)$$
$$\chi_{-a,a} \to \frac{\sin(2\pi na/T)}{\pi n}.$$

The 1 in $\boldsymbol{e}_k$ is at position n and the function $\chi_{-a,a}$ is the characteristic function of the interval $[-a, a]$, defined by

$$\chi_{-a,a}(t) = \begin{cases} 1, & \text{if } t \in [-a, a]; \\ 0, & \text{otherwise.} \end{cases}$$

The first two pairs are easily verified, so the proofs are omitted. The case for $\chi_{-a,a}$ is very similar to the square wave, but easier to prove, and therefore also omitted.

Theorem 2.18 (Fourier series properties)**.** The mapping $f \to y_n$ is linear: if $f \to x_n$, $g \to y_n$, then

$$af + bg \to ax_n + by_n$$

For all n. Moreover, if f is real and periodic with period T, the following properties hold:

1. $y_n = \overline{y_{-n}}$ for all n.
2. If $g(t) = f(-t)$ and $f \to y_n$, then $g \to \overline{y_n}$. In particular,
   (a) if $f(t) = f(-t)$ (i.e. f is symmetric), then all y_n are real, so that b_n are zero and the Fourier series is a cosine series.
   (b) if $f(t) = -f(-t)$ (i.e. f is antisymmetric), then all y_n are purely imaginary, so that the a_n are zero and the Fourier series is a sine series.
3. If $g(t) = f(t - d)$ (i.e. g is the function f delayed by d) and $f \to y_n$, then $g \to e^{-2\pi ind/T} y_n$.
4. If $g(t) = e^{2\pi idt/T} f(t)$ with d an integer, and $f \to y_n$, then $g \to y_{n-d}$.
5. Let d be a number. If $f \to y_n$, then $f(d + t) = f(d - t)$ for all t if and only if the argument of y_n is $-2\pi nd/T$ for all n.

The last property looks a bit mysterious. We will not have use for this property before the next chapter.

Proof. The proof of linearity is left to the reader. Property 1 follows immediately by writing

$$\begin{aligned} y_n &= \frac{1}{T}\int_0^T f(t)e^{-2\pi int/T}dt = \overline{\frac{1}{T}\int_0^T f(t)e^{2\pi int/T}dt} \\ &= \overline{\frac{1}{T}\int_0^T f(t)e^{-2\pi i(-n)t/T}dt} = \overline{y_{-n}}. \end{aligned}$$

Also, if $g(t) = f(-t)$, we have that

$$\begin{aligned} \frac{1}{T}\int_0^T g(t)e^{-2\pi int/T}dt &= \frac{1}{T}\int_0^T f(-t)e^{-2\pi int/T}dt = -\frac{1}{T}\int_0^{-T} f(t)e^{2\pi int/T}dt \\ &= \frac{1}{T}\int_0^T f(t)e^{2\pi int/T}dt = \overline{y_n}. \end{aligned}$$

Property 2 follows from this, since the remaining statements here were established in Theorems 2.8, 2.11, and Exercise 2.2.8. To prove property 3, we observe that the Fourier coefficients of $g(t) = f(t-d)$ are

$$\begin{aligned} \frac{1}{T}\int_0^T g(t)e^{-2\pi int/T}dt &= \frac{1}{T}\int_0^T f(t-d)e^{-2\pi int/T}dt \\ &= \frac{1}{T}\int_0^T f(t)e^{-2\pi in(t+d)/T}dt \\ &= e^{-2\pi ind/T}\frac{1}{T}\int_0^T f(t)e^{-2\pi int/T}dt = e^{-2\pi ind/T}y_n. \end{aligned}$$

For property 4 we observe that the Fourier coefficients of $g(t) = e^{2\pi idt/T}f(t)$ are

$$\begin{aligned} \frac{1}{T}\int_0^T g(t)e^{-2\pi int/T}dt &= \frac{1}{T}\int_0^T e^{2\pi idt/T}f(t)e^{-2\pi int/T}dt \\ &= \frac{1}{T}\int_0^T f(t)e^{-2\pi i(n-d)t/T}dt = y_{n-d}. \end{aligned}$$

If $f(d+t) = f(d-t)$ for all t, we define the function $g(t) = f(t+d)$ which is symmetric about 0, so that it has real Fourier coefficients. But then the Fourier coefficients of $f(t) = g(t-d)$ are $e^{-2\pi ind/T}$ times the (real) Fourier coefficients of g by property 3. It follows that y_n, the Fourier coefficients of f, has argument $-2\pi nd/T$. The proof in the other direction follows by noting that any function where the Fourier coefficients are real must be symmetric about 0, once the Fourier series is known to converge. This proves property 5. □

From this theorem we see that there exist several cases of duality between Fourier coefficients, and the function itself:

1. Delaying a function corresponds to multiplying the Fourier coefficients with a complex exponential. Vice versa, multiplying a function with a complex exponential corresponds to delaying the Fourier coefficients.

2. Symmetry/antisymmetry for a function corresponds to the Fourier coefficients being real/purely imaginary. Vice versa, a function which is real has Fourier coefficients which are conjugate symmetric.

Note that these dualities become even more explicit if we consider Fourier series of complex functions, and not just real functions.

Exercises for Section 2.4

Ex. 1 — Define the function f with period T on $[-T/2, T/2]$ by

$$f(t) = \begin{cases} 1, & \text{if } -T/4 \leq t < T/4; \\ -1, & \text{if } |T/4| \leq t < |T/2|. \end{cases}$$

f is just the square wave, shifted with $T/4$. Compute the Fourier coefficients of f directly, and use 3. in Theorem 2.18 to verify your result.

Ex. 2 — Find a function f which has the complex Fourier series

$$\sum_{n \text{ odd}} \frac{4}{\pi(n+4)} e^{2\pi i n t/T}.$$

Hint: Attempt to use one of the properties in Theorem 2.18 on the Fourier series of the square wave.

2.5 Summary

In this chapter we have defined and studied Fourier series, which is an approximation scheme forperiodic functions using trigonometric functions. We have established the basic properties of Fourier series, and some duality relationships between the function and its Fourier series. We have also computed the Fourier series of the square wave and the triangle wave, and investigated a technique for speeding up the convergence of the Fourier series.