Part I

Fourier analysis and applications to sound processing

Chapter 1

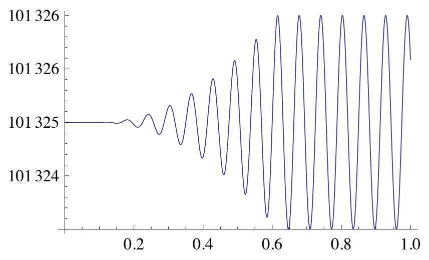
Sound

A major part of the information we receive and perceive every day is in the form of audio. Most of these sounds are transferred directly from the source to our ears, like when we have a face to face conversation with someone or listen to the sounds in a forest or a street. However, a considerable part of the sounds are generated by loudspeakers in various kinds of audio machines like cell phones, digital audio players, home cinemas, radios, television sets and so on. The sounds produced by these machines are either generated from information stored inside, or electromagnetic waves are picked up by an antenna, processed, and then converted to sound. It is this kind of sound we are going to study in this chapter. The sound that is stored inside the machines or picked up by the antennas is usually represented as *digital sound*. This has certain limitations, but at the same time makes it very easy to manipulate and process the sound on a computer.

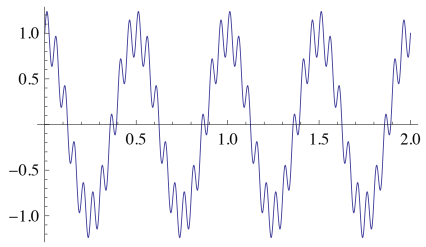
What we perceive as sound corresponds to the physical phenomenon of slight variations in air pressure near our ears. Larger variations mean louder sounds, while faster variations correspond to sounds with a higher pitch. The air pressure varies continuously with time, but at a given point in time it has a precise value. This means that sound can be considered to be a mathematical function.

Observation 1.1. A sound can be represented by a mathematical function, with time as the free variable. When a function represents a sound, it is often referred to as a *continuous signal*.

In the following we will briefly discuss the basic properties of sound: first the significance of the size of the variations, and then how many variations there are per second, the *frequency* of the sound. We also consider the important fact that any reasonable sound may be considered to be built from very simple basis sounds. Since a sound may be viewed as a function, the mathematical equivalent of this is that any decent function may be constructed from very simple basis functions. Fourier-analysis is the theoretical study of this, and in the next chapters we are going to study this from a practical and computational



(a) A sound shown in terms of air pressure



(b) A sound shown in terms of the difference from the ambient air pressure

Figure 1.1: Two examples of audio signals.

perspective. Towards the end of this chapter we also consider the basics of digital audio, and illustrate its power by performing some simple operations on digital sounds.

1.1 Loudness: Sound pressure and decibels

An example of a simple sound is shown in Figure 1.1(a) where the oscillations in air pressure are plotted agains time. We observe that the initial air pressure has the value 101 325 (we will shortly return to what unit is used here), and then the pressure starts to vary more and more until it oscillates regularly between the values 101 323 and 101 327. In the area where the air pressure is constant, no sound will be heard, but as the variations increase in size, the sound becomes louder and louder until about time $t = 0.6$ where the size of the oscillations becomes constant. The following summarises some basic facts about air pressure.

Fact 1.2 (Air pressure)**.** Air pressure is measured by the SI-unit Pa (Pascal) which is equivalent to N/m^2 (force / area). In other words, 1 Pa corresponds to the force exerted on an area of 1 m^2 by the air column above this area. The normal air pressure at sea level is 101 325 Pa.

Fact 1.2 explains the values on the vertical axis in Figure 1.1(a): The sound was recorded at the normal air pressure of 101 325 Pa. Once the sound started, the pressure started to vary both below and above this value, and after a short transient phase the pressure varied steadily between 101 324 Pa and 101 326 Pa, which corresponds to variations of size 1 Pa about the fixed value. Everyday sounds typically correspond to variations in air pressure of about 0.00002–2 Pa, while a jet engine may cause variations as large as 200 Pa. Short exposure to variations of about 20 Pa may in fact lead to hearing damage. The volcanic eruption at Krakatoa, Indonesia, in 1883, produced a sound wave with variations as large as almost 100 000 Pa, and the explosion could be heard 5000 km away.

When discussing sound, one is usually only interested in the variations in air pressure, so the ambient air pressure is subtracted from the measurement. This corresponds to subtracting 101 325 from the values on the vertical axis in Figure 1.1(a). In Figure 1.1(b) the subtraction has been performed for another sound, and we see that the sound has a slow, cos-like, variation in air pressure, with some smaller and faster variations imposed on this. This combination of several kinds of systematic oscillations in air pressure is typical for general sounds. The size of the oscillations is directly related to the loudness of the sound. We have seen that for audible sounds the variations may range from 0.00002 Pa all the way up to 100 000 Pa. This is such a wide range that it is common to measure the loudness of a sound on a logarithmic scale. Often air pressure is normalized so that it lies between -1 and 1: The value 0 then represents the ambient air pressure, while -1 and 1 represent the lowest and highest representable air pressure, respectively. The following fact box summarises the previous discussion of what a sound is, and introduces the logarithmic decibel scale.

Fact 1.3 (Sound pressure and decibels)**.** The physical origin of sound is variations in air pressure near the ear. The *sound pressure* of a sound is obtained by subtracting the average air pressure over a suitable time interval from the measured air pressure within the time interval. A square of this difference is then averaged over time, and the sound pressure is the square root of this average.

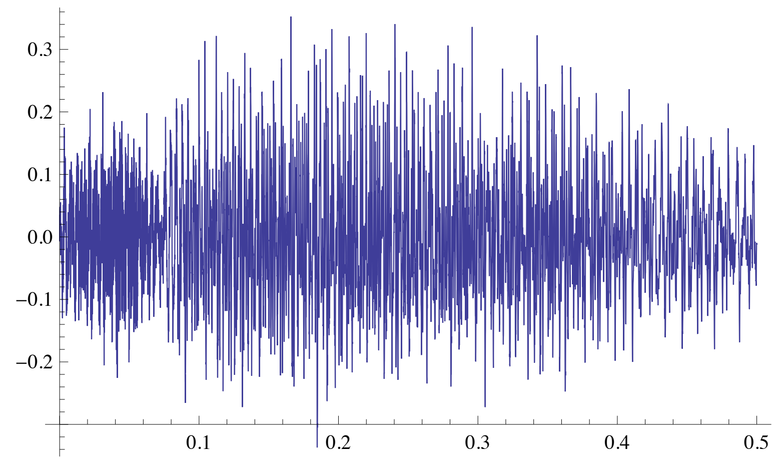
It is common to relate a given sound pressure to the smallest sound pressure that can be perceived, as a level on a decibel scale,

$$L_p = 10\log_{10}\left(\frac{p^2}{p_{\text{ref}}^2}\right) = 20\log_{10}\left(\frac{p}{p_{\text{ref}}}\right).$$

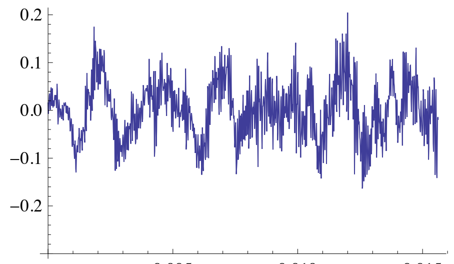
Here p is the measured sound pressure while p_{ref} is the sound pressure of a just perceivable sound, usually considered to be 0.00002 Pa.

The square of the sound pressure appears in the definition of L_p since this represents the *power* of the sound which is relevant for what we perceive as loudness.

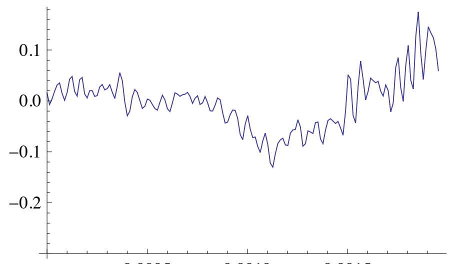
The sounds in Figure 1.1 are synthetic in that they were constructed from mathematical formulas (see Exercises 1.4.2 and 1.4.3). The sounds in Figure 1.2 on the other hand show the variation in air pressure when there is no mathematical formula involved, such as is the case for a song. In (a) there are so many oscillations that it is impossible to see the details, but if we zoom in as in (c) we can see that there is a continuous function behind all the ink. It is important to realise that in reality the air pressure varies more than this, even over the short time period in (c). However, the measuring equipment was not able to pick up those variations, and it is also doubtful whether we would be able to perceive such rapid variations.



(a) 0.5 seconds of the song



(b) the first 0.015 seconds



(c) the first 0.002 seconds

Figure 1.2: Variations in air pressure during parts of a song.

1.2 The pitch of a sound

Besides the size of the variations in air pressure, a sound has another important characteristic, namely the frequency (speed) of the variations. For most sounds the frequency of the variations varies with time, but if we are to perceive variations in air pressure as sound, they must fall within a certain range.

Fact 1.4. For a human with good hearing to perceive variations in air pressure as sound, the number of variations per second must be in the range 20–20 000.

To make these concepts more precise, we first recall what it means for a function to be periodic.

Definition 1.5. A real function f is said to be periodic with period τ if

$$f(t+\tau) = f(t)$$

for all real numbers t.

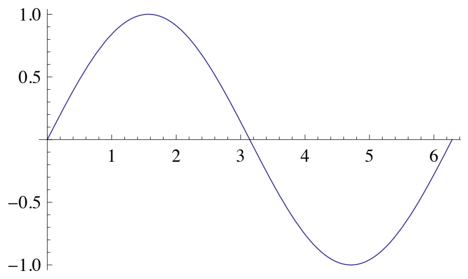
Note that all the values of a periodic function f with period τ are known if $f(t)$ is known for all t in the interval $[0,\tau)$. The prototypes of periodic functions are the trigonometric ones, and particularly $\sin t$ and $\cos t$ are of interest to us. Since $\sin(t+2\pi) = \sin t$, we see that the period of $\sin t$ is 2π and the same is true for $\cos t$.

There is a simple way to change the period of a periodic function, namely by multiplying the argument by a constant.

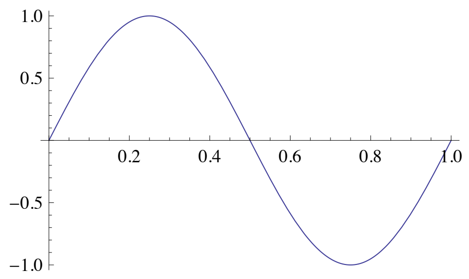
Observation 1.6 (Frequency)**.** If ν is an integer, the function $f(t) = \sin(2\pi\nu t)$ is periodic with period $\tau = 1/\nu$. When t varies in the interval $[0,1]$, this function covers a total of ν periods. This is expressed by saying that f has *frequency* ν.

Figure 1.3 illustrates observation 1.6. The function in (a) is the plain $\sin t$ which covers one period when t varies in the interval $[0,2\pi]$. By multiplying the argument by 2π, the period is squeezed into the interval $[0,1]$ so the function $\sin(2\pi t)$ has frequency $\nu = 1$. Then, by also multiplying the argument by 2, we push two whole periods into the interval $[0,1]$, so the function $\sin(2\pi 2t)$ has frequency $\nu = 2$. In (d) the argument has been multiplied by 5 — hence the frequency is 5 and there are five whole periods in the interval $[0,1]$. Note that any function on the form $\sin(2\pi\nu t + a)$ has frequency ν, regardless of the value of a.

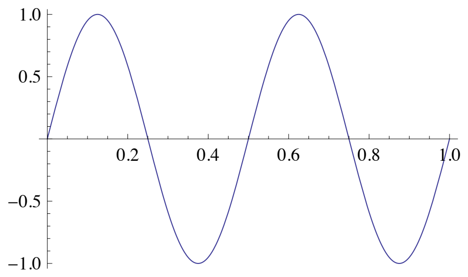
Since sound can be modelled by functions, it is reasonable to say that a sound with frequency ν is a trigonometric function with frequency ν.



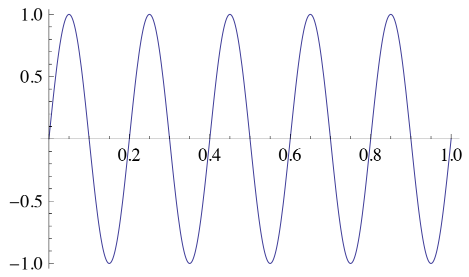
(a) $\sin t$



(b) $\sin(2\pi t)$



(c) $\sin(2\pi 2t)$



(d) $\sin(2\pi 5t)$

Figure 1.3: Versions of sin with different frequencies.

Definition 1.7. The function $\sin(2\pi\nu t)$ represents what we will call a pure tone with frequency ν. Frequency is measured in Hz (Herz) which is the same as s^{-1} (the time t is measured in seconds).

A pure tone with frequency 440 Hz sounds like this, and a pure tone with frequency 1500 Hz sounds like this.

Any sound may be considered to be a function. In the next chapter we are going to see that any reasonable function may be written as a sum of simple sin- and cos- functions with integer frequencies. When this is translated into properties of sound, we obtain an important principle.

Observation 1.8 (Decomposition of sound into pure tones)**.** Any sound f is a sum of pure tones at different frequencies. The amount of each frequency required to form f is the frequency content of f. Any sound can be reconstructed from its frequency content.

The most basic consequence of observation 1.8 is that it gives us an understanding of how any sound can be built from the simple building blocks of pure tones. This means that we can store a sound f by storing its frequency content, as an alternative to storing f itself. This also gives us a possibility for lossy compression of digital sound: It turns out that in a typical audio signal there will be most information in the lower frequencies, and some frequencies will be almost completely absent. This can be exploited for compression if we change the frequencies with small contribution a little bit and set them to 0, and then store the signal by only storing the nonzero part of the frequency content. When the sound is to be played back, we first convert the adjusted values to the adjusted frequency content back to a normal function representation with an inverse mapping.

Fact 1.9 (Basic idea behind audio compression)**.** Suppose an audio signal f is given. To compress f, perform the following steps:

1. Rewrite the signal f in a new format where frequency information becomes accessible.
2. Remove those frequencies that only contribute marginally to human perception of the sound.
3. Store the resulting sound by coding the adjusted frequency content with some lossless coding method.

This lossy compression strategy is essentially what is used in practice by commercial audio formats. The difference is that commercial software does everything in a more sophisticated way and thereby gets better compression rates. We will return to this in later chapters.

We will see later that Observation 1.8 also is the basis for many operations on sounds. The same observation also makes it possible to explain more precisely what it means that we only perceive sounds with a frequency in the range 20–20000 Hz:

Fact 1.10. Humans can only perceive variations in air pressure as sound if the Fourier series of the sound signal contains at least one sufficiently large term with frequency in the range 20–20 000 Hz.

With appropriate software it is easy to generate a sound from a mathematical function; we can 'play' the function. If we play a function like $\sin(2\pi 440t)$, we hear a pleasant sound with a very distinct pitch, as expected. There are, however, many other ways in which a function can oscillate regularly. The function in Figure 1.1(b) for example, definitely oscillates 2 times every second, but it does not have frequency 2 Hz since it is not a pure tone. This sound is also not that pleasant to listen to. We will consider two more important examples of this, which are very different from smooth, trigonometric functions.

Example 1.11. We define the *square wave* of period T as the function which repeats with period T, and is 1 on the first half of each period, and -1 on the second half. This means that we can define it as the function

$$f(t) = \begin{cases} 1, & \text{if } 0 \leq t < T/2; \\ -1, & \text{if } T/2 \leq t < T. \end{cases} \tag{1.1}$$

In Figure 1.4(a) we have plotted the square wave when $T = 1/440$. This period is chosen so that it corresponds to the pure tone we already have listened to, and you can listen to this square wave here (in Exercise 5 you will learn how to generate this sound). We hear a sound with the same pitch as $\sin(2\pi 440t)$, but note that the square wave is less pleasant to listen to: There seems to be some sharp corners in the sound, translating into a rather shrieking, piercing sound. We will later explain this by the fact that the square wave can be viewed as a sum of many frequencies, and that all the different frequencies pollute the sound so that it is not pleasant to listen to.

Example 1.12. We define the *triangle wave* of period T as the function which repeats with period T, and increases linearly from -1 to 1 on the first half of each period, and decreases linearly from 1 to -1 on the second half of each period. This means that we can define it as the function

$$f(t) = \begin{cases} 4t/T - 1, & \text{if } 0 \leq t < T/2; \\ 3 - 4t/T, & \text{if } T/2 \leq t < T. \end{cases} \tag{1.2}$$

In Figure 1.4(b) we have plotted the triangle wave when $T = 1/440$. Again, this same choice of period gives us an audible sound, and you can listen to the triangle wave here (in Exercise 5 you will learn how to generate this sound). Again you will note that the triangle wave has the same pitch as $\sin(2\pi 440t)$,

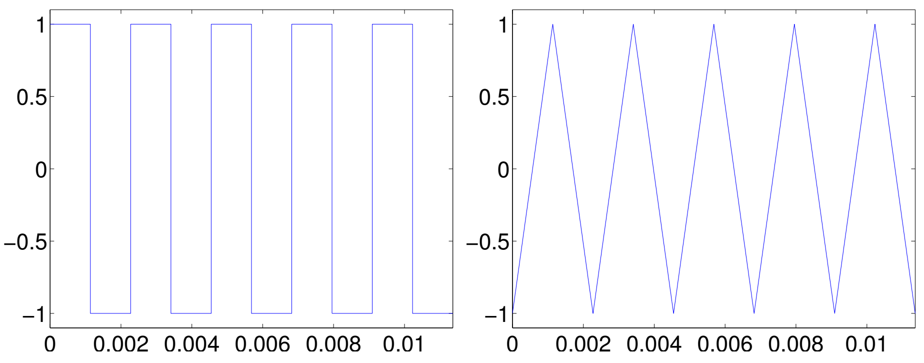

(a) The first five periods of the square wave (b) The first five periods of the triangle wave

Figure 1.4: The square wave and the triangle wave, two functions with regular oscillations, but which are not simple, trigonometric functions.

and is less pleasant to listen to than this pure tone. However, one can argue that it is somewhat more pleasant to listen to than a square wave. This will also be explained in terms of pollution with other frequencies later.

In Section 2.1 we will begin to peek behind the curtains as to why these waves sound so different, even though we recognize them as having the exact same pitch.

1.3 Digital sound

In the previous section we considered some basic properties of sound, but it was all in terms of functions defined for all time instances in some interval. On computers and various kinds of media players the sound is usually *digital* which means that the sound is represented by a large number of function values, and not by a function defined for all times in some interval.

Definition 1.13 (Digital sound)**.** A digital sound is a sequence $\boldsymbol{x} = \{x_i\}_{i=0}^{N}$ that corresponds to measurements of the air pressure of a sound f, recorded at a fixed rate of f_s (the sampling frequency or sampling rate) measurements per second, i.e.,

$$x_i = f(i/f_s), \quad \text{for } i = 0, 1; \ldots, N.$$

The measurements are often referred to as samples. The time between successive measurements is called the sampling period and is usually denoted T_s. If the sound is in stereo there will be two arrays $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, one for each channel. Measuring the sound is also referred to as sampling the sound, or analog to digital (AD) conversion.

In most cases, a digital sound is sampled from an analog (continuous) audio signal. This is usually done with a technique called Pulse Code Modulation (PCM). The audio signal is sampled at regular intervals and the sampled values stored in a suitable number format. Both the sampling frequency, and the accuracy and number format used for storing the samples, may vary for different kinds of audio, and both influence the quality of the resulting sound. For simplicity the quality is often measured by the number of bits per second, i.e., the product of the sampling rate and the number of bits (binary digits) used to store each sample. This is also referred to as the *bit rate*. For the computer to be able to play a digital sound, samples must be stored in a file or in memory on a computer. To do this efficiently, digital sound formats are used. A couple of them are described in the examples below.

In Exercise 4 you will be asked to implement a Matlab-function which plays a pure sound with a given frequency on your computer. For this you will need to know that for a pure tone with frequency f, you can obtain its samples over a period of 3 seconds with sampling rate f_s from the code

```
t=0:(1/fs):3;
sd=sin(2*pi*f*t);
```

Here the code is in MATLAB. MATLAB code will be displayed in this way throughout these notes.

Example 1.14. In the classical CD-format the audio signal is sampled 44 100 times per second and the samples stored as 16-bit integers. This works well for music with a reasonably uniform dynamic range, but is problematic when the range varies. Suppose for example that a piece of music has a very loud passage. In this passage the samples will typically make use of almost the full range of integer values, from $-2^{15}-1$ to 2^{15}. When the music enters a more quiet passage the sample values will necessarily become much smaller and perhaps only vary in the range -1000 to 1000, say. Since $2^{10} = 1024$ this means that in the quiet passage the music would only be represented with 10-bit samples. This problem can be avoided by using a floating-point format instead, but very few audio formats appear to do this.

The bit rate for CD-quality stereo sound is $44100 \times 2 \times 16$ bits/s $= 1411.2$ kb/s. This quality measure is particularly popular for lossy audio formats where the uncompressed audio usually is the same (CD-quality). However, it should be remembered that even two audio files in the same file format and with the same bit rate may be of very different quality because the encoding programs may be of different quality.

Example 1.15. For telephony it is common to sample the sound 8000 times per second and represent each sample value as a 13-bit integer. These integers are then converted to a kind of 8-bit floating-point format with a 4-bit significand. Telephony therefore generates a bit rate of 64 000 bits per second, i.e. 64 kb/s.

Newer formats with higher quality are available. Music is distributed in various formats on DVDs (DVD-video, DVD-audio, Super Audio CD) with sampling rates up to 192 000 and up to 24 bits per sample. These formats also support surround sound (up to seven channels in contrast to the two stereo channels on a CD). In the following we will assume all sound to be digital. Later we will return to how we reconstruct audible sound from digital sound.

1.4 Simple operations on digital sound

Simple operations and computations with digital sound can be done in any programming environment. Let us take a look at how this can be done. From Definition 1.13, digital sound is just an array of sample values $\boldsymbol{x} = (x_i)_{i=0}^{N-1}$, together with the sample rate f_s. Performing operations on the sound therefore amounts to doing the appropriate computations with the sample values and the sample rate. The most basic operation we can perform on a sound is simply playing it, and if we are working with sound we need a mechanism for doing this.

1.4.1 Playing a sound

You may already have listened to pure tones, square waves and triangle waves in the last section. The corresponding sound files were generated in a way we will describe shortly, placed in a directory available on the internet, and linked to from these notes. A program on your computer was able to play these files when you clicked on them. We will now describe how to use Matlab to play the same sounds. There we have the two functions

```
playblocking(playerobj)
playblocking(playerobj,[start stop])
```

These simply play an audio segment encapsulated by the object `playerobj` (we will shortly see how we can construct such an object from given audio samples and sampling rate). `playblocking` means that the method playing the sound will block until it has finished playing. We will have use for this functionality later on, since we may play sounds in successive order. With the first function the entire audio segment is played. With the second function the playback starts at sample `start`, and ends at sample `stop`. These functions are just software interfaces to the sound card in your computer. It basically sends the array of sound samples and sample rate to the sound card, which uses some method for reconstructing the sound to an analog sound signal. This analog signal is then sent to the loudspeakers and we hear the sound.

Fact 1.16. The basic command in a programming environment that handles sound takes as input an array of sound samples $\boldsymbol{x}$ and a sample rate s, and plays the corresponding sound through the computer's loudspeakers.

The mysterious `playerobj` object above can be obtained from the sound samples (represented by a vector `S`) and the sampling rate (`fs`) by the function:

```
playerobj=audioplayer(S,fs)
```

The sound samples can have different data types. We will always assume that they are of type `double`. MATLAB requires that they have values between -1 and 1 (i.e. these represent the range of numbers which can be played through the sound card of the computer). Also, `S` can actually be a matrix: Each column in the matrix represents a sound channel. Sounds we generate from a mathematical function on our own will typically have one only one channel, so that `S` has only one column. If `S` originates from a stereo sound file, it will have two columns.

You can create `S` on your own, and set the sampling rate to whatever value you like. However, we can also fill in the sound samples from a sound file. To do this from a file in the `wav`-format named `filename`, simply write

```
[S,fs]=wavread(filename)
```

The `wav`-format format was developed by Microsoft and IBM, and is one of the most common file formats for CD-quality audio. It uses a 32-bit integer to specify the file size at the beginning of the file which means that a WAV-file cannot be larger than 4 GB. In addition to filling in the sound samples in the vector `S`, this function also returns the sampling rate `fs` used in the file. The function

```
wavwrite(S,fs,filename)
```

can similarly be used to write the data stored in the vector `S` to the `wav`-file by the name `filename`. In the following we will both fill in the vector `S` on our own by using values from mathematical functions, as well as from a file. As an example of the first, we can listen to and write to a file the pure tone of frequency 440Hz considered above with the help of the following code:

```
antsec=3;
fs=40000;
t=linspace(0,antsec,fs*antsec);
S=sin(2*pi*440*t);
playerobj=audioplayer(S,fs);
playblocking(playerobj);
wavwrite(S,fs,'puretone440.wav');
```

The code creates a pure tone which lasts for three seconds (if you want the tone to last longer, you can change the value of the variable `antsec`). We also tell the computer that there are 40000 samples per second. This value is not coincidental, and we will return to this. In fact, the sound file for the pure tone embedded into this document was created in this way! In the same way we can listen to the square wave with the help of the following code:

```
antsec=3;
fs=44100;
```

```
samplesperperiod=round(fs/440);
oneperiod=[ones(1,round(samplesperperiod/2)) ...
           -ones(1,round(samplesperperiod/2))];
allsamples=zeros(1,antsec*440*length(oneperiod));
for k=1:(antsec*440)
  allsamples(((k-1)*length(oneperiod)+1):k*length(oneperiod))=oneperiod;
end
playerobj=audioplayer(allsamples,fs);
playblocking(playerobj);
```

The code creates 440 copies of the square wave per second by first computing the number of samples needed for one period when it is known that we should have a total of 40000 samples per second, and then constructing the samples needed for one period. In the same fashion we can listen to the triangle wave simply by replacing the code for generating the samples for one period with the following:

```
oneperiod=[linspace(-1,1,round(samplesperperiod/2)) ...
           linspace(1,-1,round(samplesperperiod/2))];
```

Instead of using the formula for the triangle wave, directly, we have used the function `linspace`.

As an example of how to fill in the sound samples from a file, the code

```
[S fs] = wavread('castanets.wav');
```

reads the file castanets.wav, and stores the sound samples in the matrix `S`. In this case there are two sound channels, so there are two columns in `S`. To work with sound from only one channel, we extract the second channel as follows:

```
x=S(:,2);
```

`wavread` returns sound samples with floating point precision. If we have made any changes to the sound samples, we need to secure that they are between -1 and 1 before we play them. If the sound samples are stored in `x`, this can be achieved as follows:

```
x = x / max(abs(x));
```

`x` can now be played just as the signals we constructed from mathematical formulas above.

It may be that some other environment than Matlab gives you the `play` functionality on your computer. Even if no environment on your computer supports such `play`-functionality at all, you may still be able to play the result of your computations if there is support for saving the sound in some standard format like mp3. The resulting file can then be played by the standard audio player on your computer.

Example 1.17 (Changing the sample rate)**.** We can easily play back a sound with a different sample rate than the standard one. If we in the code above instead wrote `fs=80000`, the sound card will assume that the time distance between neighbouring samples is half the time distance in the original. The result is that the sound takes half as long, and the frequency of all tones is doubled. For voices the result is a characteristic Donald Duck-like sound.

Conversely, the sound can be played with half the sample rate by setting `fs=20000`. Then the length of the sound is doubled and all frequencies are halved. This results in low pitch, roaring voices.

Fact 1.18. A digital sound can be played back with a double or half sample rate by replacing

```
playerobj=audioplayer(S,fs);
```

with

```
playerobj=audioplayer(S,2*fs);
```

and

```
playerobj=audioplayer(S,fs/2);
```

respectively.

The sample file `castanets.wav` played at double sampling rate sounds like this, while it sounds like this when it is played with half the sampling rate.

Example 1.19 (Playing the sound backwards)**.** At times a popular game has been to play music backwards to try and find secret messages. In the old days of analog music on vinyl this was not so easy, but with digital sound it is quite simple; we just need to reverse the samples. To do this we just loop through the array and put the last samples first.

Fact 1.20. Let $\boldsymbol{x} = (x_i)_{i=0}^{N-1}$ be the samples of a digital sound. Then the samples $\boldsymbol{y} = (y_i)_{i=0}^{N-1}$ of the reverse sound are given by

$$y_i = x_{N-i-1}, \text{ for } i = 0, 1, \dots N-1.$$

When we reverse the sound samples with Matlab, we have to reverse the elements in both sound channels. This can be performed as follows

```
sz=size(S,1);
newS=[S(sz:(-1):1,1) S(sz:(-1):1,2)];
```

Performing this on our sample file you generate a sound which sounds like this.

Example 1.21 (Adding noise)**.** To remove noise from recorded sound can be very challenging, but adding noise is simple. There are many kinds of noise, but one kind is easily obtained by adding random numbers to the samples of a sound.

Fact 1.22. Let $\boldsymbol{x}$ be the samples of a digital sound of length N. A new sound $\boldsymbol{y}$ with noise added can be obtained by adding a random number to each sample,

```
y=x+c*(2*rand(1,N)-1);
```

where `rand` is a MATLAB function that returns random numbers in the interval $[0,1]$, and c is a constant (usually smaller than 1) that dampens the noise. The effect of writing `(2*rand(1,N)-1)` is that random numbers between -1 and 1 are returned instead of random numbers between 0 and 1.

Adding noise in this way will produce a general hissing noise similar to the noise you hear on the radio when the reception is bad. As before you should add noise to both channels. Note alse that the sound samples may be outside $[-1,1]$ after adding noise, so that you should scale the samples before writing them to file. The factor c is important, if it is too large, the noise will simply drown the signal $\boldsymbol{y}$: `castanets.wav` with noise added with $c=0.4$ sounds like this, while with $c=0.1$ it sounds like this.

1.4.2 Filtering operations

Later on we will focus on particular operations on sound, where the output is constructed by combining several input elements in a particular way. We say that we *filter* the sound, and we call such operations *filtering operations*, or simply *filters*. Filters are important since they can change the frequency content in a signal in many ways. We will defer the precise definition of filters to Section 3.3, where we also will give the filters listed below a closer mathematical analysis.

Example 1.23 (Adding echo)**.** An echo is a copy of the sound that is delayed and softer than the original sound. We observe that the sample that comes m seconds before sample i has index $i-ms$ where s is the sample rate. This also makes sense even if m is not an integer so we can use this to produce delays that are less than one second. The one complication with this is that the number ms may not be an integer. We can get round this by rounding ms to the nearest integer which corresponds to adjusting the echo slightly.

Fact 1.24. Let $(\boldsymbol{x}, s)$ be a digital sound. Then the sound $\boldsymbol{y}$ with samples given by

```
y=x((d+1):N)-c*x(1:(N-d));
```

will include an echo of the original sound. Here `d=round(ms)` is the integer closest to ms, and c is a constant which is usually smaller than 1.

This is an example of a filtering operation where each output element is constructed from two input elements. As in the case of noise it is important to dampen the part that is added to the original sound, otherwise the echo will be too loud. Note also that the formula that creates the echo does not work at the beginning of the signal, and that the echo is unaudible if d is too small. You can listen to the sample file with echo added with $d = 10000$ and $c = 0.5$ here.

Example 1.25 (Reducing the treble)**.** The treble in a sound is generated by the fast oscillations (high frequencies) in the signal. If we want to reduce the treble we have to adjust the sample values in a way that reduces those fast oscillations. A general way of reducing variations in a sequence of numbers is to replace one number by the average of itself and its neighbours, and this is easily done with a digital sound signal. If we let the new sound signal be $\boldsymbol{y} = (y_i)_{i=0}^{N-1}$ we can compute it as

```
y(1)=x(1);
for t=2:(N-1)
  y(t)=(x(t-1)+x(t)+x(t+1))/3;
end
y(N)=x(N);
```

This is another example of a filtering operation, but this time three input elements are needed in order to produce an output element. Note that the vector $\{1/3, 1/3, 1/3\}$ uniquely describe how the input elements should be combined to produce the otuput. The elements in this vector are also referred to as the *filter coefficients*. Since this filter is based on forming averages it is also called a *moving average filter*.

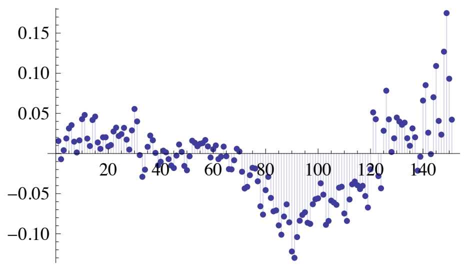
It is reasonable to let the middle sample x_i count more than the neighbours in the average, so an alternative is to compute the average by instead writing

```
  y(t)=(x(t-1)+2*x(t)+x(t+1))/4;
```

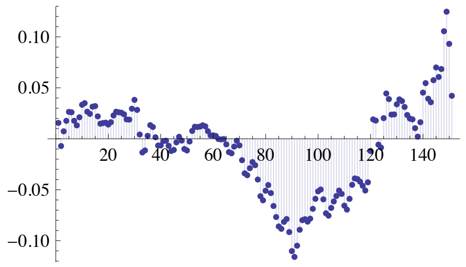
The coefficients $1, 2, 1$ here have been taken from row 2 in Pascal's triangle. It will turn out that this is a good choice of coefficients. We have not developed the tools needed to analyse the quality of filters yet, so this will be discussed later. We can also take averages of more numbers, where it will also turn out that row k of Pascals triangle also is a very good choice. The values in Pascals triangle can be computed as the coefficients of x in the expression $(1 + x)^k$, which also equal the binomial coefficients $\binom{k}{r}$ for $0 \leq r \leq k$. As an example, if we pick coefficients from row 4 of Pascals triangle instead, we would write

```
y(1)=x(1); y(2)=x(2);
for t=3:(N-2)
  y(t)=(x(t-2)+4*x(t-1)+6*x(t)+4*x(t+1)+x(t+2))/16;
end
y(N-1)=x(N-1); y(N)=x(N);
```

It will turn out that picking coefficients from a row in Pascal's triangle works better the longer the filter is:



(a) The original sound signal



(b) The result of applying the filter from row 4 of Pascal's triangle

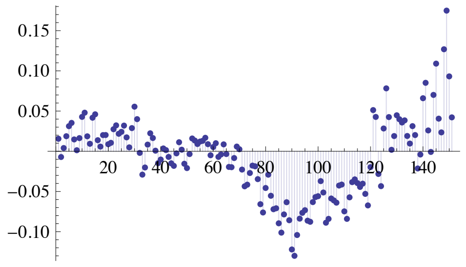
Figure 1.5: Reducing the treble.

Observation 1.26. Let $\boldsymbol{x}$ be the samples of a digital sound, and let $\{c_i\}_{i=1}^{2k+1}$ be the numbers in row $2k$ of Pascal's triangle. Then the sound with samples $\boldsymbol{y}$ given by

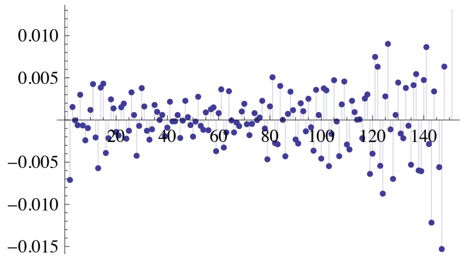
```
y=zeros(length(x));
y(1:k)=x(1:k);
for t=(k+1):(N-k)
  for j=1:(2*k+1)
    y(t)=y(t)+c(j)*x(t+k+1-j))/2^k;
  end
end
y((N-k+1):N)=x((N-k+1):N);
```

has reduced treble compared with the sound given by the samples $\boldsymbol{x}$.

An example of the result of averaging is shown in Figure 1.5. (a) shows a real sound sampled at CD-quality (44 100 samples per second). (b) shows the result of applying the averaging process by using row 4 of Pascals triangle. We see that the oscillations have been reduced, and if we play the sound it has considerably less treble. In Exercise 9 you will be asked to implement reducing the treble in the file `castanets.wav`. If you do this you should hear that the sound gets softer when you increase k: For $k = 32$ the sound will be like this, for $k = 256$



(a) The original sound signal



(b) The result of applying the filter deduced from row 4 in Pascals triangle

Figure 1.6: Reducing the bass.

it will be like this.

Example 1.27 (Reducing the bass)**.** Another common option in an audio system is reducing the bass. This corresponds to reducing the low frequencies in the sound, or equivalently, the slow variations in the sample values. It turns out that this can be accomplished by simply changing the sign of the coefficients used for reducing the treble. We can for instance change the filter described for the fourth row in Pascals triangle to

```
y(t)=(x(t-2)-4*x(t-1)+6*x(t)-4*x(t+1)+x(t+2))/16;
```

An example is shown in Figure 1.6. The original signal is shown in (a) and the result in (b). We observe that the samples in (b) oscillate much more than the samples in (a). If we play the sound in (b), it is quite obvious that the bass has disappeared almost completely.

Observation 1.28. Let $\boldsymbol{x}$ be the samples of a digital sound, and let $\{c_i\}_{i=1}^{2k+1}$ be the numbers in row $2k$ of Pascal's triangle. Then the sound with samples $\boldsymbol{y}$ given by

```
y=zeros(length(x));
y(1:k)=x(1:k);
for t=(k+1):(N-k)
```

```
  for j=1:(2*k+1)
    y(t)=x(t)+(-1)^(k+1-j)*c(j)*x(t+j-k-1))/2^k;
  end
end
y((N-k+1):N)=x((N-k+1):N);
```

has reduced bass compared to the sound given by the samples $\boldsymbol{y}$.

In Exercise 9 you will be asked to implement reducing the bass in the file `castanets.wav`. The new sound will be difficult to hear for large k, and we will explain why later. For $k = 1$ the sound will be like this, for $k = 2$ it will be like this. Even if the sound is quite low, you can hear that more of the bass has disappeared for $k = 2$.

There are also other operations we would like to perform for digital sound. For instance, it would be nice to adjust a specific set of frequencies only, so that we end up with a sound where unpleasant components of the sound have been removed. Later on we will establish mathematics which will enable us to contruct filters which have the properties that they work in desirable ways on any frequencies. It will also turn out that the filters listed above can be given a frequency interpretation: When the input sound has a given frequncy, the output sound has the same frequency.

Exercises for Section 1.4

Ex. 1 — Compute the loudness of the Krakatoa explosion on the decibel scale, assuming that the variation in air pressure peaked at 100 000 Pa.

Ex. 2 — Define the following sound signal

$$f(t) = \begin{cases} 0 & 0 \le t \le 0.2 \\ \frac{t-0.2}{0.4}\sin(2\pi 440 t) & 0.2 \le t \le 0.6 \\ \sin(2\pi 440 t) & 0.6 \le t \le 1 \end{cases}$$

This corresponds to the sound plotted in Figure 1.1(a), where the sound is unaudible in the beginning, and increases linearly in loudness over time with a given frequency until maximum loudness is avchieved. Write a Matlab program which generates this sound, and listen to it.

Ex. 3 — Find two constant a and b so that the function $f(t) = a\sin(2\pi 440 t) + b\sin(2\pi 4400 t)$ resembles the plot from Figure 1.1(b) as closely as possible. Generate the samples of this sound, and listen to it with Matlab.

Ex. 4 — Let us write some code so that we can experiment with different pure sounds

a. Write a function

```
function playpuresound(f)
```

which generates the samples over a period of 3 seconds for a pure tone with frequency f, with sampling frequency $f_s = 2.5f$ (we will explain this value later).

b. Use the function `playpuresound` to listen to pure sounds of frequency 440Hz and 1500Hz, and verify that they are the same as the sounds you already have listened to in this section.

c. How high frequencies are you able to hear with the function `playpuresound`? How low frequencies are you able to hear?

Ex. 5 — Write functions

```
function playsquare(T)
function playtriangle(T)
```

which plays the square wave of Example 1.11 and the triangle wave of Example 1.12, respectively, where T is given by the parameter. In your code, let the samples of the waves be taken at a frequency of 40000 samples per second. Verify that you generate the same sounds as you played in these examples when you set $T = \frac{1}{440}$.

Ex. 6 — In this exercise we will experiment as in the first examples of this section.

a. Write a function

```
function playdifferentfs()
```

which plays the sound samples of `castanets.wav` with the same sample rate as the original file, then with twice the sample rate, and then half the sample rate. You should start with reading the file into a matrix (as explained in this section). Are the sounds the same as those you heard in Example 1.17?

b. Write a function

```
function playreverse()
```

which plays the sound samples of `castanets.wav` backwards. Is the sound the same as the one you heard in Example 1.19?

c. Write the new sound samples from b. to a new `wav`-file, as described above, and listen to it with your favourite mediaplayer.

Ex. 7 — In this exercise, we will experiment with adding noise to a signal.

a. Write a function

```
function playnoise(c)
```

which plays the sound samples of `castanets.wav` with noise added for the damping constant c as described above. Your code should add noise to both channels of the sound, and scale the sound samples so that they are between -1 and 1.

b. With your program, generate the two sounds played in Example 1.21, and verify that they are the same as those you heard.

c. Listen to the sound samples with noise added for different values of c. For which range of c is the noise audible?

Ex. 8 — In this exercise, we will experiment with adding echo to a signal.

a. Write a function

```
function playwithecho(c,d)
```

which plays the sound samples of `castanets.wav` with echo added for damping constant c and delay d as described in Example 1.23.

b. Generate the sound from Example 1.23, and verify that it is the same as the one you heard there.

c. Listen to the sound samples for different values of d and c. For which range of d is the echo distinguisible from the sound itself? How low can you choose c in order to still hear the echo?

Ex. 9 — In this exercise, we will experiment with increasing and reducing the treble and bass in a signal as in examples 1.25 and 1.27.

a. Write functions

```
function reducetreble(k)
function reducebass(k)
```

which reduces bass and treble in the ways described above for the sound from the file `castanets.wav`, and plays the result, when row number $2k$ in Pascal' triangle is used to construct the filters. Look into the Matlab function `conv` to help you to find the values in Pascal's triangle.

b. Generate the sounds you heard in examples 1.25 and 1.27, and verify that they are the same.

c. In your code, it will not be necessary to scale the values after reducing the treble, i.e. the values are already between -1 and 1. Explain why this is the case.

d. How high must k be in order for you to hear difference from the actual sound? How high can you choose k and still recognize the sound at all?

1.5 Compression of sound and the MP3 standard

Digital audio first became commonly available when the CD was introduced in the early 1980s. As the storage capacity and processing speeds of computers increased, it became possible to transfer audio files to computers and both play and manipulate the data, in ways such as in the previous section. However, audio was represented by a large amount of data and an obvious challenge was how to reduce the storage requirements. Lossless coding techniques like Huffman and Lempel-Ziv coding were known and with these kinds of techniques the file size could be reduced to about half of that required by the CD format. However, by allowing the data to be altered a little bit it turned out that it was possible to reduce the file size down to about ten percent of the CD format, without much loss in quality. The MP3 audio format takes advantage of this.

MP3, or more precisely *MPEG-1 Audio Layer 3*, is part of an audio-visual standard called MPEG. MPEG has evolved over the years, from MPEG-1 to MPEG-2, and then to MPEG-4. The data on a DVD disc can be stored with either MPEG-1 or MPEG-2, while the data on a bluray-disc can be stored with either MPEG-2 or MPEG-4. MP3 was developed by Philips, CCETT (Centre commun d'études de télévision et télécommunications), IRT (Institut für Rundfunktechnik) and Fraunhofer Society, and became an international standard in 1991. Virtually all audio software and music players support this format. MP3 is just a sound format and does not specify the details of how the encoding should be done. As a consequence there are many different MP3 encoders available, of varying quality. In particular, an encoder which works well for higher bit rates (high quality sound) may not work so well for lower bit rates.

With MP3, the sound samples are transformed using methods we will go through in the next section. A frequency analysis of the sound is the basis for this transformation. Based on this frequency analysis, the sound is split into frequency bands, each band corresponding to a particular frequency range. With MP3, 32 frequency bands are used. Based on the frequency analysis, the encoder uses what is called a *psycho-acoustic model* to compute the significance of each band for the human perception of the sound. When we hear a sound, there is a mechanical stimulation of the ear drum, and the amount of stimulus is directly related to the size of the sample values of the digital sound. The movement of the ear drum is then converted to electric impulses that travel to the brain where they are perceived as sound. The perception process uses a transformation of the sound so that a steady oscillation in air pressure is perceived as a sound with a fixed frequency. In this process certain kinds of perturbations of the sound are hardly noticed by the brain, and this is exploited in lossy audio compression.

More precisely, when the psycho-acoustic model is applied to the frequency content resulting from our frequency analysis, *scale factors* and *masking thresh-*

olds are assigned for each band. The computed masking thresholds have to do with a phenomenon called *masking effects*. A simple example of this is that a loud sound will make a simultaneous low sound inaudible. For compression this means that if certain frequencies of a signal are very prominent, most of the other frequencies can be removed, even when they are quite large. If the sounds are below the masking threshold, it is simply ommited by the encoder, since the model says that the sound should be inaudible.

Masking effect is just one example of what is called a psycho-acoustic effect. Another obvious such effect regards computing the scale factors: the human auditory system can only perceive frequencies in the range 20 Hz – 20 000 Hz. An obvious way to do compression is therefore to remove frequencies outside this range, although there are indications that these frequencies may influence the listening experience inaudibly. The computed scaling factors tell the encoder about the precision to be used for each frequency band: If the model decides that one band is very important for our perception of the sound, it assigns a big scale factor to it, so that more effort is put into encoding it by the encoder (i.e. it uses more bits to encode this band).

Using appropriate scale factors and masking thresholds provide compression, since bits used to encode the sound are spent on parts important for our perception. Developing a useful psycho-acoustic model requires detailed knowledge of human perception of sound. Different MP3 encoders use different such models, so that may produce very different results, worse or better.

The information remaining after frequency analysis and using a psycho-acoustic model is coded efficiently with (a variant of) Huffman coding. MP3 supports bit rates from 32 to 320 kb/s and the sampling rates 32, 44.1, and 48 kHz. The format also supports variable bit rates (the bit rate varies in different parts of the file). An MP3 encoder also stores metadata about the sound, such as the title of the audio piece, album and artist name and other relevant data.

MP3 too has evolved in the same way as MPEG, from MP1 to MP2, and to MP3, each one more sophisticated than the other, providing better compression. MP3 is not the latest development of audio coding in the MPEG family: AAC (Advanced Audio Coding) is presented as the successor of MP3 by its principal developer, Fraunhofer Society, and can achieve better quality than MP3 at the same bit rate, particularly for bit rates below 192 kb/s. AAC became well known in April 2003 when Apple introduced this format (at 128 kb/s) as the standard format for their iTunes Music Store and iPod music players. AAC is also supported by many other music players, including the most popular mobile phones.

The technologies behind AAC and MP3 are very similar. AAC supports more sample rates (from 8 kHz to 96 kHz) and up to 48 channels. AAC uses the same transformation as MP3, but AAC processes 1 024 samples at a time. AAC also uses much more sophisticated processing of frequencies above 16 kHz and has a number of other enhancements over MP3. AAC, as MP3, uses Huffman coding for efficient coding of the transformed values. Tests seem quite conclusive that AAC is better than MP3 for low bit rates (typically below 192 kb/s), but for higher rates it is not so easy to differentiate between the two formats. As for MP3 (and the other formats mentioned here), the quality of an AAC file depends crucially on the quality of the encoding program.

There are a number of variants of AAC, in particular AAC Low Delay (AAC-LD). This format was designed for use in two-way communication over a network, for example the internet. For this kind of application, the encoding (and decoding) must be fast to avoid delays (a delay of at most 20 ms can be tolerated).

1.6 Summary

We discussed the basic question of what is sound is, and concluded that sound could be modeled as a sum of frequency components. We discussed meaningful operations of sound, such as adjust the bass and treble, adding echo, or adding noise. We also gave an introduction to the MP3 standard for compression of sound.

Chapter 2

Fourier analysis for periodic functions: Fourier series

In Chapter 1 we identified audio signals with functions and discussed informally the idea of decomposing a sound into basis sounds to make its frequency content available. In this chapter we will make this kind of decomposition precise by discussing how a given function can be expressed in terms of the basic trigonometric functions. This is similar to Taylor series where functions are approximated by combinations of polynomials. But it is also different from Taylor series because polynomials are different from polynomials, and the approximations are computed in a very different way. The theory of approximation of functions with trigonometric functions is generally referred to as *Fourier analysis*. This is a central tool in practical fields like image and signal processing, but it also an important field of research within pure mathematics. We will only discuss Fourier analysis for functions defined on a finite interval and for finite sequences (vectors), but Fourier analysis may also be applied to functions defined on the whole real line and to infinite sequences.

Perhaps a bit surprising, linear algebra is a very useful tool in Fourier analysis. This is because the sets of functions involved are vector spaces, both of finite and infinite dimension. Therefore many of the tools from your linear algebra course will be useful, in a situation that at first may seem far from matrices and vectors.

2.1 Basic concepts

The basic idea of Fourier series is to approximate a given function by a combination of simple cos and sin functions. This means that we have to address at least three questions:

1. How general do we allow the given function to be?

2. What exactly are the combinations of cos and sin that we use for the approximations?
3. How do we determine the approximation?

Each of these questions will be answered in this section.

We have already indicated that the functions we consider are defined on an interval, and without much loss of generality we assume this interval to be $[0, T]$, where T is some positive number. Note that any function f defined on $[0, T]$ gives rise to a related function defined on the whole real line, by simply gluing together copies of f. The result is a periodic function with period T that agrees with f on $[0, T]$.

We have to make some more restrictions. Mostly we will assume that f is continuous, but the theory can also be extended to functions which are only Riemann-integrable, more precisely, that the square of the function is integrable.

Definition 2.1 (Continuous and square-integrable functions)**.** The set of continuous, real functions defined on an interval $[0, T]$ is denoted $C[0, T]$.

A real function f defined on $[0, T]$ is said to be square integrable if f^2 is Riemann-integrable, i.e., if the Riemann integral of f^2 on $[0, T]$ exists,

$$\int_0^T f(t)^2\, dt < \infty.$$

The set of all square integrable functions on $[0, T]$ is denoted $L^2[0, T]$.

The sets of continuous and square-integrable functions can be equippped with an inner-product, a generalisation of the so-called dot-product for vectors.

Theorem 2.2. Both $L^2[0, T]$ and $C[0, T]$ are vector spaces. Moreover, if the two functions f and g lie in $L^2[0, T]$ (or in $C[0, T]$), then the product fg is also in $L^2[0, T]$ (or in $C[0, T]$). Moreover, both spaces are inner product spaces[1], with inner product[2] defined by

$$\langle f, g\rangle = \frac{1}{T}\int_0^T f(t)g(t)\, dt, \tag{2.1}$$

and associated norm

$$\|f\| = \sqrt{\frac{1}{T}\int_0^T f(t)^2 dt}. \tag{2.2}$$

The mysterious factor $1/T$ is included so that the constant function $f(t) = 1$ has norm 1, i.e., its role is as a normalizing factor.

Definition 2.1 and Theorem 2.2 answer the first question above, namely how general do we allow our functions to be. Theorem 2.2 also gives an indication of how we are going to determine approximations—we are going to use inner products. We recall from linear algebra that the projection of a function f onto a subspace W with respect to an inner product $\langle \cdot, \cdot \rangle$ is the function $g \in W$ which minimizes $\|f - g\|$, which we recognise as the error[3]. This projection is therefore also called a best approximation of f from W and is characterised by the fact that the error should be orthogonal to the subspace W, i.e., we should have

$$\langle f, g \rangle = 0, \quad \text{for all } g \in W.$$

More precisely, if $\boldsymbol{\phi} = \{\phi_i\}_{i=1}^m$ is an orthogonal basis for W, then the best approximation g is given by

$$g = \sum_{i=1}^{m} \frac{\langle f, \phi_i \rangle}{\langle \phi_i, \phi_i \rangle} \phi_i. \tag{2.3}$$

The error $\|f - g\|$ in the approximation is often referred to as the *least square error*.

We have now answered the second of our primary questions. What is left is a description of the subspace W of trigonometric functions. This space is spanned by the pure tones we discussed in Chapter 1.

Definition 2.3 (Fourier series)**.** Let $V_{N,T}$ be the subspace of $C[0, T]$ spanned by the set of functions given by

$$\begin{aligned} \mathcal{D}_{N,T} = \{&1, \cos(2\pi t/T), \cos(2\pi 2t/T), \cdots, \cos(2\pi N t/T), \\ &\sin(2\pi t/T), \sin(2\pi 2t/T), \cdots, \sin(2\pi N t/T)\}. \end{aligned} \tag{2.4}$$

The space $V_{N,T}$ is called the N'th order Fourier space. The Nth-order Fourier series approximation of f, denoted f_N, is defined as the best approximation of f from $V_{N,T}$ with respect to the inner product defined by (2.1).

The space $V_{N,T}$ can be thought of as the space spanned by the pure tones of frequencies $1/T$, $2/T$, ..., N/T, and the Fourier series can be thought of as linear combination of all these pure tones. From our discussion in Chapter 1, we see that if N is sufficiently large, we get a space which can be used to approximate most sounds in real life. The approximation f_N of a sound f from a space $V_{N,T}$ can also serve as a compressed version if many of the coefficients can be set to 0 without the error becomingg too big.

Note that all the functions in the set $\mathcal{D}_{N,T}$ are periodic with period T, but most have an even shorter period. More precisely, $\cos(2\pi n t/T)$ has period T/n, and frequency n/T. In general, the term *fundamental frequency* is used to denote the lowest frequency of a given periodic function.

Definition 2.3 characterises the Fourier series. The next lemma gives precise expressions for the coefficients.

[3]See Section 6.3 in [7] for a review of projections and least squares approximations.

Theorem 2.4. The set $\mathcal{D}_{N,T}$ is an orthogonal basis for $V_{N,T}$. In particular, the dimension of $V_{N,T}$ is $2N+1$, and if f is a function in $L^2[0,T]$, we denote by $a_0,\ldots,a_N$ and $b_1,\ldots,b_N$ the coordinates of f_N in the basis $\mathcal{D}_{N,T}$, i.e.

$$f_N(t) = a_0 + \sum_{n=1}^{N} \left(a_n \cos(2\pi nt/T) + b_n \sin(2\pi nt/T)\right). \tag{2.5}$$

The $a_0,\ldots,a_N$ and $b_1,\ldots,b_N$ are called the (real) Fourier coefficients of f, and they are given by

$$a_0 = \langle f, 1\rangle = \frac{1}{T}\int_0^T f(t)\,dt, \tag{2.6}$$

$$a_n = 2\big\langle f, \cos(2\pi nt/T)\big\rangle = \frac{2}{T}\int_0^T f(t)\cos(2\pi nt/T)\,dt \quad \text{for } n \ge 1, \tag{2.7}$$

$$b_n = 2\langle f, \sin(2\pi nt/T)\rangle = \frac{2}{T}\int_0^T f(t)\sin(2\pi nt/T)\,dt \quad \text{for } n \ge 1. \tag{2.8}$$

Proof. To prove orthogonality, assume first that $m \neq n$. We compute the inner product

$$\begin{aligned}
&\langle \cos(2\pi mt/T), \cos(2\pi nt/T)\rangle \\
&= \frac{1}{T}\int_0^T \cos(2\pi mt/T)\cos(2\pi nt/T)dt \\
&= \frac{1}{2T}\int_0^T (\cos(2\pi mt/T + 2\pi nt/T) + \cos(2\pi mt/T - 2\pi nt/T)) \\
&= \frac{1}{2T}\left[\frac{T}{2\pi(m+n)}\sin(2\pi(m+n)t/T) + \frac{T}{2\pi(m-n)}\sin(2\pi(m-n)t/T)\right]_0^T \\
&= 0.
\end{aligned}$$

Here we have added the two identities $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$ together to obtain an expression for $\cos(2\pi mt/T)\cos(2\pi nt/T)dt$ in terms of $\cos(2\pi mt/T + 2\pi nt/T)$ and $\cos(2\pi mt/T - 2\pi nt/T)$. By testing all other combinations of sin and cos also, we obtain the orthogonality of all functions in $\mathcal{D}_{N,T}$ in the same way.

We find the expressions for the Fourier coefficients from the general formula (2.3). We first need to compute the following inner products of the basis functions,

$$\begin{aligned}
\langle \cos(2\pi mt/T), \cos(2\pi mt/T)\rangle &= \frac{1}{2} \\
\langle \sin(2\pi mt/T), \sin(2\pi mt/T)\rangle &= \frac{1}{2} \\
\langle 1, 1\rangle &= 1,
\end{aligned}$$

which are easily derived in the same way as above. The orthogonal decomposition theorem (2.3) now gives

$$
\begin{aligned}
f_N(t) &= \frac{\langle f, 1\rangle}{\langle 1, 1\rangle}1 + + \sum_{n=1}^{N} \frac{\langle f, \cos(2\pi nt/T)\rangle}{\langle \cos(2\pi nt/T), \cos(2\pi nt/T)\rangle}\cos(2\pi nt/T) \\
&+ \sum_{n=1}^{N} \frac{\langle f, \sin(2\pi nt/T)\rangle}{\langle \sin(2\pi nt/T), \sin(2\pi nt/T)\rangle}\sin(2\pi nt/T) \\
&= \frac{\frac{1}{T}\int_0^T f(t)dt}{1} + \sum_{n=1}^{N} \frac{\frac{1}{T}\int_0^T f(t)\cos(2\pi nt/T)dt}{\frac{1}{2}}\cos(2\pi nt/T) \\
&+ \sum_{n=1}^{N} \frac{\frac{1}{T}\int_0^T f(t)\sin(2\pi nt/T)dt}{\frac{1}{2}}\sin(2\pi nt/T) \\
&= \frac{1}{T}\int_0^T f(t)dt + \sum_{n=1}^{N}\left(\frac{2}{T}\int_0^T f(t)\cos(2\pi nt/T)dt\right)\cos(2\pi nt/T) \\
&+ \sum_{n=1}^{N}\left(\frac{2}{T}\int_0^T f(t)\sin(2\pi nt/T)dt\right)\sin(2\pi nt/T).
\end{aligned}
$$

The relations (2.6)- (2.8) now follow by comparison with (2.5). □

Since f is a function in time, and the a_n, b_n represent contributions from different frequencies, the Fourier series can be thought of as a change of coordinates, from what we vaguely can call the *time domain*, to what we can call the *frequency domain* (or *Fourier domain*). We will call the basis $\mathcal{D}_{N,T}$ the *N'th order Fourier basis* for $V_{N,T}$. We note that $\mathcal{D}_{N,T}$ is not an orthonormal basis; it is only orthogonal.

In the signal processing literature, Equation (2.5) is known as *the synthesis equation*, since the original function f is synthesized as a sum of trigonometric functions. Similarly, equations (2.6)- (2.8) are called *analysis equations.*

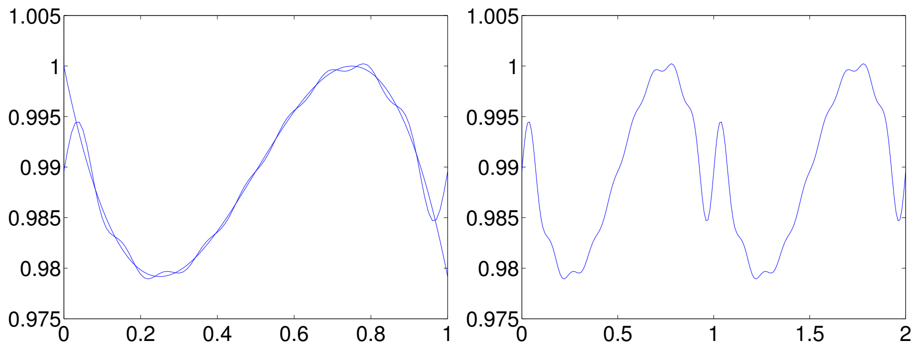
A major topic in harmonic analysis is to state conditions on f which guarantees the convergence of its Fourier series. We will not discuss this in detail here, since it turns out that, by choosing N large enough, any reasonable periodic function can be approximated arbitrarily well by its Nth-order Fourier series approximation. More precisely, we have the following result for the convergence of the Fourier series, stated without proof.

Theorem 2.5 (Convergence of Fourier series)**.** Suppose that f is periodic with period T, and that

1. f has a finite set of discontinuities in each period.

2. f contains a finite set of maxima and minima in each period.

3. $\int_0^T |f(t)|dt < \infty$.



(a) The function and its Fourier series (b) The Fourier series on a larger interval

Figure 2.1: The cubic polynomial $f(x) = -\frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{3}{16}x + 1$ on the interval $[0, 1]$, together with its Fourier series approximation from $V_{9,1}$.

Then we have that $\lim_{N\to\infty} f_N(t) = f(t)$ for all t, except at those points t where f is not continuous.

The conditions in Theorem 2.5 are called the Dirichlet conditions for the convergence of the Fourier series. They are just one example of conditions that ensure the convergence of the Fourier series. There also exist much more general conditions that secure convergence — these can require deep mathematical theory, depending on the generality.

An illustration of Theorem 2.5 is shown in Figure 2.1 where the cubic polynomial $f(x) = -\frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{3}{16}x + 1$ is approximated by a 9th order Fourier series. The trigonometric approximation is periodic with period 1 so the approximation becomes poor at the ends of the interval since the cubic polynomial is not periodic. The approximation is plotted on a larger interval in Figure 2.1(b), where its periodicity is clearly visible.

Example 2.6. Let us compute the Fourier coefficients of the square wave, as defined by (1.1) in Example 1.11. If we first use (2.6) we obtain

$$a_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{T}\int_0^{T/2} dt - \frac{1}{T}\int_{T/2}^T dt = 0.$$

Using (2.7) we get

$$\begin{aligned}
a_n &= \frac{2}{T}\int_0^T f(t)\cos(2\pi nt/T)dt \\
&= \frac{2}{T}\int_0^{T/2}\cos(2\pi nt/T)dt - \frac{2}{T}\int_{T/2}^{T}\cos(2\pi nt/T)dt \\
&= \frac{2}{T}\left[\frac{T}{2\pi n}\sin(2\pi nt/T)\right]_0^{T/2} - \frac{2}{T}\left[\frac{T}{2\pi n}\sin(2\pi nt/T)\right]_{T/2}^{T} \\
&= \frac{2}{T}\frac{T}{2\pi n}((\sin(n\pi)-\sin 0)-(\sin(2n\pi)-\sin(n\pi)) = 0.
\end{aligned}$$

Finally, using (2.8) we obtain

$$\begin{aligned}
b_n &= \frac{2}{T}\int_0^T f(t)\sin(2\pi nt/T)dt \\
&= \frac{2}{T}\int_0^{T/2}\sin(2\pi nt/T)dt - \frac{2}{T}\int_{T/2}^{T}\sin(2\pi nt/T)dt \\
&= \frac{2}{T}\left[-\frac{T}{2\pi n}\cos(2\pi nt/T)\right]_0^{T/2} + \frac{2}{T}\left[\frac{T}{2\pi n}\cos(2\pi nt/T)\right]_{T/2}^{T} \\
&= \frac{2}{T}\frac{T}{2\pi n}((-\cos(n\pi)+\cos 0)+(\cos(2n\pi)-\cos(n\pi))) \\
&= \frac{2(1-\cos(n\pi)}{n\pi} \\
&= \begin{cases} 0, & \text{if } n \text{ is even;} \\ 4/(n\pi), & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

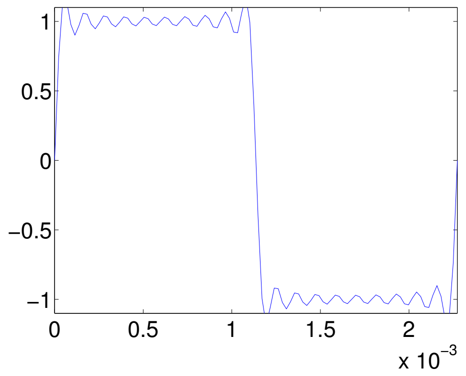
In other words, only the b_n-coefficients with n odd in the Fourier series are nonzero. From this it is clear that the Fourier series is

$$\frac{4}{\pi}\sin(2\pi t/T) + \frac{4}{3\pi}\sin(2\pi 3t/T) + \frac{4}{5\pi}\sin(2\pi 5t/T) + \frac{4}{7\pi}\sin(2\pi 7t/T) + \cdots .$$

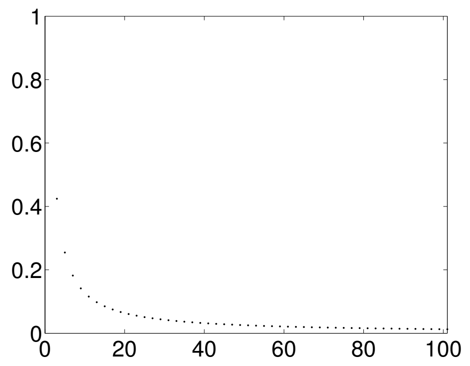
With $N = 20$, there are 10 trigonometric terms in this sum. The corresponding Fourier series can be plotted on the same interval with the following code.

```
t=0:(1/fs):3;
y=zeros(1,length(t));
for n=1:2:19
  y = y + (4/(n*pi))*sin(2*pi*n*t/T);
end
plot(t,y)
```

In Figure 2.2(a) we have plotted the Fourier series of the square wave when $T = 1/440$, and when $N = 20$. In Figure 2.2(b) we have also plotted the values



(a) The Fourier series for $N = 20$



(b) Values of the first 100 Fourier coefficients b_n

Figure 2.2: The Fourier series of the square wave of Example 2.6

of the first 100 Fourier coefficients b_n, to see that they actually converge to zero. This is clearly necessary in order for the Fourier series to converge.

Even though f oscillates regularly between -1 and 1 with period T, the discontinuities mean that it is far from the simple $\sin(2\pi t/T)$ which corresponds to a pure tone of frequency $1/T$. From Figure 2.2(b) we see that the dominant coefficient in the Fourier series is b_1, which tells us how much there is of the pure tone $\sin(2\pi t/T)$ in the square wave. This is not surprising since the square wave oscillates T times every second as well, but the additional nonzero coefficients pollute the pure sound. As we include more and more of these coefficients, we gradually approach the square wave, as shown for $N = 20$.

There is a connection between how fast the Fourier coefficients go to zero, and how we percieve the sound. A pure sine sound has only one nonzero coefficient, while the square wave Fourier coefficents decrease as $1/n$, making the sound less pleasant. This explains what we heard when we listened to the sound in Example 1.11. Also, it explains why we heard the same pitch as the pure tone, since the first frequency in the Fourier series has the same frequency as the pure tone we listened to, and since this had the highest value.

The Fourier series approximations of the square wave can be played with the `play` function, just as the square wave itself. For $N = 1$ and with $T = 1/440$ as above, it sounds like this. This sounds exactly like the pure sound with frequency 440Hz, as noted above. For $N = 5$ the Fourier series approximation sounds like this, and for $N = 9$ it sounds like this. Indeed these sounds are more like the square wave itself, and as we increase N we can hear how introduction of more frequencies gradually pollutes the sound more and more. In Exercise 7 you will be asked to write a program which verifies this.

Example 2.7. Let us also compute the Fourier coefficients of the triangle wave, as defined by (1.2) in Example 1.12. We now have

$$a_0 = \frac{1}{T}\int_0^{T/2} \frac{4}{T}\left(t - \frac{T}{4}\right) dt + \frac{1}{T}\int_{T/2}^{T} \frac{4}{T}\left(\frac{3T}{4} - t\right) dt.$$

Instead of computing this directly, it is quicker to see geometrically that the graph of f has as much area above as below the x-axis, so that this integral must be zero. Similarly, since f is symmetric about the midpoint $T/2$, and $\sin(2\pi nt/T)$ is antisymmetric about $T/2$, we have that $f(t)\sin(2\pi nt/T)$ also is antisymmetric about $T/2$, so that

$$\int_0^{T/2} f(t)\sin(2\pi nt/T)dt = -\int_{T/2}^{T} f(t)\sin(2\pi nt/T)dt.$$

This means that, for $n \geq 1$,

$$b_n = \frac{2}{T}\int_0^{T/2} f(t)\sin(2\pi nt/T)dt + \frac{2}{T}\int_{T/2}^{T} f(t)\sin(2\pi nt/T)dt = 0.$$

For the final coefficients, since both f and $\cos(2\pi nt/T)$ are symmetric about $T/2$, we get for $n \geq 1$,

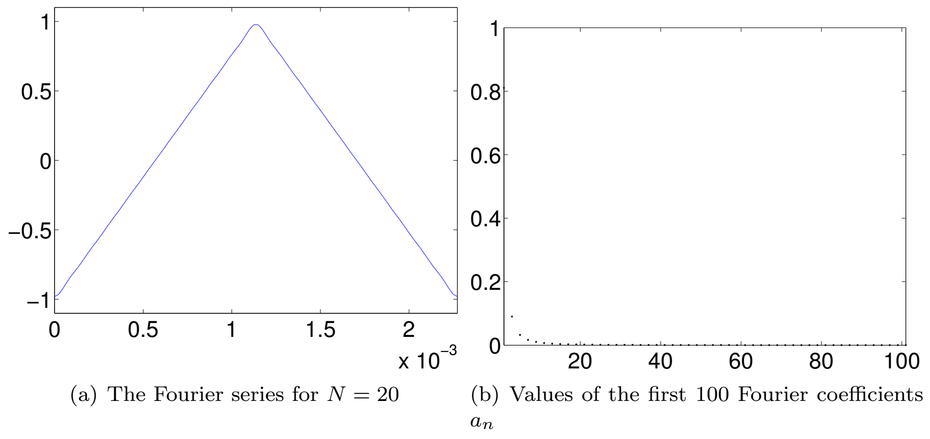
$$\begin{aligned}
a_n &= \frac{2}{T}\int_0^{T/2} f(t)\cos(2\pi nt/T)dt + \frac{2}{T}\int_{T/2}^{T} f(t)\cos(2\pi nt/T)dt \\
&= \frac{4}{T}\int_0^{T/2} f(t)\cos(2\pi nt/T)dt = \frac{4}{T}\int_0^{T/2} \frac{4}{T}\left(t - \frac{T}{4}\right)\cos(2\pi nt/T)dt \\
&= \frac{16}{T^2}\int_0^{T/2} t\cos(2\pi nt/T)dt - \frac{4}{T}\int_0^{T/2} \cos(2\pi nt/T)dt \\
&= \frac{4}{n^2\pi^2}(\cos(n\pi) - 1) \\
&= \begin{cases} 0, & \text{if } n \text{ is even;} \\ -8/(n^2\pi^2), & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

where we have dropped the final tedious calculations (use integration by parts). From this it is clear that the Fourier series of the triangle wave is

$$-\frac{8}{\pi^2}\cos(2\pi t/T) - \frac{8}{3^2\pi^2}\cos(2\pi 3t/T) - \frac{8}{5^2\pi^2}\cos(2\pi 5t/T) - \frac{8}{7^2\pi^2}\cos(2\pi 7t/T) + \cdots.$$

In Figure 2.3 we have repeated the plots used for the square wave, for the triangle wave. As before, we have used $T = 1/440$. The figure clearly shows that the Fourier series coefficients decay much faster.

We can play different Fourier series approximations of the triangle wave, just as those for the square wave. For $N = 1$ and with $T = 1/440$ as above, it sounds like this. Again, this sounds exactly like the pure sound with frequency 440Hz. For $N = 5$ the Fourier series approximation sounds like this, and for $N = 9$ it



(a) The Fourier series for $N = 20$ (b) Values of the first 100 Fourier coefficients a_n

Figure 2.3: The Fourier series of the triangle wave of Example 2.7

sounds like this. Again these sounds are more like the triangle wave itself, and as we increase N we can hear that introduction of more frequencies pollutes the sound. However, since the triangle wave Fourier coefficients decrease as $1/n^2$ instead of $1/n$ as for the square wave, the sound is, although unpleasant due to pollution by many frequencies, not as unpleasant as the square wave. Also, it converges faster to the triangle wave itself, as also can be heard. In Exercise 7 you will be asked to write a program which verifies this.

From the previous examples we understand how we can use the Fourier coefficients to analyse or improve the sound: Noise in a sound often corresponds to the presence of some high frequencies with large coefficients, and by removing these, we remove the noise. For example, we could set all the coefficients except the first one to zero. This would change the unpleasant square wave to the pure tone $\sin 2\pi 440t$, which we started our experiments with.

2.1.1 Fourier series for symmetric and antisymmetric functions

In Example 2.6 we saw that the Fourier coefficients b_n vanished, resulting in a sine-series for the Fourier series. Similarly, in Example 2.7 we saw that a_n vanished, resulting in a cosine-series. This is not a coincident, and is captured by the following result, since the square wave was defined so that it was antisymmetric about 0, and the triangle wave so that it was symmetric about 0.

Theorem 2.8 (Symmetry and antisymmetry)**.** If f is antisymmetric about 0 (that is, if $f(-t) = -f(t)$ for all t), then $a_n = 0$, so the Fourier series is actually a sine-series. If f is symmetric about 0 (which means that $f(-t) = f(t)$ for all t), then $b_n = 0$, so the Fourier series is actually a cosine-series.

Proof. Note first that we can write

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(2\pi nt/T)dt \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(2\pi nt/T)dt,$$

i.e. we can change the integration bounds from $[0, T]$ to $[-T/2, T/2]$. This follows from the fact that all $f(t)$, $\cos(2\pi nt/T)$ and $\sin(2\pi nt/T)$ are periodic with period T.

Suppose first that f is symmetric. We obtain

$$\begin{aligned}
b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(2\pi nt/T)dt \\
&= \frac{2}{T}\int_{-T/2}^{0} f(t)\sin(2\pi nt/T)dt + \frac{2}{T}\int_{0}^{T/2} f(t)\sin(2\pi nt/T)dt \\
&= \frac{2}{T}\int_{-T/2}^{0} f(t)\sin(2\pi nt/T)dt - \frac{2}{T}\int_{0}^{-T/2} f(-t)\sin(-2\pi nt/T)dt \\
&= \frac{2}{T}\int_{-T/2}^{0} f(t)\sin(2\pi nt/T)dt - \frac{2}{T}\int_{-T/2}^{0} f(t)\sin(2\pi nt/T)dt = 0.
\end{aligned}$$

where we have made the substitution $u = -t$, and used that sin is antisymmetric. The case when f is antisymmetric can be proved in the same way, and is left as an exercise. □

In fact, the connection between symmetric and antisymmetric functions, and sine- and cosine series can be made even stronger by observing the following:

1. Any cosine series $a_0 + \sum_{n=1}^{N} a_n \cos(2\pi nt/T)$ is a symmetric function.
2. Any sine series $\sum_{n=1}^{N} b_n \sin(2\pi nt/T)$ is an antisymmetric function.
3. Any periodic function can be written as a sum of a symmetric and antisymmetric function by writing
$$f(t) = \frac{f(t) + f(-t)}{2} + \frac{f(t) - f(-t)}{2}. \tag{2.9}$$
4. If $f_N(t) = a_0 + \sum_{n=1}^{N}(a_n \cos(2\pi nt/T) + b_n \sin(2\pi nt/T))$, then
$$\begin{aligned}
\frac{f_N(t) + f_N(-t)}{2} &= a_0 + \sum_{n=1}^{N} a_n \cos(2\pi nt/T) \\
\frac{f_N(t) - f_N(-t)}{2} &= \sum_{n=1}^{N} b_n \sin(2\pi nt/T).
\end{aligned}$$

Exercises for Section 2.1

Ex. 1 — Find a function f which is Riemann-integrable on $[0,T]$, and so that $\int_0^T f(t)^2 dt$ is infinite.

Ex. 2 — Given the two Fourier spaces V_{N_1,T_1}, V_{N_2,T_2}. Find necessary and sufficient conditions in order for $V_{N_1,T_1} \subset V_{N_2,T_2}$.

Ex. 3 — Prove the second part of Theorem 2.8, i.e. show that if f is antisymmetric about 0 (i.e. $f(-t) = -f(t)$ for all t), then $a_n = 0$, i.e. the Fourier series is actually a sine-series.

Ex. 4 — Find the Fourier series coefficients of the periodic functions with period T defined by being $f(t) = t$, $f(t) = t^2$, and $f(t) = t^3$, on $[0,T]$.

Ex. 5 — Write down difference equations for finding the Fourier coefficients of $f(t) = t^{k+1}$ from those of $f(t) = t^k$, and write a program which uses this recursion. Use the program to verify what you computed in Exercise 4.

Ex. 6 — Use the previous exercise to find the Fourier series for $f(x) = -\frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{3}{16}x + 1$ on the interval $[0,1]$. Plot the 9th order Fourier series for this function. You should obtain the plots from Figure 2.1.

Ex. 7 — Let us write programs so that we can listen to the Fourier approximations of the square wave and the triangle wave.

a. Write functions

```
function playsquaretrunk(T,N)
function playtriangletrunk(T,N)
```

which plays the order N Fourier approximation of the square wave and the triangle wave, respectively, for three seconds. Verify that you can generate the sounds you played in examples 2.6 and 2.7.

b. For these Fourier approximations, how high must you choose N for them to be indistuingishable from the square/triangle waves themselves? Also describe how the characteristics of the sound changes when n increases.

2.2 Complex Fourier series

In Section 2.1 we saw how a function can be expanded in a series of sines and cosines. These functions are related to the complex exponential function via Eulers formula

$$e^{ix} = \cos x + i \sin x$$

where i is the imaginary unit with the property that $i^2 = -1$. Because the algebraic properties of the exponential function are much simpler than those of the cos and sin, it is often an advantage to work with complex numbers, even though the given setting is real numbers. This is definitely the case in Fourier analysis. More precisely, we would like to make the substitutions

$$\cos(2\pi nt/T) = \frac{1}{2}\left(e^{2\pi int/T} + e^{-2\pi int/T}\right) \tag{2.10}$$

$$\sin(2\pi nt/T) = \frac{1}{2i}\left(e^{2\pi int/T} - e^{-2\pi int/T}\right) \tag{2.11}$$

in Definition 2.3. From these identities it is clear that the set of complex exponential functions $e^{2\pi int/T}$ also is a basis of periodic functions (with the same period) for $V_{N,T}$. We may therefore reformulate Definition 2.3 as follows:

Definition 2.9 (Complex Fourier basis)**.** We define the set of functions

$$\mathcal{F}_{N,T} = \{e^{-2\pi iNt/T}, e^{-2\pi i(N-1)t/T}, \cdots, e^{-2\pi it/T}, \tag{2.12}$$

$$1, e^{2\pi it/T}, \cdots, e^{2\pi i(N-1)t/T}, e^{2\pi iNt/T}\}, \tag{2.13}$$

and call this the order N complex Fourier basis for $V_{N,T}$.

The function $e^{2\pi int/T}$ is also called a pure tone with frequency n/T, just as for sines and cosines. We would like to show that these functions also are orthogonal. To show this, we need to say more on the inner product we have defined by (2.1). A weakness with this definition is that we have assumed real functions f and g, so that this can not be used for the complex exponential functions $e^{2\pi int/T}$. For general complex functions we will extend the definition of the inner product as follows:

$$\langle f, g\rangle = \frac{1}{T}\int_0^T f\bar{g}\,dt. \tag{2.14}$$

The associated norm now becomes

$$\|f\| = \sqrt{\frac{1}{T}\int_0^T |f(t)|^2 dt}. \tag{2.15}$$

The motivation behind Equation 2.14, where we have conjugated the second function, lies in the definition of an *inner product for vector spaces over complex* *numbers*. From before we are used to vector spaces over real numbers, but vector spaces over complex numbers are defined through the same set of axioms as for real vector spaces, only replacing real numbers with complex numbers. For complex vector spaces, the axioms defining an inner product are the same as for real vector spaces, except for that the axiom

$$\langle f, g\rangle = \langle g, f\rangle \tag{2.16}$$

is replaced with the axiom

$$\langle f, g\rangle = \overline{\langle g, f\rangle}, \tag{2.17}$$

i.e. a conjugation occurs when we switch the order of the functions. This new axiom can be used to prove the property $\langle f, cg\rangle = \bar{c}\langle f, g\rangle$, which is a somewhat different property from what we know for real inner product spaces. This property follows by writing

$$\langle f, cg\rangle = \overline{\langle cg, f\rangle} = \overline{c\langle g, f\rangle} = \bar{c}\overline{\langle g, f\rangle} = \bar{c}\langle f, g\rangle.$$

Clearly the inner product 2.14 satisfies Axiom 2.17. With this definition it is quite easy to see that the functions $e^{2\pi i n t/T}$ are orthonormal. Using the orthogonal decomposition theorem we can therefore write

$$\begin{aligned} f_N(t) &= \sum_{n=-N}^{N} \frac{\langle f, e^{2\pi i n t/T}\rangle}{\langle e^{2\pi i n t/T}, e^{2\pi i n t/T}\rangle} e^{2\pi i n t/T} = \sum_{n=-N}^{N} \langle f, e^{2\pi i n t/T}\rangle e^{2\pi i n t/T} \\ &= \sum_{n=-N}^{N} \left(\frac{1}{T}\int_0^T f(t)e^{-2\pi i n t/T} dt \right) e^{2\pi i n t/T}. \end{aligned}$$

We summarize this in the following theorem, which is a version of Theorem 2.4 which uses the complex Fourier basis:

Theorem 2.10. We denote by $y_{-N}, \dots, y_0, \dots, y_N$ the coordinates of f_N in the basis $\mathcal{F}_{N,T}$, i.e.

$$f_N(t) = \sum_{n=-N}^{N} y_n e^{2\pi i n t/T}. \tag{2.18}$$

The y_n are called the complex Fourier coefficients of f, and they are given by.

$$y_n = \langle f, e^{2\pi i n t/T}\rangle = \frac{1}{T}\int_0^T f(t)e^{-2\pi i n t/T} dt. \tag{2.19}$$

If we reorder the real and complex Fourier bases so that the two functions $\{\cos(2\pi n t/T), \sin(2\pi n t/T)\}$ and $\{e^{2\pi i n t/T}, e^{-2\pi i n t/T}\}$ have the same index in the bases, equations (2.10)-(2.11) give us that the change of basis matrix[4] from

[4]See Section 4.7 in [7], to review the mathematics behind change of basis.

$\mathcal{D}_{N,T}$ to $\mathcal{F}_{N,T}$, denoted $P_{\mathcal{F}_{N,T}\leftarrow\mathcal{D}_{N,T}}$, is represented by repeating the matrix

$$\frac{1}{2}\begin{pmatrix} 1 & 1/i \\ 1 & -1/i \end{pmatrix}$$

along the diagonal (with an additional 1 for the constant function 1). In other words, since a_n, b_n are coefficients relative to the real basis and y_n, y_{-n} the corresponding coefficients relative to the complex basis, we have for $n > 0$,

$$\begin{pmatrix} y_n \\ y_{-n} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1/i \\ 1 & -1/i \end{pmatrix}\begin{pmatrix} a_n \\ b_n \end{pmatrix}.$$

This can be summarized by the following theorem:

Theorem 2.11 (Change of coefficients between real and complex Fourier bases)**.** The complex Fourier coefficients y_n and the real Fourier coefficients a_n, b_n of a function f are related by

$$\begin{aligned} y_0 &= a_0, \\ y_n &= \frac{1}{2}(a_n - ib_n), \\ y_{-n} &= \frac{1}{2}(a_n + ib_n), \end{aligned}$$

for $n = 1, \ldots, N$.

Combining with Theorem 2.8, Theorem 2.11 can help us state properties of complex Fourier coefficients for symmetric- and antisymmetric functions. We look into this in Exercise 8.

Due to the somewhat nicer formulas for the complex Fourier coefficients when compraed to the real Fourier coefficients, we will write most Fourier series in complex form in the following.

Exercises for Section 2.2

Ex. 1 — Show that the complex functions $e^{2\pi int/T}$ are orthonormal.

Ex. 2 — Repeat Exercise 2.1.4, computing the complex Fourier series instead of the real Fourier series.

Ex. 3 — Show that both $\cos^n t$ and $\sin^n t$ are in $V_{N,T}$, and find an expression for their complex Fourier coefficients.

Ex. 4 — Consider a sum of two complex exponentials. When is their sum also periodic? What is the fundamental period of the sum if the sum also is periodic?

Ex. 5 — Compute the complex Fourier coefficients of the square wave using Equation 2.19, i.e. repeat the calculations from Example 2.6 for the complex case. Use Theorem 2.11 to verify your result.

Ex. 6 — Repeat Exercise 5 for the triangle wave.

Ex. 7 — Use Equation 2.19 to compute the complex Fourier coefficients of the periodic functions with period T defined by, respectively, $f(t) = t$, $f(t) = t^2$, and $f(t) = t^3$, on $[0, T]$. Use Theorem 2.11 to verify you calculations from Exercise 4.

Ex. 8 — In this exercise we will prove a version of Theorem 2.8 for complex Fourier coefficients.

a. If f is symmetric about 0, show that y_n is real, and that $y_{-n} = y_n$.
b. If f is antisymmetric about 0, show that the y_n are purely imaginary, $y_0 = 0$, and that $y_{-n} = -y_n$.
c. Show that $\sum_{n=-N}^{N} y_n e^{2\pi int/T}$ is symmetric when $y_{-n} = y_n$ for all n, and rewrite it as a cosine-series.
d. Show that $\sum_{n=-N}^{N} y_n e^{2\pi int/T}$ is antisymmetric when $y_0 = 0$ and $y_{-n} = -y_n$ for all n, and rewrite it as a sine-series.

2.3 Rate of convergence for Fourier series

We have earlier mentioned criteria which guarantee that the Fourier series converges. Another important topic is the rate of convergence of the Fourier series, given that it converges. If the series converges quickly, we may only need a few terms in the Fourier series to obtain a reasonable approximation, meaning that good Fourier series approximations can be computed quickly. We have already seen examples which illustrate convergence rates that appear to be different: The square wave seemed to have very slow convergence rate near the discontinuities, while the triangle wave did not seem to have the same problem.

Before discussing results concerning convergence rates we consider a simple lemma which will turn out to be useful.

Lemma 2.12. If the complex Fourier coefficients of f are y_n and f is differentiable, then the Fourier coefficients of $f'(t)$ are $\frac{2\pi in}{T} y_n$.

Proof. The Fourier coefficients of $f'(t)$ are

$$\begin{aligned}\frac{1}{T}\int_0^T f'(t)e^{-2\pi int/T}dt &= \frac{1}{T}\left(\left[f(t)e^{-2\pi int/T}\right]_0^T + \frac{2\pi in}{T}\int_0^T f(t)e^{-2\pi int/T}dt\right)\\ &= \frac{2\pi in}{T}y_n.\end{aligned}$$

where the second equation was obtained from integration by parts. □

If we turn this around, we note that the Fourier coefficients of $f(t)$ are $T/(2\pi in)$ times those of $f'(t)$. If f is s times differentiable, we can repeat this argument to show that the Fourier coefficients of $f(t)$ are $\big(T/(2\pi in)\big)^s$ times those of $f^{(s)}(t)$. In other words, the Fourier coefficients of a function which is many times differentiable decay to zero very fast.

Observation 2.13. The Fourier series converges quickly when the function is many times differentiable.

An illustration is found in examples 2.6 and 2.7, where we saw that the Fourier series coefficients for the triangle wave converged more quickly to zero than those of the square wave. This is explained by the fact that the square wave is discontinuous, while triangle wave is continuous with a discontinuous first derivative.

Very often, the slow convergence of a Fourier series is due to some discontinuity of (a derivative of) the function at a given point. In this case a strategy to speed up the convergence of the Fourier series could be to create an extension of the function which is continuous, if possible, and use the Fourier series of this new function instead. With the help of the following definition, we will show that this strategy works, at least in cases where there is only one single point of discontinuity (for simplicity we have assumed that the discontinuity is at 0).

Definition 2.14 (Symmetric extension of a function)**.** Let f be a function defined on $[0,T]$. The symmetric extension of f denotes the function $\breve{f}$ defined on $[0,2T]$ by

$$\breve{f}(t) = \begin{cases} f(t), & \text{if } 0 \le t \le T;\\ f(2T-t), & \text{if } T < t \le 2T.\end{cases}$$

Clearly $\breve{f}(0) = \breve{f}(2T)$, so when f is continuous, it can be periodically extended to a continuous function with period $2T$, contrary to the function f we started with. Also, $\breve{f}$ keeps the characteristics of f, since they are equal on $[0,T]$. Also, $\breve{f}$ is clearly a symmetric function, so that it can be expressed as a cosine-series. The Fourier coefficients of the two functions are related.

Theorem 2.15. The complex Fourier coefficients y_n of f, and the cosine-coefficients a_n of $\breve{f}$ are related by $a_{2n} = y_n + y_{-n}$.

Proof. The $2n$th complex Fourier coefficient of $\breve{f}$ is

$$\begin{aligned}&\frac{1}{2T}\int_0^{2T} \breve{f}(t)e^{-2\pi i 2nt/(2T)}dt \\ &= \frac{1}{2T}\int_0^{T} f(t)e^{-2\pi int/T}dt + \frac{1}{2T}\int_T^{2T} f(2T-t)e^{-2\pi int/T}dt.\end{aligned}$$

Substituting $u = 2T - t$ in the second integral we see that this is

$$\begin{aligned}&= \frac{1}{2T}\int_0^{T} f(t)e^{-2\pi int/T}dt - \frac{1}{2T}\int_T^{0} f(u)e^{2\pi inu/T}du \\ &= \frac{1}{2T}\int_0^{T} f(t)e^{-2\pi int/T}dt + \frac{1}{2T}\int_0^{T} f(t)e^{2\pi int/T}dt \\ &= \frac{1}{2}y_n + \frac{1}{2}y_{-n}.\end{aligned}$$

Therefore we have $a_{2n} = y_n - y_{-n}$. □

This result is not enough to obtain the entire Fourier series of $\breve{f}$, but at least it gives us half of it.

Example 2.16. Let f be the function with period T defined by $f(t) = 2t/T - 1$ for $0 \leq t < T$. In each period the function increases linearly from 0 to 1. Because f is discontinuous at the boundaries between the periods, we would except the Fourier series to converge slowly. Since the function is antisymmetric, the coefficients a_n are zero, and we compute b_n as

$$\begin{aligned}b_n &= \frac{2}{T}\int_0^{T} \frac{2}{T}\left(t - \frac{T}{2}\right)\sin(2\pi nt/T)dt = \frac{4}{T^2}\int_0^{T}\left(t - \frac{T}{2}\right)\sin(2\pi nt/T)dt \\ &= \frac{4}{T^2}\int_0^{T} t\sin(2\pi nt/T)dt - \frac{2}{T}\int_0^{T}\sin(2\pi nt/T)dt \\ &= -\frac{2}{\pi n},\end{aligned}$$

so that the Fourier series is

$$-\frac{2}{\pi}\sin(2\pi t/T) - \frac{2}{2\pi}\sin(2\pi 2t/T) - \frac{2}{3\pi}\sin(2\pi 3t/T) - \frac{2}{4\pi}\sin(2\pi 4t/T) - \cdots,$$

which indeed converges slowly to 0. Let us now instead consider the symmetrization of f. Clearly this is the triangle wave with period $2T$, and the Fourier series of this is

$$\begin{aligned}&-\frac{8}{\pi^2}\cos(2\pi t/(2T)) - \frac{8}{3^2\pi^2}\cos(2\pi 3t/(2T)) - \frac{8}{5^2\pi^2}\cos(2\pi 5t/(2T)) \\ &-\frac{8}{7^2\pi^2}\cos(2\pi 7t/(2T)) + \cdots.\end{aligned}$$

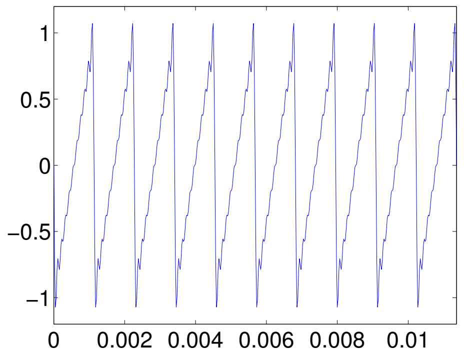


Figure 2.4: The Fourier series for $N = 10$ for the function in Example 2.16

Comparing the two series, we see that the coefficient at frequency n/T in the first series has value $-2(n\pi)$, while in the second series it has value

$$-\frac{8}{(2n)^2\pi^2} = -\frac{2}{n^2\pi^2}.$$

The second series clearly converges faster than the first. Also, we could have obtained half of the second set of coefficients from the first, by using Theorem 2.15.

If we use $T = 1/880$, the symmetrization will be the square wave of Example 2.7. Its Fourier series for $N = 10$ is shown in Figure 2.3(b) and the Fourier series for for f $N = 20$ is shown in Figure 2.4. The value $N = 10$ is used since this corresponds to the same frequencies as the previous figure for $N = 20$. It is clear from the plot that the Fourier series of f is not a very good approximation. However, we cannot differentiate between the Fourier series and the function itself for the triangle wave.

2.4 Some properties of Fourier series

We will end this section by establishing some important properties of the Fourier series, in particular the Fourier coefficients for some important functions. In these lists, we will use the notation $f \to y_n$ to indicate that y_n is the n'th Fourier coefficient of $f(t)$.

Theorem 2.17 (Fourier series pairs)**.** The functions 1, $e^{2\pi int/T}$, and $\chi_{-a,a}$

have the Fourier coefficients

$$\begin{aligned} 1 &\to \boldsymbol{e}_0 = (1, 0, 0, 0\ldots,) \\ e^{2\pi i n t/T} &\to \boldsymbol{e}_k = (0, 0, \ldots, 1, 0, 0, \ldots) \\ \chi_{-a,a} &\to \frac{\sin(2\pi n a/T)}{\pi n}. \end{aligned}$$

The 1 in $\boldsymbol{e}_k$ is at position n and the function $\chi_{-a,a}$ is the characteristic function of the interval $[-a, a]$, defined by

$$\chi_{-a,a}(t) = \begin{cases} 1, & \text{if } t \in [-a, a]; \\ 0, & \text{otherwise.} \end{cases}$$

The first two pairs are easily verified, so the proofs are omitted. The case for $\chi_{-a,a}$ is very similar to the square wave, but easier to prove, and therefore also omitted.

Theorem 2.18 (Fourier series properties)**.** The mapping $f \to y_n$ is linear: if $f \to x_n$, $g \to y_n$, then

$$af + bg \to ax_n + by_n$$

For all n. Moreover, if f is real and periodic with period T, the following properties hold:

1. $y_n = \overline{y_{-n}}$ for all n.
2. If $g(t) = f(-t)$ and $f \to y_n$, then $g \to \overline{y_n}$. In particular,
   (a) if $f(t) = f(-t)$ (i.e. f is symmetric), then all y_n are real, so that b_n are zero and the Fourier series is a cosine series.
   (b) if $f(t) = -f(-t)$ (i.e. f is antisymmetric), then all y_n are purely imaginary, so that the a_n are zero and the Fourier series is a sine series.
3. If $g(t) = f(t - d)$ (i.e. g is the function f delayed by d) and $f \to y_n$, then $g \to e^{-2\pi i n d/T} y_n$.
4. If $g(t) = e^{2\pi i d t/T} f(t)$ with d an integer, and $f \to y_n$, then $g \to y_{n-d}$.
5. Let d be a number. If $f \to y_n$, then $f(d + t) = f(d - t)$ for all t if and only if the argument of y_n is $-2\pi n d/T$ for all n.

The last property looks a bit mysterious. We will not have use for this property before the next chapter.

Proof. The proof of linearity is left to the reader. Property 1 follows immediately by writing

$$\begin{aligned}
y_n &= \frac{1}{T}\int_0^T f(t)e^{-2\pi int/T}dt = \overline{\frac{1}{T}\int_0^T f(t)e^{2\pi int/T}dt} \\
&= \overline{\frac{1}{T}\int_0^T f(t)e^{-2\pi i(-n)t/T}dt} = \overline{y_{-n}}.
\end{aligned}$$

Also, if $g(t) = f(-t)$, we have that

$$\begin{aligned}
\frac{1}{T}\int_0^T g(t)e^{-2\pi int/T}dt &= \frac{1}{T}\int_0^T f(-t)e^{-2\pi int/T}dt = -\frac{1}{T}\int_0^{-T} f(t)e^{2\pi int/T}dt \\
&= \frac{1}{T}\int_0^T f(t)e^{2\pi int/T}dt = \overline{y_n}.
\end{aligned}$$

Property 2 follows from this, since the remaining statements here were established in Theorems 2.8, 2.11, and Exercise 2.2.8. To prove property 3, we observe that the Fourier coefficients of $g(t) = f(t-d)$ are

$$\begin{aligned}
\frac{1}{T}\int_0^T g(t)e^{-2\pi int/T}dt &= \frac{1}{T}\int_0^T f(t-d)e^{-2\pi int/T}dt \\
&= \frac{1}{T}\int_0^T f(t)e^{-2\pi in(t+d)/T}dt \\
&= e^{-2\pi ind/T}\frac{1}{T}\int_0^T f(t)e^{-2\pi int/T}dt = e^{-2\pi ind/T}y_n.
\end{aligned}$$

For property 4 we observe that the Fourier coefficients of $g(t) = e^{2\pi idt/T}f(t)$ are

$$\begin{aligned}
\frac{1}{T}\int_0^T g(t)e^{-2\pi int/T}dt &= \frac{1}{T}\int_0^T e^{2\pi idt/T}f(t)e^{-2\pi int/T}dt \\
&= \frac{1}{T}\int_0^T f(t)e^{-2\pi i(n-d)t/T}dt = y_{n-d}.
\end{aligned}$$

If $f(d+t) = f(d-t)$ for all t, we define the function $g(t) = f(t+d)$ which is symmetric about 0, so that it has real Fourier coefficients. But then the Fourier coefficients of $f(t) = g(t-d)$ are $e^{-2\pi ind/T}$ times the (real) Fourier coefficients of g by property 3. It follows that y_n, the Fourier coefficients of f, has argument $-2\pi nd/T$. The proof in the other direction follows by noting that any function where the Fourier coefficients are real must be symmetric about 0, once the Fourier series is known to converge. This proves property 5. □

From this theorem we see that there exist several cases of duality between Fourier coefficients, and the function itself:

1. Delaying a function corresponds to multiplying the Fourier coefficients with a complex exponential. Vice versa, multiplying a function with a complex exponential corresponds to delaying the Fourier coefficients.

2. Symmetry/antisymmetry for a function corresponds to the Fourier coefficients being real/purely imaginary. Vice versa, a function which is real has Fourier coefficients which are conjugate symmetric.

Note that these dualities become even more explicit if we consider Fourier series of complex functions, and not just real functions.

Exercises for Section 2.4

Ex. 1 — Define the function f with period T on $[-T/2, T/2]$ by

$$f(t) = \begin{cases} 1, & \text{if } -T/4 \leq t < T/4; \\ -1, & \text{if } |T/4| \leq t < |T/2|. \end{cases}$$

f is just the square wave, shifted with $T/4$. Compute the Fourier coefficients of f directly, and use 3. in Theorem 2.18 to verify your result.

Ex. 2 — Find a function f which has the complex Fourier series

$$\sum_{n \text{ odd}} \frac{4}{\pi(n+4)} e^{2\pi int/T}.$$

Hint: Attempt to use one of the properties in Theorem 2.18 on the Fourier series of the square wave.

2.5 Summary

In this chapter we have defined and studied Fourier series, which is an approximation scheme forperiodic functions using trigonometric functions. We have established the basic properties of Fourier series, and some duality relationships between the function and its Fourier series. We have also computed the Fourier series of the square wave and the triangle wave, and investigated a technique for speeding up the convergence of the Fourier series.

Chapter 3

Fourier analysis for vectors

In Chapter 2 we saw how a function defined on an interval can be decomposed into a linear combination of sines and cosines, or equivalently, a linear combination of complex exponential functions. However, this kind of decomposition is not very convenient from a computational point of view. The coefficients are given by integrals that in most cases cannot be evaluated exactly, so some kind of numerical integration technique would have to be applied.

In this chapter our starting point is simply a vector of finite dimension. Our aim is then to decompose this vector in terms of linear combinations of vectors built from complex exponentials. This simply amounts to multiplying the original vector by a matrix, and there are efficient algorithms for doing this. It turns out that these algorithms can also be used for computing good approximations to the continuous Fourier series in Chapter 2.

Recall from Chapter 1 that a digital sound is simply a sequence of numbers, in other words, a vector. An algorithm for decomposing a vector into combinations of complex exponentials therefore corresponds to an algorithm for decomposing a digital sound into a combination of pure tones.

3.1 Basic ideas

We start by recalling what a digital sound is and by establishing some notation and terminology.

Fact 3.1. A digital sound is a finite sequence (or equivalently a vector) $\boldsymbol{x}$ of numbers, together with a number (usually an integer) f_s, the sample rate, which denotes the number of measurements of the sound per second. The length of the vector is usually assumed to be N, and it is indexed from 0 to $N-1$. Sample k is denoted by x_k, i.e.,

$$\boldsymbol{x} = (x_k)_{k=0}^{N-1}.$$

Note that this indexing convention for vectors is not standard in mathematics and is different from what we have used before. Note in particular that MATLAB indexes vectors from 1, so algorithms given here must be adjusted appropriately.

We also need the standard inner product and norm for complex vectors. At the outset our vectors will have real components, but we are going to perform Fourier analysis with complex exponentials which will often result in complex vectors.

Definition 3.2. For complex vectors of length N the Euclidean inner product is given by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{k=0}^{N-1} x_k \overline{y_k}. \tag{3.1}$$

The associated norm is

$$\|\boldsymbol{x}\| = \sqrt{\sum_{k=0}^{N-1} |x_k|^2}. \tag{3.2}$$

In the previous chapter we saw that, using a Fourier series, a function with period T could be approximated by linear combinations of the functions (the pure tones) $\{e^{2\pi i n t/T}\}_{n=0}^{N}$. This can be generalised to vectors (digital sounds), but then the pure tones must of course also be vectors.

Definition 3.3 (Fourier analysis for vectors)**.** In Fourier analysis of vectors, a vector $\boldsymbol{x} = (x_0, \ldots, x_{N-1})$ is represented as a linear combination of the N vectors

$$\boldsymbol{\phi}_n = \frac{1}{\sqrt{N}} \left(1, e^{2\pi i n/N}, e^{2\pi i 2n/N}, \ldots, e^{2\pi i k n/N}, \ldots, e^{2\pi i n(N-1)/N}\right).$$

These vectors are called the normalised complex exponentials or the pure digital tones of order N. The whole collection $\mathcal{F}_N = \{\boldsymbol{\phi}_n\}_{n=0}^{N}$ is called the N-point Fourier basis.

The following lemma shows that the vectors in the Fourier basis are orthogonal, so they do indeed form a basis.

Lemma 3.4. The normalised complex exponentials $\{\boldsymbol{\phi_n}\}_{n=0}^{N-1}$ of order N form an orthonormal basis in $\mathbb{R}^N$.

Proof. Let n_1 and n_2 be two distinct integers in the range $[0, N-1]$. The inner product of $\boldsymbol{\phi}_{n_1}$ and $\boldsymbol{\phi}_{n_2}$ is then given by

$$\begin{aligned}
N\langle \boldsymbol{\phi}_{n_1}, \boldsymbol{\phi}_{n_2} \rangle &= \langle e^{2\pi i n_1 k/N}, e^{2\pi i n_2 k/N} \rangle \\
&= \sum_{k=0}^{N-1} e^{2\pi i n_1 k/N} e^{-2\pi i n_2 k/N} \\
&= \sum_{k=0}^{N-1} e^{2\pi i (n_1 - n_2) k/N} \\
&= \frac{1 - e^{2\pi i (n_1 - n_2)}}{1 - e^{2\pi i (n_1 - n_2)/N}} \\
&= 0.
\end{aligned}$$

In particular, this orthogonality means that the the complex exponentials form a basis. And since we also have $\langle \boldsymbol{\phi}_n, \boldsymbol{\phi}_n \rangle = 1$ it is in fact an orthonormal basis. □

Note that the normalising factor $\frac{1}{\sqrt{N}}$ was not present for pure tones in the previous chapter. Also, the normalising factor $\frac{1}{T}$ from the last chapter is not part of the definition of the inner product in this chapter. These are small differences which have to do with slightly different notation for functions and vectors, and which will not cause confusion in what follows.

3.2 The Discrete Fourier Transform

Fourier analysis for finite vectors is focused around mapping a given vector from the standard basis to the Fourier basis, performing some operations on the Fourier representation, and then changing the result back to the standard basis. The Fourier matrix, which represents this change of basis, is therefore of crucial importance, and in this section we study some of its basic properties. We start by defining the Fourier matrix.

Definition 3.5 (Discrete Fourier Transform)**.** The change of coordinates from the standard basis of $\mathbb{R}^N$ to the Fourier basis $\mathcal{F}_N$ is called the discrete Fourier transform (or DFT). The $N \times N$ matrix F_N that represents this change of basis is called the (N-point) Fourier matrix. If $\boldsymbol{x}$ is a vector in R^N, its coordinates $\boldsymbol{y} = (y_0, y_1, \ldots, y_{N-1})$ relative to the Fourier basis are called the Fourier coefficients of $\boldsymbol{x}$, in other words $\boldsymbol{y} = F_N \boldsymbol{x}$). The DFT of $\boldsymbol{x}$ is sometimes denoted by $\hat{\boldsymbol{x}}$.

We will normally write $\boldsymbol{x}$ for the given vector in $\mathbb{R}^N$, and $\boldsymbol{y}$ for the DFT of this vector. In applied fields, the Fourier basis vectors are also called *synthesis vectors*, since they can be used used to "synthesize" the vector $\boldsymbol{x}$, with weights provided by the DFT coefficients $\boldsymbol{y} = (y_n)_{n=0}^{N-1}$. To be more precise, we have that the change of coordinates performed by the DFT can be written as

$$\boldsymbol{x} = y_0\boldsymbol{\phi}_0 + y_1\boldsymbol{\phi}_1 + \cdots + y_{N-1}\boldsymbol{\phi}_{N-1} = \begin{pmatrix}\boldsymbol{\phi}_0 & \boldsymbol{\phi}_1 & \cdots & \boldsymbol{\phi}_{N-1}\end{pmatrix}\boldsymbol{y} = F_N^{-1}\boldsymbol{y}, \quad (3.3)$$

where we have used the inverse of the defining relation $\boldsymbol{y} = F_N\boldsymbol{x}$, and that the $\boldsymbol{\phi}_n$ are the columns in F_N^{-1} (this follows from the fact that F_N^{-1} is the change of coordinates matrix from the Fourier basis to the standard basis, and the Fourier basis vectors are clearly the columns in this matrix). Equation (3.3) is also called the synthesis equation.

Let us also find the matrix F_N itself. From Lemma 3.4 we know that the columns of F_N^{-1} are orthonormal. If the matrix was real, it would have been called orthogonal, and the inverse matrix could be obtained by transposing. F_N^{-1} is complex however, and it is easy to see that the conjugation present in the definition of the inner product (3.1) translates into that the inverse of a complex matrix with orthonormal columns is given by the matrix where the entries are both transposed and conjugated. Let us denote the conjugated transpose of T by T^H, and say that a complex matrix is unitary when $T^{-1} = T^H$. From our discussion it is clear that F_N^{-1} is a unitary matrix, i.e. its inverse, F_N, is its conjugate transpose. Moreover since F_N^{-1} is symmetric, its inverse is in fact just its conjugate,

$$F_N = \overline{F_N^{-1}}.$$

Theorem 3.6. The Fourier matrix F_N is the unitary $N \times N$-matrix with entries given by

$$(F_N)_{nk} = \frac{1}{\sqrt{N}}e^{-2\pi i nk/N},$$

for $0 \leq n, k \leq N-1$.

Note that in the signal processing literature, it is not common to include the normalizing factor $1/\sqrt{N}$ in the definition of the DFT. From our more mathematical point of view this is useful since it makes the Fourier matrix unitary.

In practical applications of Fourier analysis one typically applies the DFT, performs some operations on the coefficients, and then maps the result back using the inverse Fourier matrix. This inverse transformation is so common that it deserves a name of its own.

Definition 3.7 (IDFT)**.** If $\boldsymbol{y} \in \mathbb{R}^N$ the vector $\boldsymbol{x} = (F_N)^H\boldsymbol{y}$ is referred to as the inverse discrete Fourier transform or (IDFT) of $\boldsymbol{y}$.

That $\boldsymbol{y}$ is the DFT of $\boldsymbol{x}$ and $\boldsymbol{x}$ is the IDFT of $\boldsymbol{y}$ can also be expressed in component form

$$x_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} y_n e^{2\pi i n k/N}, \tag{3.4}$$

$$y_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k e^{-2\pi i n k/N}. \tag{3.5}$$

In applied fields such as signal processing, it is more common to state the DFT and IDFT in these component forms, rather than in the matrix forms $\boldsymbol{x} = (F_N)^H \boldsymbol{y}$ and $\boldsymbol{y} = F_N \boldsymbol{y}$.

Let us use now see how these formulas work out in practice by considering some examples.

Example 3.8 (DFT on a square wave)**.** Let us attempt to apply the DFT to a signal $\boldsymbol{x}$ which is 1 on indices close to 0, and 0 elsewhere. Assume that

$$x_{-L} = \ldots = x_{-1} = x_0 = x_1 = \ldots = x_L = 1,$$

while all other values are 0. This is similar to a square wave, with some modifications: First of all we assume symmetry around 0, while the square wave of Example 1.11 assumes antisymmetry around 0. Secondly the values of the square wave are now 0 and 1, contrary to -1 and 1 before. Finally, we have a different proportion of where the two values are assumed. Nevertheless, we will also refer to the current digital sound as a square wave.

Since indices with the DFT are between 0 an $N-1$, and since $\boldsymbol{x}$ is assumed to have period N, the indices $[-L, L]$ where our signal is 1 translates to the indices $[0, L]$ and $[N-L, N-1]$ (i.e., it is 1 on the first and last parts of the vector). Elsewhere our signal is zero. Since $\sum_{k=N-L}^{N-1} e^{-2\pi i n k/N} = \sum_{k=-L}^{-1} e^{-2\pi i n k/N}$ (since $e^{-2\pi i n k/N}$ is periodic with period N), the DFT of $\boldsymbol{x}$ is

$$\begin{aligned}
y_n &= \frac{1}{\sqrt{N}} \sum_{k=0}^{L} e^{-2\pi i n k/N} + \frac{1}{\sqrt{N}} \sum_{k=N-L}^{N-1} e^{-2\pi i n k/N} \\
&= \frac{1}{\sqrt{N}} \sum_{k=0}^{L} e^{-2\pi i n k/N} + \frac{1}{\sqrt{N}} \sum_{k=-L}^{-1} e^{-2\pi i n k/N} \\
&= \frac{1}{\sqrt{N}} \sum_{k=-L}^{L} e^{-2\pi i n k/N} \\
&= \frac{1}{\sqrt{N}} e^{2\pi i n L/N} \frac{1 - e^{-2\pi i n(2L+1)/N}}{1 - e^{-2\pi i n/N}} \\
&= \frac{1}{\sqrt{N}} e^{2\pi i n L/N} e^{-\pi i n(2L+1)/N} e^{\pi i n/N} \frac{e^{\pi i n(2L+1)/N} - e^{-\pi i n(2L+1)/N}}{e^{\pi i n/N} - e^{-\pi i n/N}} \\
&= \frac{1}{\sqrt{N}} \frac{\sin(\pi n(2L+1)/N)}{\sin(\pi n/N)}.
\end{aligned}$$

This computation does in fact also give us the IDFT of the same vector, since the IDFT just requires a change of sign in all the exponents. From this example we see that, in order to represent $\boldsymbol{x}$ in terms of frequency components, all components are actually needed. The situation would have been easier if only a few frequencies were needed.

Example 3.9. In most cases it is difficult to compute a DFT by hand, due to the entries $e^{-2\pi ink/N}$ in the matrices, which typically can not be represented exactly. The DFT is therefore usually calculated on a computer only. However, in the case $N = 4$ the calculations are quite simple. In this case the Fourier matrix takes the form

$$F_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}.$$

We now can compute the DFT of a vector like $(1, 2, 3, 4)^T$ simply as

$$F_4\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+2+3+4 \\ 1-2i-3+4i \\ 1-2+3-4 \\ 1+2i-3-4i \end{pmatrix} = \begin{pmatrix} 5 \\ -1+i \\ -1 \\ -1-i \end{pmatrix}.$$

Example 3.10 (Direct implementation of the DFT)**.** MATLAB supports complex arithmetic, so the DFT can be implemented very simply and directly by the code

```
function y=DFTImpl(x)
  N=length(x);
  FN=zeros(N);
  for n=1:N
    FN(n,:)=exp(-2*pi*1i*(n-1)*(0:(N-1))/N)/sqrt(N);
  end
  y=FN*x;
```

Note that n has been replaced by $n-1$ in this code since n runs from 1 to N (array indices must start at 1 in MATLAB).

A direct implementation of the IDFT, which we could call `IDFTImpl` can be done similarly. Multiplying a full $N \times N$ matrix by a vector requires roughly N^2 arithmetic operations. The DFT algorithm above will therefore take a long time when N becomes moderately large, particularly in MATLAB. It turns out that if N is a power of 2, there is a much more efficient algorithm for computing the DFT which we will study in a later chapter. MATLAB also has a built-in implementation of the DFT which uses such an efficient algorithm.

The DFT has properties which are very similar to those of Fourier series, as they were listed in Theorem 2.18. The following theorem sums this up:

Theorem 3.11 (DFT properties)**.** Let $\boldsymbol{x}$ be a real vector of length N. The DFT has the following properties:

1. $(\widehat{\boldsymbol{x}})_{N-n} = \overline{(\widehat{\boldsymbol{x}})_n}$ for $0 \leq n \leq N-1$.
2. If $\boldsymbol{z}$ is the vector with the components of $\boldsymbol{x}$ reversed so that $z_k = x_{N-k}$ for $0 \leq k \leq N-1$, then $\widehat{\boldsymbol{z}} = \overline{\widehat{\boldsymbol{x}}}$. In particular,
   (a) if $x_k = x_{N-k}$ for all n (so $\boldsymbol{x}$ is symmetric), then $\widehat{\boldsymbol{x}}$ is a real vector.
   (b) if $x_k = -x_{N-k}$ for all k (so $\boldsymbol{x}$ is antisymmetric), then $\widehat{\boldsymbol{x}}$ is a purely imaginary vector.
3. If d is an integer and $\boldsymbol{z}$ is the vector with components $z_k = x_{k-d}$ (the vector $\boldsymbol{x}$ with its elements delayed by d), then $(\widehat{\boldsymbol{z}})_n = e^{-2\pi i d n/N}(\widehat{\boldsymbol{x}})_n$.
4. If d is an integer and $\boldsymbol{z}$ is the vector with components $z_k = e^{2\pi i d k/N} x_k$, then $(\widehat{\boldsymbol{z}})_n = (\widehat{\boldsymbol{x}})_{n-d}$.
5. Let d be a multiple of $1/2$. Then the following are equivalent:
   (a) $x_{d+k} = x_{d-k}$ for all k so that $d+k$ and $d-k$ are integers (in other words $\boldsymbol{x}$ is symmetric about d).
   (b) The argument of $(\widehat{\boldsymbol{x}})_n$ is $-2\pi d n/N$ for all n.

Proof. The methods used in the proof are very similar to those used in the proof of Theorem 2.18. From the definition of the DFT we have

$$\begin{aligned}(\widehat{\boldsymbol{x}})_{N-n} &= \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{-2\pi i k(N-n)/N} x_k = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{2\pi i k n/N} x_k \\ &= \overline{\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{-2\pi i k n/N} x_k} = \overline{(\widehat{\boldsymbol{x}})_n}\end{aligned}$$

which proves property 1. To prove property 2, we write

$$\begin{aligned}(\widehat{\boldsymbol{z}})_n &= \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} z_k e^{-2\pi i k n/N} = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} x_{N-k} e^{-2\pi i k n/N} \\ &= \frac{1}{\sqrt{N}}\sum_{u=1}^{N} x_u e^{-2\pi i (N-u) n/N} = \frac{1}{\sqrt{N}}\sum_{u=0}^{N-1} x_u e^{2\pi i u n/N} \\ &= \overline{\frac{1}{\sqrt{N}}\sum_{u=0}^{N-1} x_u e^{-2\pi i u n/N}} = \overline{(\widehat{\boldsymbol{x}})_n}.\end{aligned}$$

If $\boldsymbol{x}$ is symmetric it follows that $\boldsymbol{z} = \boldsymbol{x}$, so that $(\widehat{\boldsymbol{x}})_n = \overline{(\widehat{\boldsymbol{x}})_n}$. Therefore $\boldsymbol{x}$ must be real. The case of antisymmetry follows similarly.

To prove property 3 we observe that

$$\begin{aligned}(\widehat{\boldsymbol{z}})_n &= \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} x_{k-d}e^{-2\pi ikn/N} = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} x_k e^{-2\pi i(k+d)n/N} \\ &= e^{-2\pi idn/N}\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} x_k e^{-2\pi ikn/N} = e^{-2\pi idn/N}\left(\widehat{\boldsymbol{x}}\right)_n .\end{aligned}$$

For the proof of property 4 we note that the DFT of $\boldsymbol{z}$ is

$$(\widehat{\boldsymbol{z}})_n = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{2\pi idk/N}x_n e^{-2\pi ikn/N} = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} x_n e^{-2\pi i(n-d)k/N} = (\widehat{\boldsymbol{x}})_{n-d} .$$

Finally, to prove property 5, we note that if d is an integer, the vector $\boldsymbol{z}$ where $\boldsymbol{x}$ is delayed by $-d$ samples satisfies the relation $(\widehat{\boldsymbol{z}})_n = e^{2\pi idn/N}(\widehat{\boldsymbol{x}})_n$ because of property 3. Since $\boldsymbol{z}$ satisfies $z_n = z_{N-n}$, we have by property 2 that $(\widehat{\boldsymbol{z}})_n$ is real, and it follows that the argument of $(\widehat{\boldsymbol{x}})_n$ is $-2\pi dn/N$. It is straightforward to convince oneself that property 5 also holds when d is not an integer also (i.e., a multiple of 1/2). $\square$

For real sequences, Property 1 says that we need to store only about one half of the DFT coefficients, since the remaining coefficients can be obtained by conjugation. In particular, when N is even, we only need to store $y_0, y_1, \ldots, y_{N/2}$, since the other coefficients can be obtained by conjugating these.

3.2.1 Connection between the DFT and Fourier series

So far we have focused on the DFT as a tool to rewrite a vector in terms of digital, pure tones. In practice, the given vector $\boldsymbol{x}$ will often be sampled from some real data given by a function $f(t)$. We may then talk about the frequency content of the vector $\boldsymbol{x}$ and the frequency content of f and ask ourselves how these are related. More precisely, what is the relationship between the Fourier coefficients of f and the DFT of $\boldsymbol{x}$?

In order to study this, assume for simplicity that f is a sum of finitely many frequencies. This means that there exists an M so that f is equal to its Fourier approximation f_M,

$$f(t) = f_M(t) = \sum_{n=-M}^{M} z_n e^{2\pi int/T}, \tag{3.6}$$

where z_n is given by

$$z_n = \frac{1}{T}\int_0^T f(t)e^{-2\pi int/T}\,dt.$$

We recall that in order to represent the frequency n/T fully, we need the corresponding exponentials with both positive and negative arguments, i.e., both $e^{2\pi int/T}$ and $e^{-2\pi int/T}$.

Fact 3.12. Suppose f is given by its Fourier series (3.6). Then the total frequency content for the frequency n/T is given by the two coefficients z_n and z_{-n}.

Suppose that the vector $\boldsymbol{x}$ contains values sampled uniformly from f at N points,

$$x_k = f(kT/N), \quad \text{for } k = 0, 1, \ldots, N-1. \tag{3.7}$$

The vector $\boldsymbol{x}$ can be expressed in terms of its DFT $\boldsymbol{y}$ as

$$x_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} y_n e^{2\pi i n k/N}. \tag{3.8}$$

If we evaluate f at the sample points we have

$$f(kT/N) = \sum_{n=-M}^{M} z_n e^{2\pi i n k/N}, \tag{3.9}$$

and a comparison now gives

$$\sum_{n=-M}^{M} z_n e^{2\pi i n k/N} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} y_n e^{2\pi i n k/N} \quad \text{for } k = 0, 1, \ldots, N-1.$$

Exploiting the fact that both $\boldsymbol{y}$ and the complex exponentials are periodic with period N, and assuming that we take N samples with N odd, we can rewrite this as

$$\sum_{n=-M}^{M} z_n e^{2\pi i n k/N} = \frac{1}{\sqrt{N}} \sum_{n=-(N-1)/2}^{(N-1)/2} y_n e^{2\pi i n k/N}.$$

This is a matrix relation on the form $G\boldsymbol{z} = H\boldsymbol{y}/\sqrt{N}$, where

1. G is the $N \times (2M+1)$-matrix with entries $\frac{1}{\sqrt{N}} e^{2\pi i n k/N}$,
2. H is the $N \times N$-matrix with entries $\frac{1}{\sqrt{N}} e^{2\pi i n k/N}$.

In Exercise 6 you will be asked to show that $G^H G = I_{2M+1}$, and that $G^H H = \begin{pmatrix} I & \mathbf{0} \end{pmatrix}$, when $N \geq 2M+1$. Thus, if we choose the number of sample points N so that $N \geq 2M+1$, multiplying with G^H on both sides in $G\boldsymbol{z} = H\boldsymbol{y}/\sqrt{N}$ gives us that

$$\boldsymbol{z} = \begin{pmatrix} I & \mathbf{0} \end{pmatrix} \left(\frac{1}{\sqrt{N}} \boldsymbol{y} \right),$$

i.e. $\boldsymbol{z}$ consists of the first $2M+1$ elements in $\boldsymbol{y}/\sqrt{N}$. Setting $N = 2M+1$ we can summarize this.

Proposition 3.13 (Relation between Fourier coefficients and DFT coefficients)**.** Let f be a Fourier series

$$f(t) = \sum_{n=-M}^{M} z_n e^{2\pi int/T},$$

on the interval $[0, T]$ and let $N = 2M + 1$ be an odd integer. Suppose that $\boldsymbol{x}$ is sampled from f by

$$x_k = f(kT/N), \quad \text{for } k = 0, 1, \ldots, N-1.$$

and let $\boldsymbol{y}$ be the DFT of $\boldsymbol{x}$. Then $\boldsymbol{z} = \boldsymbol{y}/\sqrt{N}$, and the total contribution to f from frequency n/T, where n is an integer in the range $0 \le n \le M$, is given by y_n and y_{N-n}.

We also need a remark on what we should interpret as high and low frequency contributions, when we have applied a DFT. The low "frequency contribution" in f is the contribution from

$$e^{-2\pi iLt/T}, \ldots, e^{-2\pi it/T}, 1, e^{2\pi it/T}, \ldots, e^{2\pi iLt/T}$$

in f, i.e. $\sum_{n=-L}^{L} z_n e^{2\pi int/T}$. This means that low frequencies correspond to indices n so that $-L \le n \le L$. However, since DFT coefficients have indices between 0 and $N-1$, low frequencies correspond to indices n in $[0, L] \cup [N-L, N-1]$. If we make the same argument for high frequencies, we see that they correspond to DFT indices near $N/2$:

Observation 3.14 (DFT indices for high and low frequencies)**.** When $\boldsymbol{y}$ is the DFT of $\boldsymbol{x}$, the low frequencies in $\boldsymbol{x}$ correspond to the indices in $\boldsymbol{y}$ near 0 and N. The high frequencies in $\boldsymbol{x}$ correspond to the indices in $\boldsymbol{y}$ near $N/2$.

We will use this observation in the following example, when we use the DFT to distinguish between high and low frequencies in a sound.

Example 3.15 (Using the DFT to adjust frequencies in sound)**.** Since the DFT coefficients represent the contribution in a sound at given frequencies, we can listen to the different frequencies of a sound by adjusting the DFT coefficients. Let us first see how we can listen to the lower frequencies only. As explained, these correspond to DFT-indices n in $[0, L] \cup [N-L, N-1]$. In MATLAB these have indices from 1 to L+1, and from $N-L$+1 to N. The remaining frequencies, i.e. the higher frequencies which we want to eliminate, thus have MATLAB-indices between $L+2$ and $N-L$. We can now perform a DFT, eliminate high frequencies by setting the corresponding frequencies to zero, and perform an inverse DFT to recover the sound signal with these frequencies eliminated. With the help of the DFT implementation from Example 3.10, all this can be achieved with the following code:

```
y=DFTImpl(x);
y((L+2):(N-L))=zeros(N-(2*L+1),1);
newx=IDFTImpl(y);
```

To test this in practice, we also need to obtain the actual sound samples. If we use our sample file `castanets.wav`, you will see that the code runs very slowly. In fact it seems to never complete. The reason is that `DFTImpl` attempts to construct a matrix F_N with as many rows and columns as there are sound samples in the file, and there are just too many samples, so that F_N grows too big, and matrix multiplication with it gets too time-consuming. We will shortly see much better strategies for applying the DFT to a sound file, but for now we will simply attempt instead to split the sound file into smaller blocks, each of size $N = 32$, and perform the code above on each block. It turns out that this is less time-consuming, since big matrices are avoided. You will be spared the details for actually splitting the sound file into blocks: you can find the function `playDFTlower(L)` which performs this splitting, sets the relevant frequency components to 0, and plays the resulting sound samples. If you try this for $L = 7$ (i.e. we keep only 15 of the DFT coefficients) the result sounds like this. You can hear the disturbance in the sound, but we have not lost that much even if more than half the DFT coefficients are dropped. If we instead try $L = 3$ the result will sound like this. The quality is much poorer now. However we can still recognize the song, and this suggests that most of the frequency information is contained in the lower frequencies.

Similarly we can listen to high frequencies by including only DFT coefficients with index close to $\frac{N}{2}$. The function `playDFThigher(L)` sets all DFT coefficients to zero, except for those with indices $\frac{N}{2} - L, \ldots, \frac{N}{2}, \ldots, \frac{N}{2} + L$. Let us verify that there is less information in the higher frequencies by trying the same values for L as above for this function. For $L = 7$ (i.e. we keep only the middle 15 DFT coefficients) the result sounds like this, for $L = 3$ the result sounds like this. Both sounds are quite unrecognizable, confirming that most information is contained in the lower frequencies.

Note that there may be a problem in the previous example: for each block we compute the frequency representation of the values in that block. But the frequency representation may be different when we take all the samples into consideration. In other words, when we split into blocks, we can't expect that we exactly eliminate all the frequencies in question. This is a common problem in signal processing theory, that one in practice needs to restrict to smaller segments of samples, but that this restriction may have undesired effects in terms of the frequencies in the output.

3.2.2 Interpolation with the DFT

There are two other interesting facets to Theorem 3.13, besides connecting the DFT and the Fourier series: The first has to do with interpolation: The theorem enables us to find (unique) trigonometric functions which interpolate (pass through) a set of data points. We have in elementary calculus courses seen how to determine a polynomial of degree $N-1$ that interpolates a set of N data points — such polynomials are called interpolating polynomials. The following result tells how we can find an interpolating trigonometric function using the DFT.

Corollary 3.16 (Interpolation with the Fourier basis)**.** Let f be a function defined on the interval $[0,T]$, and let $\boldsymbol{x}$ be the sampled sequence given by

$$x_k = f(kT/N) \quad \text{for } k = 0, 1, \ldots, N-1.$$

There is exactly one linear combination $g(t)$ on the form

$$g(t) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} y_n e^{2\pi i n t/T}$$

which satisfies the conditions

$$g(kT/N) = f(kT/N), \quad k = 0, 1, \ldots, N-1$$

and its coefficients are determined by the DFT $\boldsymbol{y} = \hat{\boldsymbol{x}}$ of $\boldsymbol{x}$.

The proof for this follows by inserting $t = 0$, $t = T/N$, $t = 2T/N$, $\ldots$, $t = (N-1)T/N$ in the equation $f(t) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} y_n e^{2\pi i n t/T}$ to arrive at the equations

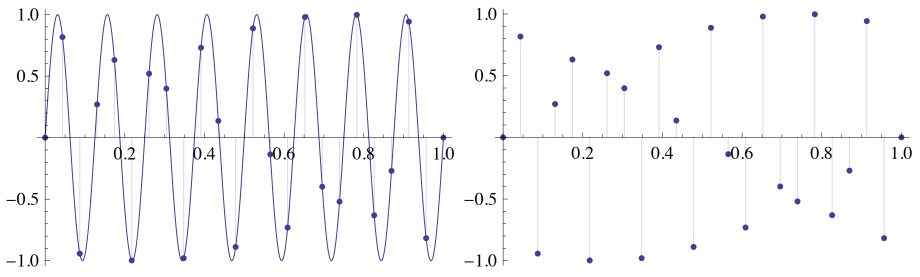
$$f(kT/N) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} y_n e^{2\pi i n k/N} \qquad 0 \le k \le N-1.$$

This gives us an equation system for finding the y_n with the invertible Fourier matrix as coefficient matrix, and the result follows.

3.2.3 Sampling and reconstruction with the DFT

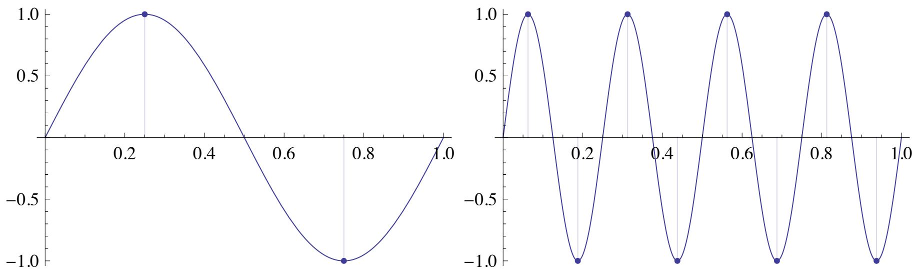
The second interesting facet to Theorem 3.13 has to do with when reconstruction of a function from its sample values is possible. An example of sampling a function is illustrated in Figure 3.1. From Figure 3.1(b) it is clear that some information is lost when we discard everything but the sample values. There may however be an exception to this, if we assume that the function satisfies some property. Assume that f is equal to a finite Fourier series. This means that f can be written on the form (3.6), so that the highest frequency in the signal is bounded by M/T. Such functions also have their own name:

Definition 3.17 (Band-limited functions)**.** A function f is said to be *band-limited* if there exists a number ν so that f does not contain frequencies higher than ν.



(a) (b)

Figure 3.1: An example of sampling. Figure (a) shows how the samples are picked from underlying continuous time function. Figure (b) shows what the samples look like on their own.



(a) (b)

Figure 3.2: Sampling the function $\sin 2\pi t$ with two points, and the function $\sin 2\pi 4t$ with eight points.

Our analysis prior to Theorem 3.13 states that all periodic, band-limited functions can be reconstructed exactly from their samples, using the DFT, as long as the number of samples is $N \geq 2M+1$, taken uniformly over a period. Moreover, the DFT is central in the reconstruction formula. We say that we reconstruct f from its samples. Dividing by T we get $\frac{N}{T} \geq \frac{2M+1}{T}$, which states that the sampling frequency ($f_s = N/T$ is the number of samples per second) should be bigger than two times the highest frequency (M/T). In Figure 3.2 we try to get some intuition on this by considering some pure tones. In Figure (a) we consider one period of $\sin 2\pi t$, and see that we need at least two sample points in $[0,1]$, since one point would clearly be too little. This translates directly into having at least eight sample points in Figure (b) where the function is $\sin 2\pi 4t$, which has four periods in the interval $[0,1]$.

Let us restate the reconstruction of f without the DFT. The reconstruction formula was

$$f(t) = \frac{1}{\sqrt{N}} \sum_{n=-M}^{M} y_n e^{2\pi i n t/T}.$$

If we here substitute $\boldsymbol{y} = F_N \boldsymbol{x}$ we get that this equals

$$\begin{aligned}
&\frac{1}{N} \sum_{n=-M}^{M} \sum_{k=0}^{N-1} x_k e^{-2\pi i n k/N} e^{2\pi i n t/T} \\
&= \sum_{k=0}^{N-1} \frac{1}{N} \left(\sum_{n=-M}^{M} x_k e^{2\pi i n (t/T - k/N)} \right) \\
&= \sum_{k=0}^{N-1} \frac{1}{N} e^{-2\pi i M (t/T - k/N)} \frac{1 - e^{2\pi i (2M+1)(t/T - k/N)}}{1 - e^{2\pi i (t/T - k/N)}} x_k \\
&= \sum_{k=0}^{N-1} \frac{1}{N} \frac{\sin(\pi(t - kT_s)/T_s)}{\sin(\pi(t - kT_s)/T)} f(kT_s),
\end{aligned}$$

where we have substituted $N = T/T_s$ (deduced from $T = NT_s$ with T_s being the sampling period). Let us summarize our findings as follows:

Theorem 3.18 (Sampling theorem and the ideal interpolation formula for periodic functions)**.** Let f be a periodic function with period T, and assume that f has no frequencies higher than νHz. Then f can be reconstructed exactly from its samples $f(0), \ldots, f((N-1)T_s)$ (where T_s is the sampling period and $N = \frac{T}{T_s}$ is the number of samples per period) when the sampling rate $F_s = \frac{1}{T_s}$ is bigger than 2ν. Moreover, the reconstruction can be performed through the formula

$$f(t) = \sum_{k=0}^{N-1} f(kT_s) \frac{1}{N} \frac{\sin(\pi(t - kT_s)/T_s)}{\sin(\pi(t - kT_s)/T)}. \tag{3.10}$$

Formula (3.10) is also called the ideal interpolation formula for periodic functions. Such formulas, where one reconstructs a function based on a weighted sum of the sample values, are more generally called *interpolation formulas*. We will return to other interpolation formulas later, which have different properties.

Note that f itself may not be equal to a finite Fourier series, and reconstruction is in general not possible then. Interpolation as performed in Section 3.2.2 is still possible, however, but the $g(t)$ we obtain from Corollary 3.16 may be different from $f(t)$.

Exercises for Section 3.2

Ex. 1 — Compute the 4 point DFT of the vector $(2, 3, 4, 5)^T$.

Ex. 2 — As in Example 3.9, state the exact cartesian form of the Fourier matrix for the cases $N = 6$, $N = 8$, and $N = 12$.

Ex. 3 — Let $\boldsymbol{x}$ be the vector with entries $x_k = c^k$. Show that the DFT of $\boldsymbol{x}$ is given by the vector with components

$$y_n = \frac{1}{\sqrt{N}} \frac{1 - c^N}{1 - ce^{-2\pi in/N}}$$

for $n = 0, \ldots, N-1$.

Ex. 4 — If $\boldsymbol{x}$ is complex, Write the DFT in terms of the DFT on real sequences. Hint: Split into real and imaginary parts, and use linearity of the DFT.

Ex. 5 — As in Example 3.10, write a function

```
function x=IDFTImpl(y)
```

which computes the IDFT.

Ex. 6 — Let G be the $N \times (2M+1)$-matrix with entries $\frac{1}{\sqrt{N}} e^{2\pi ink/N}$, and H the $N \times N$-matrix with entries $\frac{1}{\sqrt{N}} e^{2\pi ink/N}$. Show that $G^H G = I_{2M+1}$ and that $G^H H = \begin{pmatrix} I & \mathbf{0} \end{pmatrix}$ when $N \geq 2M+1$. Write also down an expression for $G^H G$ when $N < 2M+1$, to show that it is in general different from the identity matrix.

3.3 Operations on vectors: filters

In Chapter 1 we defined some operations on digital sounds, which we loosely referred to as filters. One example was the averaging filter

$$z_n = \frac{1}{4}(x_{n-1} + 2x_n + x_{n+1}), \quad \text{for } n = 0, 1, \ldots, N-1 \tag{3.11}$$

of Example 1.25 where $\boldsymbol{x}$ denotes the input vector and $\boldsymbol{z}$ the output vector. Before we state the formal definition of filters, let us consider Equation (3.11) in some more detail to get more intuition about filters.

As before we assume that the input vector is periodic with period N, so that $x_{n+N} = x_n$. Our first observation is that the output vector $\boldsymbol{z}$ is also periodic with period N since

$$z_{n+N} = \frac{1}{4}(x_{n+N-1} + 2x_{n+N} + x_{n+N+1}) = \frac{1}{4}(x_{n-1} + 2x_n + x_{n+1}) = z_n.$$

The filter is also clearly a linear transformation and may therefore be represented by an $N \times N$ matrix S that maps the vector $\boldsymbol{x} = (x_0, x_1, \dots, x_{N-1})$ to the vector $\boldsymbol{z} = (z_0, z_1, \dots, z_{N-1})$, i.e., we have $\boldsymbol{z} = S\boldsymbol{x}$. To find S we note that for $1 \le n \le N-2$ it is clear from Equation (3.11) that row n has the value $1/4$ in column $n-1$, the value $1/2$ in column n, and the value $1/4$ in column $n+1$. For row 0 we must be a bit more careful, since the index -1 is outside the legal range of the indices. This is where the periodicity helps us out so that

$$z_0 = \frac{1}{4}(x_{-1} + 2x_0 + x_1) = \frac{1}{4}(x_{N-1} + 2x_0 + x_1) = \frac{1}{4}(2x_0 + x_1 + x_{N-1}).$$

From this we see that row 0 has the value $1/4$ in columns 1 and $N-1$, and the value $1/2$ in column 0. In exactly the same way we can show that row $N-1$ has the entry $1/4$ in columns 0 and $N-2$, and the entry $1/2$ in column $N-1$. In summary, the matrix of the averaging filter is given by

$$S = \frac{1}{4}\begin{pmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 2 \end{pmatrix}. \tag{3.12}$$

A matrix on this form is called a Toeplitz matrix. Such matrices are very popular in the literature and have many applications. The general definition may seem complicated, but is in fact quite straightforward:

Definition 3.19 (Toeplitz matrices)**.** An $N \times N$-matrix S is called a Toeplitz matrix if its elements are constant along each diagonal. More formally, $S_{k,l} = S_{k+s,l+s}$ for all nonnegative integers k, l, and s such that both $k+s$ and $l+s$ lie in the interval $[0, N-1]$. A Toeplitz matrix is said to be circulant if in addition

$$S_{(k+s) \bmod N, (l+s) \bmod N} = S_{k,l}$$

for all integers k, l in the interval $[0, N-1]$, and all s (Here mod denotes the remainder modulo N).

As the definition says, a Toeplitz matrix is constant along each diagonal, while the additional property of being circulant means that each row and column of the matrix 'wraps over' at the edges. It is quite easy to check that the matrix S given by Equation (3.12) satisfies Definition 3.19 and is a circulant Toeplitz matrix. A Toeplitz matrix is uniquely identified by the values on its nonzero diagonals, and a circulant Toeplitz matrix is uniquely identified by the $N/2$ diagonals above or on the main diagonal, and the $N/2$ diagonals below the main diagonal. We will encounter Toeplitz matrices also in other contexts in these notes.

In Chapter 1, the operations we loosely referred to as filters, such as formula (3.11), could all be written on the form

$$z_n = \sum_k t_k x_{n-k}. \tag{3.13}$$

Many other operations are also defined in this way. The values t_k will be called *filter coefficients*. The range of k is not specified, but is typically an interval around 0, since z_n usually is calculated by combining x_ks with indices close to n. Both positive and negative indices are allowed. As an example, for formula (3.11) k ranges over $-1, 0$, and 1, and we have that $t_{-1} = t_1 = 1/4$, and $t_0 = 1/2$. By following the same argument as above, the following is clear:

Proposition 3.20. Any operation defined by Equation (3.13) is a linear transformation which transforms a vector of period N to another of period N. It may therefore be represented by an $N \times N$ matrix S that maps the vector $\boldsymbol{x} = (x_0, x_1, \ldots, x_{N-1})$ to the vector $\boldsymbol{z} = (z_0, z_1, \ldots, z_{N-1})$, i.e., we have $\boldsymbol{z} = S\boldsymbol{x}$. Moreover, the matrix S is a circulant Toeplitz matrix, and the first column $\boldsymbol{s}$ of this matrix is given by

$$s_k = \begin{cases} t_k, & \text{if } 0 \le k < N/2; \\ t_{k-N} & \text{if } N/2 \le k \le N-1. \end{cases} \tag{3.14}$$

In other words, the first column of S can be obtained by placing the coefficients in (3.13) with positive indices at the beginning of $\boldsymbol{s}$, and the coefficients with negative indices at the end of $\boldsymbol{s}$.

This proposition will be useful for us, since it explains how to pass from the form (3.13), which is most common in practice, to the matrix form S.

Example 3.21. Let us apply Proposition 3.20 on the operation defined by formula (3.11):

1. for $k = 0$ Equation 3.14 gives $s_0 = t_0 = 1/2$.
2. For $k = 1$ Equation 3.14 gives $s_1 = t_1 = 1/4$.
3. For $k = N - 1$ Equation 3.14 gives $s_{N-1} = t_{-1} = 1/4$.

For all k different from 0, 1, and $N - 1$, we have that $s_k = 0$. Clearly this gives the matrix in Equation (3.12).

Proposition 3.20 is also useful when we have a circulant Toeplitz matrix S, and we want to find filter coefficients t_k so that $\boldsymbol{z} = S\boldsymbol{x}$ can be written as in Equation (3.13):

Example 3.22. Consider the matrix

$$S = \begin{pmatrix} 2 & 1 & 0 & 3 \\ 3 & 2 & 1 & 0 \\ 0 & 3 & 2 & 1 \\ 1 & 0 & 3 & 2 \end{pmatrix}.$$

This is a circulant Toeplitz matrix with $N = 4$, and we see that $s_0 = 2$, $s_1 = 3$, $s_2 = 0$, and $s_3 = 1$. The first equation in (3.14) gives that $t_0 = s_0 = 2$, and $t_1 = s_1 == 3$. The second equation in (3.14) gives that $t_{-2} = s_2 = 0$, and $t_{-1} = s_3 = 1$. By including only the t_k which are nonzero, the operation can be written as

$$z_n = t_{-1}x_{n-(-1)} + t_0 x_n + t_1 x_{n-1} + t_2 x_{n-2} = x_{n+1} + 2x_0 + 3x_{n-1}.$$

3.3.1 Formal definition of filters and frequency response

Let us now define filters formally, and establish their relationship to Toeplitz matrices. We have seen that a sound can be decomposed into different frequency components, and we would like to define filters as operations which adjust these frequency components in a predictable way. One such example is provided in Example 3.15, where we simply set some of the frequency components to 0. The natural starting point is to require for a filter that the output of a pure tone is a pure tone with the same frequency.

Definition 3.23 (Digital filters and frequency response)**.** A linear transformation $S : \mathbb{R}^N \mapsto \mathbb{R}^N$ is a said to be a digital filter, or simply a filter, if it maps any Fourier vector in $\mathbb{R}^N$ to a multiple of itself. In other words, for any integer n in the range $0 \leq n \leq N-1$ there exists a value $\lambda_{S,n}$ so that

$$S(\phi_n) = \lambda_{S,n}\phi_n, \tag{3.15}$$

i.e., the N Fourier vectors are the eigenvectors of S. The vector of (eigen)values $\boldsymbol{\lambda}_S = (\lambda_{S,n})_{n=0}^{N-1}$ is often referred to as the frequency response of S.

We will identify the linear transformation S with its matrix relative to the standard basis. Since the Fourier basis vectors are orthogonal vectors, S is clearly orthogonally diagonalizable. Since also the Fourier basis vectors are the columns in $(F_N)^H$, we have that

$$S = F_N^H D F_N \tag{3.16}$$

whenever S is a digital filter, where D has the frequency response (i.e. the eigenvalues) on the diagonal[1]. In particular, if S_1 and S_2 are digital filters, we

[1]Recall that the orthogonal diagonalization of S takes the form $S = PDP^T$, where P contains as columns an orthonormal set of eigenvectors, and D is diagonal with the eigenvectors listed on the diagonal (see Section 7.1 in [7]).

can write $S_1 = F_N^H D_1 F_N$ and $S_2 = F_N^H D_2 F_N$, so that

$$S_1 S_2 = F_N^H D_1 F_N F_N^H D_2 F_N = F_N^H D_1 D_2 F_N.$$

Since $D_1 D_2 = D_2 D_1$ for any diagonal matrices, we get the following corollary:

Corollary 3.24. All digital filters commute, i.e. if S_1 and S_2 are digital filters, $S_1 S_2 = S_2 S_1$.

There are several equivalent characterizations of a digital filter. The first one was stated above in terms of the definition through eigenvectors and eigenvalues. The next characterization helps us prove that the operations from Chapter 1 actually are filters.

Theorem 3.25. A linear transformation S is a digital filter if and only if it is a circulant Toeplitz matrix.

Proof. That S is a filter is equivalent to the fact that $S = (F_N)^H D F_N$ for some diagonal matrix D. We observe that the entry at position (k, l) in S is given by

$$S_{k,l} = \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi ikn/N} \lambda_{S,n} e^{-2\pi inl/N} = \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi(k-l)n/N} \lambda_{S,n}.$$

Another entry on the same diagonal (shifted s rows and s columns) is

$$\begin{aligned} S_{(k+s) \bmod N, (l+s) \bmod N} &= \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i((k+s) \bmod N - (l+s) \bmod N)n/N} \lambda_{S,n} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i(k-l)n/N} \lambda_{S,n} = S_{k,l}, \end{aligned}$$

which proves that S is a circulant Toeplitz matrix. □

In particular, operations defined by (3.13) are digital filters, when restricted to vectors with period N. The following results enables us to compute the eigenvalues/frequency response easily, so that we do not need to form the characteristic polynomial and find its roots:

Theorem 3.26. Any digital filter is uniquely characterized by the values in the first column of its matrix. Moreover, if $\boldsymbol{s}$ is the first column in S, the frequency response of S is given by

$$\boldsymbol{\lambda}_S = \sqrt{N} F_N \boldsymbol{s}. \tag{3.17}$$

Conversely, if we know the frequency response $\boldsymbol{\lambda}_S$, the first column $\boldsymbol{s}$ of S is given by

$$\boldsymbol{s} = \frac{1}{\sqrt{N}} (F_N)^H \boldsymbol{\lambda}_S. \tag{3.18}$$

Proof. If we replace S by $(F_N)^H D F_N$ we find that

$$F_N \boldsymbol{s} = F_N S \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = F_N F_N^H D F_N \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = D F_N \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \frac{1}{\sqrt{N}} D \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

where we have used the fact that the first column in F_N has all entries equal to $1/\sqrt{N}$. But the the diagonal matrix D has all the eigenvalues of S on its diagonal, and hence the last expression is the vector of eigenvalues $\boldsymbol{\lambda}_S$, which proves (3.17). Equation (3.18) follows directly by applying the inverse DFT to (3.17). □

Since the first column $\boldsymbol{s}$ characterizes the filter S uniquely, one often refers to S by the vector $\boldsymbol{s}$. The first column $\boldsymbol{s}$ is also called the *impulse response*. This name stems from the fact that we can write $\boldsymbol{s} = S\boldsymbol{e}_0$, i.e., the vector $\boldsymbol{s}$ is the output (often called response) to the vector $\boldsymbol{e}_0$ (often called an impulse).

Example 3.27. The identity matrix is a digital filter since $I = (F_N)^H I F_N$. Since $\boldsymbol{e}_0 = S\boldsymbol{e}_0$, it has impulse response $\boldsymbol{s} = \boldsymbol{e}_0$. Its frequency response has 1 in all components and therefore preserves all frequencies, as expected.

Equations (3.16), (3.17), and (3.18) are important relations between the matrix- and frequency representations of a filter. We see that the DFT is a crucial ingredient in these relations. A consequence is that, once you recognize a matrix as circulant Toeplitz, you do not need to make the tedious calculation of eigenvectors and eigenvalues which you are used to. Let us illustrate this with an example.

Example 3.28. Let us compute the eigenvalues and eigenvectors of the simple matrix

$$S = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}.$$

It is straightforward to compute the eigenvalues and eigenvectors of this matrix the way you learnt in your first course in linear algebra. However, this matrix is also a circulant Toeplitz matrix, so that we can also use the results in this section to compute the eigenvalues and eigenvectors. Since here $N = 2$, we have that $e^{2\pi i n k/N} = e^{\pi i n k} = (-1)^{nk}$. This means that the Fourier basis vectors are $(1,1)/\sqrt{2}$ and $(1,-1)/\sqrt{2}$, which also are the eigenvectors of S. The eigenvalues are the frequency response of S, which can be obtained as

$$\sqrt{N} F_N \boldsymbol{s} = \sqrt{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

The eigenvalues are thus 3 and 5. You could have obtained the same result with Matlab. Note that Matlab may not return the eigenvectors exactly as the Fourier basis vectors, since the eigenvectors are not unique (the multiple of an eigenvector is still an eigenvector). In this case Matlab may for instance switch the signs of the eigenvectors. We have no control over what Matlab actually chooses to do, since it uses some underlying numerical algorithm for computing eigenvectors which we can't influence.

In signal processing, the frequency content of a vector (i.e., its DFT) is also referred to as its spectrum. This may be somewhat confusing from a linear algebra perspective, because in this context the term spectrum is used to denote the eigenvalues of a matrix. But because of Theorem 3.26 this is not so confusing after all if we interpret the spectrum of a vector (in signal processing terms) as the spectrum of the corresponding digital filter (in linear algebra terms).

3.3.2 Some properties of the frequency response

Equation (3.17) states that the frequency response can be written as

$$\lambda_{S,n} = \sum_{k=0}^{N-1} s_k e^{-2\pi i n k/N}, \quad \text{for } n = 0, 1, \ldots, N-1, \tag{3.19}$$

where s_k are the components of the impulse response $\boldsymbol{s}$.

Example 3.29. When only few of the coefficients s_k are nonzero, it is possible to obtain nice expressions for the frequency response. To see this, let us compute the frequency response of the filter defined from formula (3.11). We saw that the first column of the corresponding Toeplitz matrix satisfied $s_0 = 1/2$, and $s_{N-1} = s_1 = 1/4$. The frequency response is thus

$$\begin{aligned}\lambda_{S,n} &= \frac{1}{2}e^0 + \frac{1}{4}e^{-2\pi i n/N} + \frac{1}{4}e^{-2\pi i n(N-1)/N}\\ &= \frac{1}{2}e^0 + \frac{1}{4}e^{-2\pi i n/N} + \frac{1}{4}e^{2\pi i n/N} = \frac{1}{2} + \frac{1}{2}\cos(2\pi n/N).\end{aligned}$$

If we make the substitution $\omega = 2\pi n/N$ in the formula for $\lambda_{S,n}$, we may interpret the frequency response as the values on a continuous function on $[0, 2\pi)$.

Theorem 3.30. The function $\lambda_S(\omega)$ defined on $[0, 2\pi)$ by

$$\lambda_S(\omega) = \sum_k t_k e^{-ik\omega}, \tag{3.20}$$

where t_k are the filter coefficients of S, satisfies

$$\lambda_{S,n} = \lambda_S(2\pi n/N) \text{ for } n = 0, 1, \ldots, N-1$$

for any N. In other words, regardless of N, the frequency reponse lies on the curve λ_S.

Proof. For any N we have that

$$\begin{aligned}
\lambda_{S,n} &= \sum_{k=0}^{N-1} s_k e^{-2\pi ink/N} = \sum_{0\leq k<N/2} s_k e^{-2\pi ink/N} + \sum_{N/2\leq k\leq N-1} s_k e^{-2\pi ink/N} \\
&= \sum_{0\leq k<N/2} t_k e^{-2\pi ink/N} + \sum_{N/2\leq k\leq N-1} t_{k-N} e^{-2\pi ink/N} \\
&= \sum_{0\leq k<N/2} t_k e^{-2\pi ink/N} + \sum_{-N/2\leq k\leq -1} t_k e^{-2\pi in(k+N)/N} \\
&= \sum_{0\leq k<N/2} t_k e^{-2\pi ink/N} + \sum_{-N/2\leq k\leq -1} t_k e^{-2\pi ink/N} \\
&= \sum_{-N/2\leq k<N/2} t_k e^{-2\pi ink/N} = \lambda_S(\omega).
\end{aligned}$$

where we have used Equation (3.14). □

Both $\lambda_S(\omega)$ and $\lambda_{S,n}$ will be referred to as frequency responses in the following. When there is a need to distinguish the two we will call $\lambda_{S,n}$ the *vector frequency response*, and $\lambda_S(\omega)$) the *continuous frequency response*. ω is also called *angular frequency*.

The difference in the definition of the continuous- and the vector frequency response lies in that one uses the filter coefficients t_k, while the other uses the impulse response s_k. While these contain the same values, they are stored differently. Had we used the impulse response to define the continuous frequency response, we would have needed to compute $\sum_{k=0}^{N-1} s_k e^{-\pi i\omega}$, which does not converge when $N \to \infty$ (although it gives the right values at all points $\omega = 2\pi n/N$ for all N)! The filter coefficcients avoid this convergence problem, however, since we assume that only t_k with $|k|$ small are nonzero. In other words, filter coefficients are used in the definition of the continuous frequency response so that we can find a continuous curve where we can find the vector frequency response values for all N.

The frequency response contains the important characteristics of a filter, since it says how it behaves for the different frequencies. When analyzing a filter, we therefore often plot the frequency response. Often we plot only the absolute value (or the magnitude) of the frequency response, since this is what explains how each frequency is amplified or attenuated. Since λ_S is clearly periodic with period 2π, we may restrict angular frequency to the interval $[0, 2\pi)$. The conclusion in Observation 3.14 was that the low frequencies in a vector correspond to DFT indices close to 0 and $N-1$, and high frequencies correspond to DFT indices close to $N/2$. This observation is easily translated to a statement about angular frequencies:

Observation 3.31. When plotting the frequency response on $[0, 2\pi)$, angular frequencies near 0 and 2π correspond to low frequencies, angular frequencies near π correspond to high frequencies

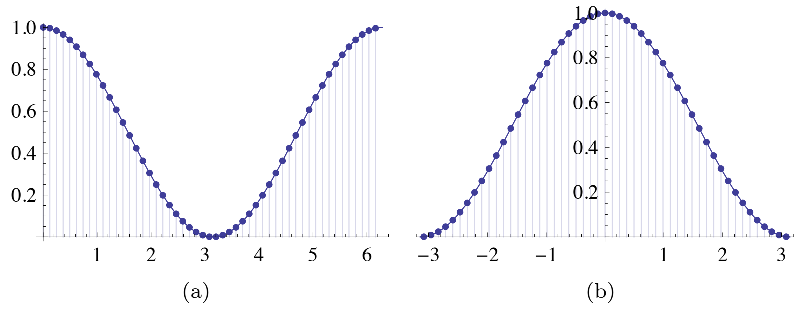


Figure 3.3: The (absolute value of the) frequency response of the smoothing filter in Example 1.25 which we discussed at the beginning of this section.

λ_S may also be viewed as a function defined on the interval $[-\pi, \pi)$. Plotting on $[-\pi, \pi]$ is often done in practice, since it makes clearer what corresponds to lower frequencies, and what corresponds to higher frequencies:

Observation 3.32. When plotting the frequency response on $[-\pi, \pi)$, angular frequencies near 0 correspond to low frequencies, angular frequencies near $\pm\pi$ correspond to high frequencies.

Example 3.33. In Example 3.29 we computed the vector frequency response of the filter defined in formula (3.11). The filter coefficients are here $t_{-1} = 1/4$, $t_0 = 1/2$, and $t_1 = 1/4$. The continuous frequency response is thus

$$\lambda_S(\omega) = \frac{1}{4}e^{i\omega} + \frac{1}{2} + \frac{1}{4}e^{-i\omega} = \frac{1}{2} + \frac{1}{2}\cos\omega.$$

Clearly this matches the computation from Example 3.29. Figure 3.3 shows plots of this frequency response, plotted on the intervals $[0, 2\pi)$ and $[-\pi, \pi)$. Both the continuous frequency response and the vector frequency response for $N = 51$ are shown. Figure (b) shows clearly how the high frequencies are softened by the filter.

Since the frequency response is essentially a DFT, it inherits several properties from Theorem 3.11.

Theorem 3.34. The frequency response has the properties:

1. The continuous frequency response satisfies $\lambda_S(-\omega) = \overline{\lambda_S(\omega)}$.
2. If S is a digital filter, S^T is also a digital filter. Morever, if the frequency response of S is $\lambda_S(\omega)$, then the frequency response of S^T is $\overline{\lambda_S(\omega)}$.

3. If S is symmetric, λ_S is real. Also, if S is antisymmetric (the element on the opposite side of the diagonal is the same, but with opposite sign), λ_S is purely imaginary.

4. If S_1 and S_2 are digital filters, then $S_1 S_2$ also is a digital filter, and $\lambda_{S_1 S_2}(\omega) = \lambda_{S_1}(\omega)\lambda_{S_2}(\omega)$.

Proof. Property 1. and 3. follow directly from Theorem 3.11. Transposing a matrix corresponds to reversing the first colum of the matrix and thus also the filter coefficients. Due to this Property 2. also follows from Theorem 3.11. The last property follows in the same was as we showed that filters commute:

$$S_1 S_2 = (F_N)^H D_1 F_N (F_N)^H D_2 F_N = (F_N)^H D_1 D_2 F_N.$$

The frequency response of $S_1 S_2$ is thus obtained by multiplying the frequency responses of S_1 and S_2. $\square$

In particular the frequency response may not be real, although this was the case in the first example of this section. Theorem 3.34 applies both for the vector- and continuous frequency response. Also, clearly $S_1 + S_2$ is a filter when S_1 and S_2 are. The set of all filters is thus a vector space, which also is closed under multiplication. Such a space is called an *algebra*. Since all filters commute, this algebra is also called a *commutative algebra*.

Example 3.35. Assume that the filters S_1 and S_2 have the frequency responses $\lambda_{S_1}(\omega) = \cos(2\omega)$, $\lambda_{S_2}(\omega) = 1+3\cos\omega$. Let us see how we can use Theorem 3.34 to compute the filter coefficients and the matrix of the filter $S = S_1 S_2$. We first notice that, since both frequency responses are real, all S_1, S_2, and $S = S_1 S_2$ are symmetric. We rewrite the frequency responses as

$$\begin{aligned}
\lambda_{S_1}(\omega) &= \frac{1}{2}(e^{2i\omega} + e^{-2i\omega}) = \frac{1}{2}e^{2i\omega} + \frac{1}{2}e^{-2i\omega} \\
\lambda_{S_2}(\omega) &= 1 + \frac{3}{2}(e^{i\omega} + e^{-i\omega}) = \frac{3}{2}e^{i\omega} + 1 + \frac{3}{2}e^{-i\omega}.
\end{aligned}$$

We now get that

$$\begin{aligned}
\lambda_{S_1 S_2}(\omega) = \lambda_{S_1}(\omega)\lambda_{S_2}(\omega) &= \left(\frac{1}{2}e^{2i\omega} + \frac{1}{2}e^{-2i\omega}\right)\left(\frac{3}{2}e^{i\omega} + 1 + \frac{3}{2}e^{-i\omega}\right) \\
&= \frac{3}{4}e^{3i\omega} + \frac{1}{2}e^{2i\omega} + \frac{3}{4}e^{i\omega} + \frac{3}{4}e^{-i\omega} + \frac{1}{2}e^{-2i\omega} + \frac{3}{4}e^{-3i\omega}
\end{aligned}$$

From this expression we see that the filter coefficients of S are $t_{\pm 1} = 3/4$, $t_{\pm 2} = 1/2$, $t_{\pm 3} = 3/4$. All other filter coefficients are 0. Using Theorem 3.20, we get that $s_1 = 3/4$, $s_2 = 1/2$, and $s_3 = 3/4$, while $s_{N-1} = 3/4$, $s_{N-2} = 1/2$, and $s_{N-3} = 3/4$ (all other s_k are 0). This gives us the matrix representation of S.

3.3.3 Assembling the filter matrix and compact notation

Let us return to how we first defined a filter in Equation (3.13):

$$z_n = \sum_k t_k x_{n-k}.$$

As mentioned, the range of k may not be specified. In some applications in signal processing there may in fact be infinitely many nonzero t_k. However, when $\boldsymbol{x}$ is assumed to have period N, we may as well assume that the k's range over an interval of length N (else many of the t_k's can be added together to simplify the formula). Also, any such interval can be chosen. It is common to choose the interval so that it is centered around 0 as much as possible. For this, we can choose for instance $[\lfloor N/2 \rfloor - N + 1, \lfloor N/2 \rfloor]$. With this choice we can write Equation (3.13) as

$$z_n = \sum_{k=\lfloor N/2 \rfloor - N+1}^{\lfloor N/2 \rfloor} t_k x_{n-k}. \tag{3.21}$$

The index range in this sum is typically even smaller, since often much less than N of the t_k are nonzero (For Equation (3.11), there were only three nonzero t_k). In such cases one often uses a more compact notation for the filter:

Definition 3.36 (Compact notation for filters)**.** Let $k_{\text{min}} \le 0$, $k_{\text{max}} \ge 0$ be the smallest and biggest index of a filter coefficient in Equation (3.21) so that $t_k \neq 0$ (if no such values exist, let $k_{\text{min}} = 0$, or $k_{\text{max}} = 0$), i.e.

$$z_n = \sum_{k=k_{min}}^{k_{max}} t_k x_{n-k}. \tag{3.22}$$

We will use the following compact notation for S:

$$S = \{t_{k_{min}}, \ldots, t_{-1}, \underline{t_0}, t_1, \ldots, t_{k_{max}}\}.$$

In other words, the entry with index 0 has been underlined, and only the nonzero t_k's are listed. By the length of S, denoted $l(S)$, we mean the number $k_{max} - k_{min}$.

One seldom writes out the matrix of a filter, but rather uses this compact notation. Note that the length of S can also be written as the number of nonzero filter coefficients minus 1. $l(S)$ thus follows the same convention as the degree of a polynomial: It is 0 if the polynomial is constant (i.e. one nonzero filter coefficient).

Example 3.37. Using the compact notation for a filter, we would write $S = \{1/4, \underline{1/2}, 1/4\}$ for the filter given by formula (3.11)). For the filter

$$z_n = x_{n+1} + 2x_0 + 3x_{n-1}$$

from Example 3.22, we would write $S = \{1, \underline{2}, 3\}$.

Equation (3.13) is also called the convolution of the two vectors $\boldsymbol{t}$ and $\boldsymbol{x}$. Convolution is usually defined without the assumption that the vectors are periodic, and without any assumption on their lengths (i.e. they may be sequences of inifinite length):

Definition 3.38 (Convolution of vectors)**.** By the *convolution* of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ we mean the vector $\boldsymbol{x} * \boldsymbol{y}$ defined by

$$(\boldsymbol{x} * \boldsymbol{y})_n = \sum_k x_k y_{n-k}. \tag{3.23}$$

In other words, applying a filter S corresponds to convolving the filter coefficients of S with the input. If both $\boldsymbol{x}$ and $\boldsymbol{y}$ have infinitely many nonzero entries, the sum is an infinite one, which may diverge. For the filters we look at, we always have a finite number of nonzero entries t_k, so we never have this convergence problem since the sum is a finite one. MATLAB has the built-in function `conv` for convolving two vectors of finite length. This function does not indicate which indices the elements of the returned vector belongs to, however. Exercise 11 explains how one may keep track of these indices.

Since the number of nonzero filter coefficients is typically much less than N (the period of the input vector), the matrix S have many entries which are zero. Multiplication with such matrices requires less additions and multiplications than for other matrices: If S has k nonzero filter coefficients, S has Nk nonzero entries, so that kN multiplications and $(k-1)N$ additions are needed to compute $S\boldsymbol{x}$. This is much less than the N^2 multiplications and $(N-1)N$ additions needed in the general case. Perhaps more important is that we need not form the entire matrix, we can perform the matrix multiplication directly in a loop. Exercise 10 investigates this further. For large N we risk running into out of memory situations if we had to form the entire matrix.

3.3.4 Some examples of filters

We have now established the basic theory of filters, so it is time to study some specific examples. Many of the filters below were introduced in Section 1.4.

Example 3.39 (Time delay filters)**.** The simplest possible type of Toeplitz matrix is one where there is only one nonzero diagonal. Let us define the Toeplitz matrix E_d as the one which has first column equal to $\boldsymbol{e}_d$. In the notation above, and when $d > 0$, this filter can also be written as $S = \{\underline{0}, \ldots, 1\}$ where the 1 occurs at position d. We observe that

$$(E_d\boldsymbol{x})_n = \sum_{k=0}^{N-1} (E_d)_{n,k}\, x_k = \sum_{k=0}^{N-1} (E_d)_{(n-k) \bmod N, 0}\, x_k = x_{(n-d) \bmod N},$$

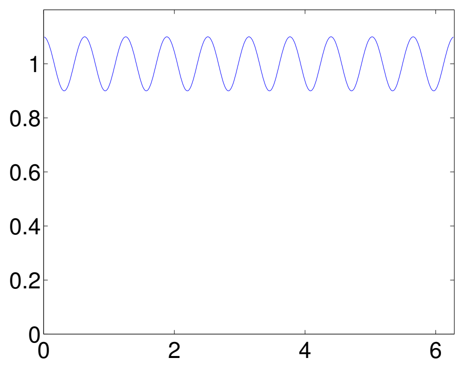


Figure 3.4: The frequency response of a filter which adds an echo with damping factor $c = 0.1$ and delay $d = 10$.

since only when $(n-k) \bmod N = d$ do we have a contribution in the sum. It is thus clear that multiplication with E_d delays a vector by d samples, in a circular way. For this reason E_d is also called a *time delay filter*. The frequency response of the time delay filter is clearly the function $\lambda_S(\omega) = e^{-id\omega}$, which has magnitude 1. This filter therefore does not change the magnitude of the different frequencies.

Example 3.40 (Adding echo)**.** In Example 1.23 we encountered a filter which could be used for adding echo to sound. Using our compact filter notation this can be written as

$$S = \{\underline{1}, 0, \ldots, 0, c\},$$

where the damping factor c appears after the delay d. The frequency response of this is $\lambda_S(\omega) = 1 + ce^{-id\omega}$. This frequency response is not real, which means that the filter is not symmetric. In Figure 3.4 we have plotted the magnitude of this frequency response with $c = 0.1$ and $d = 10$. We see that the response varies between 0.9 and 1.1, so that adding exho changes frequencies according to the damping factor c. The deviation from 1 is controlled by the damping factor c. Also, we see that the oscillation in the frequency response, as visible in the plot, is controlled by the delay d.

Previously we have claimed that some operations, such as averaging the samples, can be used for adjusting the bass and the treble of a sound. Theorem 3.25 supports this, since the averaging operations we have defined correspond to circulant Toeplitz matrices, which are filters which adjust the frequencies as dictated by the frequency response. Below we will analyze the frequency response of the corresponsing filters, to verify that it works as we have claimed for the frequencies corresponding to bass and treble in sound.

Example 3.41 (Reducing the treble)**.** In Example 1.25 we encountered the moving average filter

$$S = \left\{ \frac{1}{3}, \underline{\frac{1}{3}}, \frac{1}{3} \right\}.$$

This could be used for reducing the treble in a sound. If we set $N = 4$, the corresponding circulant Toeplitz matrix for the filter is

$$S = \frac{1}{3} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

The frequency response is $\lambda_S(\omega) = (e^{i\omega} + 1 + e^{-i\omega})/3 = (1 + 2\cos(\omega))/3$. More generally, if the filter is $\boldsymbol{s} = (1, \cdots, \underline{1}, \cdots, 1)/(2L+1)$, where there is symmetry around 0, we recognize this as $\boldsymbol{x}/(2L+1)$, where $\boldsymbol{x}$ is a vector of ones and zeros, as defined in Example 3.8. From that example we recall that

$$\boldsymbol{x} = \frac{1}{\sqrt{N}} \frac{\sin(\pi n(2L+1)/N)}{\sin(\pi n/N)},$$

so that the frequency response of S is

$$\lambda_{S,n} = \frac{1}{2L+1} \frac{\sin(\pi n(2L+1)/N)}{\sin(\pi n/N)},$$

and

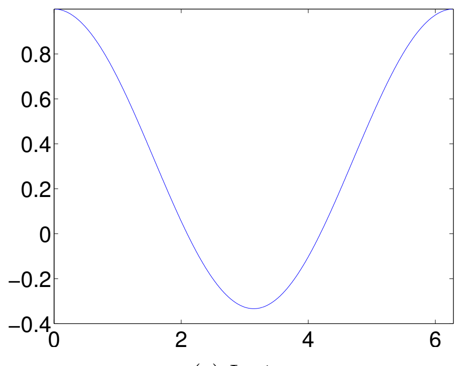
$$\lambda_S(\omega) = \frac{1}{2L+1} \frac{\sin((2L+1)\omega/2)}{\sin(\omega/2)}.$$

We clearly have

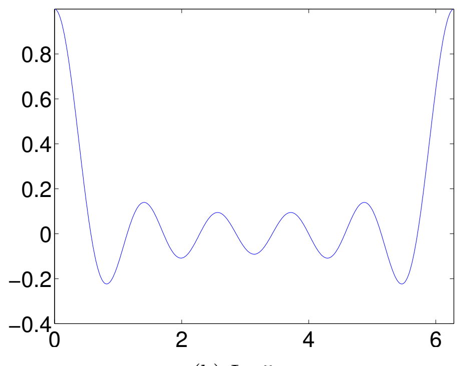
$$0 \leq \frac{1}{2L+1} \frac{\sin((2L+1)\omega/2)}{\sin(\omega/2)} \leq 1,$$

so this frequency response approaches 1 as $\omega \to 0^+$. The frequency response thus peaks at 0, and it is clear that this peak gets narrower and narrower as L increases, i.e. we use more and more samples in the averaging process. This appeals to our intuition that this kind of filters smooths the sound by keeping only lower frequencies. In Figure 3.5 we have plotted the frequency response for moving average filters with $L = 1$, $L = 5$, and $L = 20$. We see, unfortunately, that the frequency response is far from a filter which keeps some frequencies unaltered, while annihilating others (this is a desirable property which is refered to as being a *bandpass filter*): Although the filter distinguishes between high and low frequencies, it slightly changes the small frequencies. Moreover, the higher frequencies are not annihilated, even when we increase L to high values.

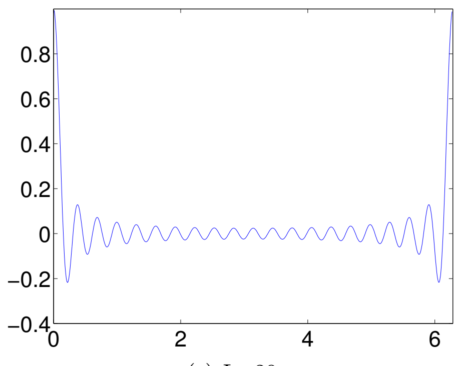
In the previous example we mentioned a filter which kept some frequencies unaltered, and annihilated others. This is a desirable property for filters, so let us give names to such filters:



(a) L=1



(b) L=5



(c) L=20

Figure 3.5: The frequency response of moving average filters of different length.

Definition 3.42. A filter S is called

1. a *lowpass filter* if $\lambda_S(\omega) \approx 1$ when ω is close to 0, and $\lambda_S(\omega) \approx 0$ when ω is close to π (i.e. S keeps low frequencies and annhilates high frequencies),
2. a *highpass filter* if $\lambda_S(\omega) \approx 1$ when ω is close to π, and $\lambda_S(\omega) \approx 0$ when ω is close to 0 (i.e. S keeps high frequencies and annhilates low frequencies),
3. a *bandpass filter* if $\lambda_S(\omega) \approx 1$ within some interval $[a, b] \subset [0, 2\pi]$, and $\lambda_S(\omega) \approx 0$ outside this interval.

This definition should be considered rather vague when it comes to what we mean by "ω close to $0, \pi$", and "$\lambda_S(\omega) \approx 0, 1$": in practice, when we talk about lowpass and highpass filters, it may be that the frequency responses are still quite far from what is commonly refered to as *ideal lowpass or highpass filters*, where the frequency response only assumes the values 0 and 1 near 0 and π. The next example considers an ideal lowpass filter.

Example 3.43 (Ideal lowpass filters)**.** By definition, the ideal lowpass filter keeps frequencies near 0, and removes frequencies near π. In Chapter 1 we mentioned that we were not able to find the filter coefficients for such a filter. We now have the theory in place in order to achieve this: In Example 3.15 we implemented the ideal lowpass filter with the help of the DFT. Mathematically, the code was equivalent to computing $(F_N)^H D F_N$ where D is the diagonal matrix with the entries $0, \ldots, L$ and $N-L, \ldots, N-1$ being 1, the rest being 0. Clearly this is a digital filter, with frequency response as stated. If the filter should keep the angular frequencies $|\omega| \leq \omega_c$ only, where ω_c describes the highest frequency we should keep, we should choose L so that $\omega_c = 2\pi L/N$. In Example 3.8 we computed the DFT of this vector, and it followed from Theorem 3.11 that the IDFT of this vector equals its DFT. This means that we can find the filter coefficients by using Equation (3.18): Since the IDFT was $\frac{1}{\sqrt{N}} \frac{\sin(\pi k(2L+1)/N)}{\sin(\pi k/N)}$, the filter coefficients are

$$\frac{1}{\sqrt{N}} \frac{1}{\sqrt{N}} \frac{\sin(\pi k(2L+1)/N)}{\sin(\pi k/N)} = \frac{1}{N} \frac{\sin(\pi k(2L+1)/N)}{\sin(\pi k/N)}.$$

This means that the filter coefficients lie as N points uniformly spaced on the curve $\frac{1}{N} \frac{\sin(\omega(2L+1)/2)}{\sin(\omega/2)}$ between 0 and π. This curve has been encountered many other places in these notes. The filter which keeps only the frequency $\omega_c = 0$ has all filter coefficients being $\frac{1}{N}$ (set $L = 1$), and when we include all frequencies (set $L = N$) we get the filter where $x_0 = 1$ and all other filter coefficients are 0. When we are between these two cases, we get a filter where s_0 is the biggest coefficient, while the others decrease towards 0 along the curve we have computed. The bigger L and N are, the quicker they decrease to zero. All filter coefficients are typically nonzero for this filter, since this curve is zero only at certain points. This is unfortunate, since it means that the filter is time-consuming to compute.

The two previous examples show an important duality between vectors which are 1 on some elements and 0 on others (also called window vectors), and the vector $\frac{1}{N}\frac{\sin(\pi k(2L+1)/N)}{\sin(\pi k/N)}$ (also called a sinc): filters of the one type correspond to frequency responses of the other type, and vice versa. The examples also show that, in some cases only the filter coefficients are known, while in other cases only the frequency response is known. In any case we can deduce the one from the other, and both cases are important.

Filters are much more efficient when there are few nonzero filter coefficients. In this respect the second example displays a problem: in order to create filters with particularly nice properties (such as being an ideal lowpass filter), one may need to sacrifice computational complexity by increasing the number of nonzero filter coefficients. The trade-off between computational complexity and desirable filter properties is a very important issue in *filter design theory.*

Example 3.44. In order to decrease the computational complexity for the ideal lowpass filter in Example 3.43, one can for instance include only the first filter coefficients, i.e. $\{\frac{1}{N}\frac{\sin(\pi k(2L+1)/N)}{\sin(\pi k/N)}\}_{k=-N_0}^{N_0}$, ignoring the last ones. Hopefully this gives us a filter where the frequency reponse is not that different from the ideal lowpass filter. In Figure 3.6 we show the corresponding frequency responses. In the figure we have set $N = 128$, $L = 32$, so that the filter removes all frequencies $\omega > \pi/2$. N_0 has been chosen so that the given percentage of all coefficients are included. Clearly the figure shows that we should be careful when we omit filter coefficients: if we drop too many, the frequency response is far away from that of an ideal bandpass filter.

Example 3.45 (Reducing the treble II)**.** Let S be the moving average filter of two elements, i.e.

$$(S\boldsymbol{x})_n = \frac{1}{2}(x_{n-1} + x_n).$$

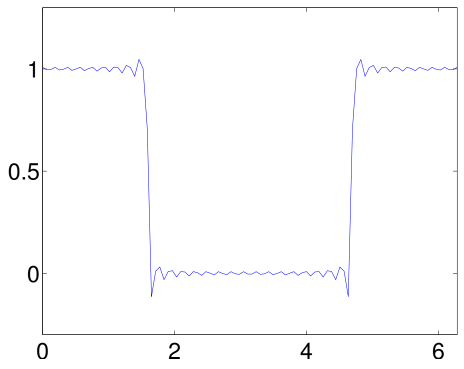
In Example 3.41 we had an odd number of filter coefficients. Here we have only two. We see that the frequency response in this case is

$$\lambda_S(\omega) = \frac{1}{2}(1 + e^{-i\omega}) = e^{-i\omega/2}\cos(\omega/2).$$

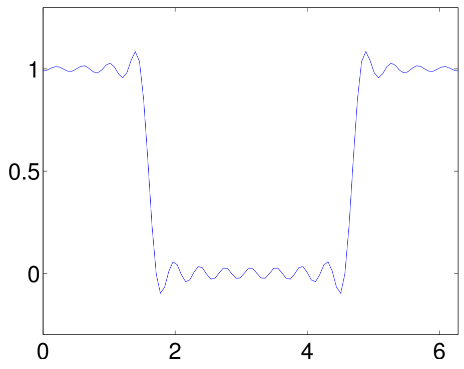
The frequency response is complex now, since the filter is not symmetric in this case. Let us now apply this filter k times, and denote by S_k the resulting filter. Theorem 3.34 gives us that the frequency response of S_k is

$$\lambda_{S^k}(\omega) = \frac{1}{2^k}(1 + e^{-i\omega})^k = e^{-ik\omega/2}\cos^k(\omega/2),$$

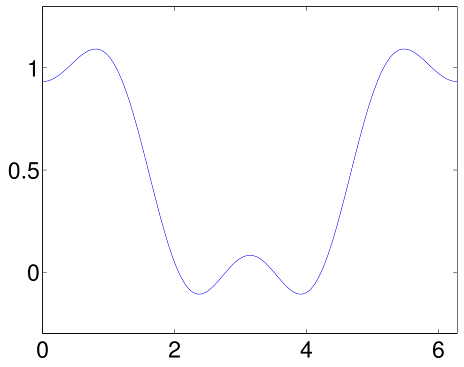
which is a polynomial in $e^{-i\omega}$ with the coefficients taken from Pascal's triangle. At least, this partially explains how filters with coefficients taken from Pascal's triangle appear, as in Example 1.25. These filters are more desirable than the



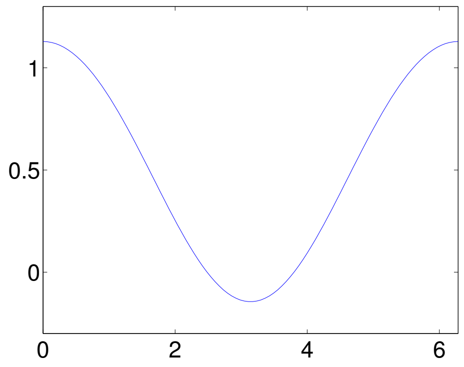
(a) all $N = 128$ filter coefficients



(b) 1/4 of all filter coefficients



(c) 1/16 of all filter coefficients



(d) 1/32 of all filter coefficients

Figure 3.6: The frequency response which results by omitting the last filter coefficients for the ideal lowpass filter.

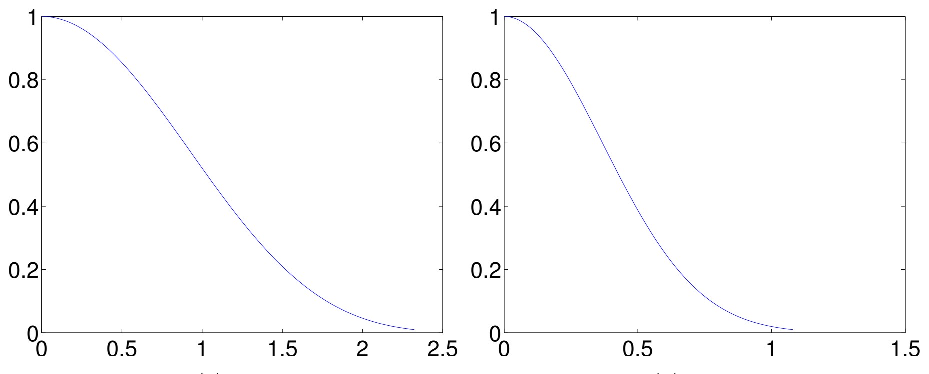


Figure 3.7: The frequency response of filters corresponding to a moving average filter convolved with itself k times.

moving average filters, and are used for smoothing abrupt changes in images and in sound. The reason is that, since we take a k'th power with k large, λ_{S^k} is more square-like near 0, i.e. it becomes more and more like a bandpass filter near 0. In Figure 3.7 we have plotted the magnitude of the frequence response when $k = 5$, and when $k = 30$. This behaviour near 0 is not so easy to see from the figure. Note that we have zoomed in on the frequency response to the area where it actually decreases to 0.

In Example 1.27 we claimed that we could obtain a bass-reducing filter by using alternating signs on the filter coefficients in a treble-reducing filter. Let us explain why this is the case. Let S be a filter with filter coefficients s_k, and let us consider the filter T with filter coefficient $(-1)^k s_k$. The frequency response of T is

$$\begin{aligned}\lambda_T(\omega) &= \sum_k (-1)^k s_k e^{-i\omega k} = \sum_k (e^{-i\pi})^k s_k e^{-i\omega k} \\ &= \sum_k e^{-i\pi k} s_k e^{-i\omega k} = \sum_k s_k e^{-i(\omega+\pi)k} = \lambda_S(\omega+\pi).\end{aligned}$$

where we have set $-1 = e^{-i\pi}$ (note that this is nothing but Property 4. in Theorem 3.11, with $d = N/2$). Now, for a lowpass filter S, $\lambda_S(\omega)$ has values near 1 when ω is close to 0 (the low frequencies), and values near 0 when ω is close to π (the high frequencies). For a highpass filter T, $\lambda_T(\omega)$ has values near 0 when ω is close to 0 (the low frequencies), and values near 1 when ω is close to π (the high frequencies). When T is obtained by adding an alternating sign to the filter coefficicents of S, The relation $\lambda_T(\omega) = \lambda_S(\omega+\pi)$ thus says that T is a highpass filter when S is a lowpass filter, and vice versa:

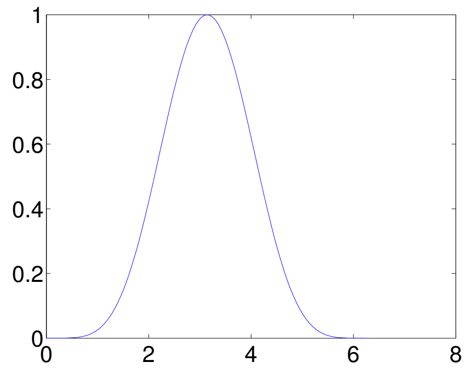


Figure 3.8: The frequency response of the bass reducing filter, which corresponds to row 5 of Pascal's triangle.

Observation 3.46. Assume that T is obtained by adding an alternating sign to the filter coefficicents of S. If S is a lowpass filter, then T is a highpass filter. If S is a highpass filter, then T is a lowpass filter.

The following example explains why this is the case.

Example 3.47 (Reducing the bass)**.** In Example 1.27 we constructed filters where the rows in Pascal's triangle appeared, but with alternating sign. The frequency response of this when using row 5 of Pascal's triangle is shown in Figure 3.8. It is just the frequency response of the corresponding treble-reducing filter shifted with π. The alternating sign can also be achieved if we write the frequency response $\frac{1}{2^k}(1+e^{-i\omega})^k$ from Example 3.45 as $\frac{1}{2^k}(1-e^{-i\omega})^k$, which corresponds to applying the filter $S(\boldsymbol{x}) = \frac{1}{2}(-x_{n-1}+x_n)$ k times.

3.3.5 Time-invariance of filters

The third characterization of digital filters we will prove is stated in terms of the following concept:

Definition 3.48 (Time-invariance)**.** A linear transformation from $\mathbb{R}^N$ to $\mathbb{R}^N$ is said to be time-invariant if, for any d, the output of the *delayed* input vector $\boldsymbol{z}$ defined by $z_n = x_{(n-d) \bmod N}$ is the delayed output vector $\boldsymbol{w}$ defined by $w_n = y_{(n-d) \bmod N}$.

We have the following result:

Theorem 3.49. A linear transformation S is a digital filter if and only if it is time-invariant.

Proof. Let $\boldsymbol{y} = S\boldsymbol{x}$, and $\boldsymbol{z}, \boldsymbol{w}$ as defined above. We have that

$$\begin{aligned} w_n = (S\boldsymbol{x})_{n-d} &= \sum_{k=0}^{N-1} S_{n-d,k} x_k \\ &= \sum_{k=0}^{N-1} S_{n,k+d} x_k = \sum_{k=0}^{N-1} S_{n,k} x_{k-d} \\ &= \sum_{k=0}^{N-1} S_{n,k} z_k = (S\boldsymbol{z})_n \end{aligned}$$

This proves that $S\boldsymbol{z} = \boldsymbol{w}$, so that S is time-invariant. □

By Example 3.39, delaying a vector with d elements corresponds to multiplication with the filter E_d. That S is time-invariant could thus also have been defined by demanding that $SE_d = E_d S$ for any d. That all filters are time invariant follows also immediately from the fact that all filters commute.

Due to Theorem 3.49, digital filters are also called LTI filters (LTI stands for Linear, Time-Invariant). By combining the definition of a digital filter with theorems 3.26, and 3.49, we get the following:

Theorem 3.50 (Characterizations of digital filters)**.** The following are equivalent characterizations of a digital filter:

1. $S = (F_N)^H D F_N$ for a diagonal matrix D, i.e. the Fourier basis is a basis of eigenvectors for S.
2. S is a circulant Toeplitz matrix.
3. S is linear and time-invariant.

3.3.6 Linear phase filters

Some filters are particularly important for applications:

Definition 3.51 (Linear phase)**.** We say that a digital filter S has linear phase if there exists some d so that $S_{d+n,0} = S_{d-n,0}$ for all n.

From Theorem 3.11 4. it follows that the argument of the frequency response at n for S is $-2\pi dn/N$. Moreover, the frequency response is real if $d = 0$, and this also corresponds to that the matrix is symmetric. One reason that linear phase filters are important for applications is that they can be more efficiently implemented than general filters. As an example, if S is symmetric around 0, we can write

$$\begin{aligned}(S\boldsymbol{x})_n &= \sum_{k=0}^{N-1} s_k x_{n-k} = \sum_{k=0}^{N/2-1} s_k x_{n-k} + \sum_{k=N/2}^{N-1} s_k x_{n-k} \\ &= \sum_{k=0}^{N/2-1} s_k x_{n-k} + \sum_{k=0}^{N/2-1} s_{k+N/2} x_{n-k-N/2} \\ &= \sum_{k=0}^{N/2-1} s_k x_{n-k} + \sum_{k=0}^{N/2-1} s_{N/2-k} x_{n-k-N/2} \\ &= \sum_{k=0}^{N/2-1} s_k x_{n-k} + \sum_{k=0}^{N/2-1} s_k x_{n+k} = \sum_{k=0}^{N/2-1} s_k (x_{n-k} + x_{n+k})\end{aligned}$$

If we compare the first and last expressions here, we need the same number of summations, but the number of multiplications needed in the latter expression has been halved. The same point can also be made about the factorization into a composition of many moving average filters of length 2 in Example 3.45. This also corresponds to a linear phase filter. Each application of a moving average filter of length 2 does not really require any multiplications, since multiplication with $\frac{1}{2}$ really corresponds to a bitshift. Therefore, the factorization of Example 3.45 removes the need for doing any multiplications at all, while keeping the number of additions the same. There is a huge computational saving in this. We will see another desirable property of linear phase filters in the next section, and we will also return to these filters later.

3.3.7 Perfect reconstruction systems

The following is easily proved, and left as exercises:

Theorem 3.52. The following hold:

1. The set of (circulant) Toeplitz matrices form a vector space.
2. If G_1 and G_2 are (circulant) Toeplitz matrices, then G_1G_2 is also a (circulant) Toeplitz matrix, and $l(G_1G_2) = l(G_1) + l(G_2)$.
3. $l(G) = 0$ if and only if G has only one nonzero diagonal.

An immediate corollary of this is in terms of what is called *perfect reconstruction systems*:

Definition 3.53 (Perfect reconstruction system)**.** By a perfect reconstruction system we mean a pair of $N \times N$-matrices (G_1, G_2) so that $G_2 G_1 = I$. For a vector $\boldsymbol{x}$ we refer to $\boldsymbol{z} = G_1 \boldsymbol{x}$ as the transformed vector. For a vector $\overline{\boldsymbol{z}}$ we refer to $\overline{\boldsymbol{x}} = G_2 \overline{\boldsymbol{z}}$ as the reconstructed vector.

The terms perfect reconstruction, transformation, and reconstruction come from signal processing, where one thinks of G_1 as a transform, and G_2 as another transform which reconstructs the input to the first transform from its output. In practice, we are interested in finding perfect reconstruction systems where the transformed $G_1\boldsymbol{x}$ is so that it is more suitable for further processing, such as compression, or playback in an audio system. One example is the DFT: We have already proved that $(F_N, (F_N)^H)$ is a perfect reconstruction system for ant N. One problem with this system is that the Fourier matrix is not sparse. Although efficient algorithms exist for the DFT, one may find systems which are quicker to compute in the transform and reconstruction steps. We are therefore in practice interested in establishing perfect reconstruction systems, where the involved matrices have particular forms. Digital filters is one such form, since these are quick to compute when there are few nonzero filter coefficients. Unfortunately, related to this we have the following corollary to Theorem 3.52:

Corollary 3.54. let G_1 and G_2 be circulant Toeplitz matrices so that (G_1, G_2) is a perfect reconstruction system. Then there exist a scalar α and an integer d so that $G_1 = \alpha E_d$ and $G_2 = \alpha^{-1} E_{-d}$, i.e. both matrices have only one nonzero diagonal, with the values being inverse of oneanother, and the diagonals being symmetric about the main diagonal.

In short, this states that there do not exist perfect reconstruction systems involving nontrivial digital filters. This sounds very bad, since filters, as we will see, represent some of the nicest operations which can be implemented. Note that, however, it may still be possible to construct such (G_1, G_2) so that $G_1 G_2$ is "close to" I. Such systems can be called "recontruction systems", and may be very important in settings where some loss in the transformation process is acceptable. We will not consider such systems here.

In a search for other perfect reconstruction systems than those given by the DFT (and DCT in the next section), we thus have to look for other matrices than those given by digital filters. In Section **??** we will see that it is possible to find such systems for the matrices we define in the next chapter, which are a natural generalization of digital filters.

Exercises for Section 3.3

Ex. 1 — Compute and plot the frequency response of the filter $S = \{1/4, \underline{1/2}, 1/4\}$. Where does the frequency response achieve its maximum and minimum value, and what are these values?

Ex. 2 — Plot the frequency response of the filter $T = \{1/4, \underline{-1/2}, 1/4\}$. Where does the frequency response achieve its maximum and minimum value, and what are these values? Can you write down a connection between this frequency response and that from Exercise 1?

Ex. 3 — Consider the two filters $S_1 = \{\underline{1}, 0, \ldots, 0, c\}$ and $S_2 = \{\underline{1}, 0, \ldots, 0, -c\}$. Both of these can be interpreted as filters which add an echo. Show that $\frac{1}{2}(S_1 + S_2) = I$. What is the interpretation of this relation in terms of echos?

Ex. 4 — In Example 1.19 we looked at time reversal as an operation on digital sound. In $\mathbb{R}^N$ this can be defined as the linear mapping which sends the vector $\boldsymbol{e}_k$ to $\boldsymbol{e}_{N-1-k}$ for all $0 \leq k \leq N-1$.

a. Write down the matrix for the time reversal linear mapping, and explain from this why time reversal is not a digital filter.
b. Prove directly that time reversal is not a time-invariant operation.

Ex. 5 — Consider the linear mapping S which keeps every second component in $\mathbb{R}^N$, i.e. $S(\boldsymbol{e}_{2k}) = \boldsymbol{e}_{2k}$, and $S(\boldsymbol{e}_{2k-1}) = \mathbf{0}$. Is S a digital filter?

Ex. 6 — A filter S_1 has the frequency response $\frac{1}{2}(1+\cos\omega)$, and another filter has the frequency response $\frac{1}{2}(1 + \cos(2\omega))$.

a. Is S_1S_2 a lowpass filter, or a highpass filter?
b. What does the filter S_1S_2 do with angular frequencies close to $\omega = \pi/2$.
c. Find the filter coefficients of S_1S_2.
   Hint: Use Theorem 3.34 to compute the frequency response of S_1S_2 first.
d. Write down the matrix of the filter S_1S_2 for $N = 8$.

Ex. 7 — Let E_{d_1} and E_{d_2} be two time delay filters. Show that $E_{d_1}E_{d_2} = E_{d_1+d_2}$ (i.e. that the composition of two time delays again is a time delay) in two different ways

a. Give a direct argument which uses no computations.
b. By using Property 3 in Theorem 3.11, i.e. by using a property for the Discrete Fourier Transform.

Ex. 8 — Let S be a digital filter. Show that S is symmetric if and only if the frequency response satisfies $s_k = s_{N-k}$ for all k.

Ex. 9 — Consider again Example 3.43. Find an expression for a filter so that only frequencies so that $|\omega - \pi| < \omega_c$ are kept, i.e. the filter should only keep angular frequencies close to π (i.e. here we construct a highpass filter).

Ex. 10 — Assume that S is a circulant Toeplitz matrix so that only

$$S_{0,0}, \ldots, S_{0,F} \text{ and } S_{0,N-E}, \ldots, S_{0,N-1}$$

are nonzero on the first row, where E, F are given numbers. When implementing this filter on a computer we need to make sure that the vector indices lie in $[0, N-1]$. Show that $y_n = (S\boldsymbol{x})_n$ can be split into the following different formulas, depending on n, to achieve this:

a. $0 \leq n < E$:

$$y_n = \sum_{k=0}^{n-1} S_{0,N+k-n} x_k + \sum_{k=n}^{F+n} S_{0,k-n} x_k + \sum_{k=N-1-E+n}^{N-1} S_{0,k-n} x_k. \quad (3.24)$$

b. $E \leq n < N - F$:

$$y_n = \sum_{k=n-E}^{n+F} S_{0,k-n} x_k. \quad (3.25)$$

c. $N - F \leq n < N$:

$$y_n = \sum_{k=0}^{n-(N-F)} S_{0,k-n} x_k + \sum_{k=n-E}^{n-1} S_{0,N+k-n} x_k + \sum_{k=n}^{N-1} S_{0,k-n} x_k. \quad (3.26)$$

These three cases give us the full implementation of the filter. This implementation is often more useful than writing down the entire matrix S, since we save computation when many of the matrix entries are zero.

Ex. 11 — In this exercise we will find out how to keep to track of the length and the start and end indices when we convolve two sequences

a. Let $\boldsymbol{g}$ and $\boldsymbol{h}$ be two sequences with finitely many nonzero elements. Show that $\boldsymbol{g}*\boldsymbol{h}$ also has finitely many nonzero elements, and show that $l(\boldsymbol{g}*\boldsymbol{h}) = l(\boldsymbol{g}) + l(\boldsymbol{h})$.

b. Find expressions for the values k_{min}, k_{max} for the filter $\boldsymbol{g} * \boldsymbol{h}$, in terms of those for the filters $\boldsymbol{g}$ and $\boldsymbol{h}$.

Ex. 12 — Write a function

```
function [gconvh gconvhmin]=filterimpl(g,gmin,h,hmin)
```

which performs the convolution of two sequences, but also keeps track of the index of the smallest nonzero coefficient in the sequences.

Ex. 13 — Consider the matrix

$$S = \begin{pmatrix} 4 & 1 & 3 & 1 \\ 1 & 4 & 1 & 3 \\ 3 & 1 & 4 & 1 \\ 1 & 4 & 1 & 3 \end{pmatrix}.$$

a. Compute the eigenvalues and eigenvectors of S using the results of this section. You should only need to perform one DFT in order to achieve this.
b. Verify the result from a. by computing the eigenvectors and eigenvalues the way you taught in your first course in linear algebra. This should be a much more tedious task.
c. Use Matlab to compute the eigenvectors and eigenvalues of S also. For some reason some of the eigenvectors seem to be different from the Fourier basis vectors, which you would expect from the theory in this section. Try to find an explanation for this.

Ex. 14 — Define the filter S by $S = \{1, 2, \underline{3}, 4, 5, 6\}$. Write down the matrix for S when $N = 8$. Plot (the magnitude of) $\lambda_S(\omega)$, and indicate the values $\lambda_{S,n}$ for $N = 8$ in this plot.

Ex. 15 — Assume that the filter S is defined by the formula

$$z_n = \frac{1}{4}x_{n+1} + \frac{1}{4}x_n + \frac{1}{4}x_{n-1} + \frac{1}{4}x_{n-2}.$$

Write down the matrix for S when $N = 8$. Compute and plot (the magnitude of) $\lambda_S(\omega)$.

Ex. 16 — Given the circulant Toeplitz matrix

$$S = \frac{1}{5}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

Write down the filter coefficients of this matrix, and use the compact notation $\{t_{k_{min}}, \ldots, t_{-1}, \underline{t_0}, t_1, \ldots, t_{k_{max}}\}$. Compute and plot (the magnitude) of $\lambda_S(\omega)$.

Ex. 17 — Assume that $S = \{\underline{1}, c, c^2, \ldots, c^k\}$. Compute and plot $\lambda_S(\omega)$ when $k = 4$ and $k = 8$. How does the choice of k influence the frequency response? How does the choice of c influence the frequency response?

Ex. 18 — Assume that S_1 and S_2 are two circulant Toeplitz matrices.

a. How can you express the eigenvalues of S_1+S_2 in terms of the eigenvalues of S_1 and S_2?
b. How can you express the eigenvalues of S_1S_2 in terms of the eigenvalues of S_1 and S_2?
c. If A and B are general matrices, can you find a formula which expresses the eigenvalues of $A + B$ and AB in terms of those of A and B? If not, can you find a counterexample to what you found in a. and b.?

Ex. 19 — In this exercise we will investigate how we can combine lowpass and highpass filters to produce other filters

a. Assume that S_1 and $S2$ are lowpass filters. What kind of filter is S_1S_2? What if both S_1 and S_2 are highpass filters?
b. Assume that one of S_1, S_2 is a highpass filter, ans that the other is a lowpass filter. What kind of filter S_1S_2 in this case?

3.4 Symmetric digital filters and the DCT

We have mentioned that most periodic functions can be approximated well by Fourier series on the form $\sum_n y_n e^{2\pi int/T}$. The convergence speed of this Fourier series may be slow however, and we mentioned that it depends on the regularity of the function. In particular, the convergence speed is higher when the function is continuous. For $f \in C[0, T]$ where $f(0) \neq f(T)$, we do not get a continuous function if we repeat the function in periods. However, we demonstrated that we could create a symmetric extension g, defined on $[0, 2T]$, so that $g(0) = g(2T)$. The Fourier series of g actually took the form of a cosine-series, and we saw that this series converged faster to g, than the Fourier series of f did to f.

In this section we will specialize this argument to vectors: by defining the symmetric extension of a vector, we will attempt to find a better approximation than we could with the Fourier basis vectors. This approach is useful for digital filters also, if the filters preserve these symmetric extensions. Let us summarize with the following idea:

Idea 3.55 (Increasing the convergence speed of the DFT)**.** Assume that we have a function f, and that we take the samples $x_k = f(kT/N)$. From $\boldsymbol{x} \in \mathbb{R}^N$ we would like to create an extension $\breve{\boldsymbol{x}}$ so that the first and last values are equal. For such an extension, the Fourier basis vectors can give a very good approximation to f.

As our candidate for the extension of $\boldsymbol{x}$, we will consider the following:

Definition 3.56 (Symmetric extension of a vector)**.** By the symmetric extension of $\boldsymbol{x} \in \mathbb{R}^N$, we mean $\breve{\boldsymbol{x}} \in \mathbb{R}^{2N}$ defined by

$$\breve{\boldsymbol{x}}_k = \begin{cases} x_k & 0 \leq k < N \\ x_{2N-1-k} & N \leq k < 2N-1 \end{cases} \tag{3.27}$$

We say that a vector in $\mathbb{R}^{2N}$ is symmetric if it can be written as the symmetric extension of a vector in $\mathbb{R}^N$.

The symmetric extension $\breve{\boldsymbol{x}}$ thus has the original vector $\boldsymbol{x}$ as its first half, and a copy of $\boldsymbol{x}$ in reverse order as its second half. Clearly, the first and last values of $\breve{\boldsymbol{x}}$ are equal. In other words, a vector in $\mathbb{R}^{2N}$ is a symmetric extension of a vector in $\mathbb{R}^N$ if and only if it is symmetric about $N - \frac{1}{2}$. Clearly also the set of symmetric extensions is a vector space. Our idea in terms of filters is the following:

Idea 3.57 (Increasing the precision of a digital filter)**.** If a filter maps a symmetric extension of one vector to a symmetric extension of another vector then it is a good approximation of an analog version in terms of Fourier series.

We will therefore be interested in finding filters which preserves symmetric extensions. We will show the following, which characterize such filters:

Theorem 3.58 (Characterization of filters which preserve symmetric extensions)**.** A digital filter S of size $2N$ preverves symmetric extensions if and only if S is symmetric. Moreover, S is uniquely characterized by its restriction to $\mathbb{R}^N$, denoted by S_r, which is given by $S_1 + (S_2)^f$, where $S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$, and where $(S_2)^f$ is the matrix S_2 with the columns reversed. Moreover, if we define

$$d_{n,N} = \begin{cases} \sqrt{\frac{1}{N}} & , n = 0 \\ \sqrt{\frac{2}{N}} & , 1 \leq n < N \end{cases}$$

and $\boldsymbol{d}_n = d_{n,N} \cos\left(2\pi \frac{n}{2N}\left(k + \frac{1}{2}\right)\right)$ for $0 \leq n \leq N-1$, then $\{\boldsymbol{d}_0, \boldsymbol{d}_1, \ldots, \boldsymbol{d}_{N-1}\}$ is an orthonormal basis of eigenvectors for S_r.

Proof. Let $\boldsymbol{z} = \hat{\breve{\boldsymbol{x}}}$. Since $\breve{\boldsymbol{x}}$ is symmetric about $N - \frac{1}{2}$, by Theorem 3.11 it follows that the argument of z_n is $-2\pi(N - \frac{1}{2})n/(2N)$. Since z_{2N-n} is the conjugate of z_n by the same theorem, it follows that $z_{2N-n} = e^{4\pi i(N-\frac{1}{2})n/(2N)} z_n = e^{-2\pi i n/(2N)} z_n$. It follows that a vector in $\mathbb{R}^{2N}$ is a symmetric extension if and only if its DFT is in the span of the vectors

$$\{\boldsymbol{e}_0, \{\boldsymbol{e}_n + e^{-2\pi i n/(2N)} \boldsymbol{e}_{2N-n}\}_{n=1}^{N-1}\}.$$

These vectors are clearly orthogonal. Their span can also be written as the span of the vectors

$$\left\{\boldsymbol{e}_0, \left\{\frac{1}{\sqrt{2}}\left(e^{\pi i n/(2N)}\boldsymbol{e}_n + e^{-\pi i n/(2N)}\boldsymbol{e}_{2N-n}\right)\right\}_{n=1}^{N-1}\right\}, \tag{3.28}$$

where the last vectors have been first multiplied with $e^{\pi i n/(2N)}$, and then normalized so that they have norm 1. Equation (3.28) now gives us an orthononormal basis for the DFT's of all symmetric extensions. Let us map these vectors back with the IDFT. We get first that

$$(F_{2N})^H(\boldsymbol{e}_0) = \left(\frac{1}{\sqrt{2N}}, \frac{1}{\sqrt{2N}}, \ldots, \frac{1}{\sqrt{2N}}\right) = \frac{1}{\sqrt{2N}} \cos\left(2\pi \frac{0}{2N}\left(k + \frac{1}{2}\right)\right).$$

We also get that

$$\begin{aligned}
&(F_{2N})^H\left(\frac{1}{\sqrt{2}}\left(e^{\pi i n/(2N)}\boldsymbol{e}_n + e^{-\pi i n/(2N)}\boldsymbol{e}_{2N-n}\right)\right)\\
&= \frac{1}{\sqrt{2}}\left(e^{\pi i n/(2N)}\frac{1}{\sqrt{2N}}e^{2\pi i n k/(2N)} + e^{-\pi i n/(2N)}\frac{1}{\sqrt{2N}}e^{2\pi i(2N-n)k/(2N)}\right)\\
&= \frac{1}{\sqrt{2}}\left(e^{\pi i n/(2N)}\frac{1}{\sqrt{2N}}e^{2\pi i n k/(2N)} + e^{-\pi i n/(2N)}\frac{1}{\sqrt{2N}}e^{-2\pi i n k/(2N)}\right)\\
&= \frac{1}{2\sqrt{N}}\left(e^{2\pi i(n/(2N))(k+1/2)} + e^{-2\pi i(n/(2N))(k+1/2)}\right)\\
&= \frac{1}{\sqrt{N}}\cos\left(2\pi\frac{n}{2N}\left(k + \frac{1}{2}\right)\right).
\end{aligned}$$

Since F_{2N} is unitary, and thus preserves the scalar product, this means that

$$\left\{\frac{1}{\sqrt{2N}}\cos\left(2\pi\frac{0}{2N}\left(k+\frac{1}{2}\right)\right), \left\{\frac{1}{\sqrt{N}}\cos\left(2\pi\frac{n}{2N}\left(k+\frac{1}{2}\right)\right)\right\}_{n=1}^{N-1}\right\} \tag{3.29}$$

is an orthonormal basis for the set of symmetric extensions in $\mathbb{R}^{2N}$. We have that

$$\begin{aligned}
&S\left(\cos\left(2\pi\frac{n}{2N}\left(k+\frac{1}{2}\right)\right)\right)\\
&=S\left(\frac{1}{2}\left(e^{2\pi i(n/(2N))(k+1/2)}+e^{-2\pi i(n/(2N))(k+1/2)}\right)\right)\\
&=\frac{1}{2}\left(e^{\pi in/(2N)}S\left(e^{2\pi ink/(2N)}\right)+e^{-\pi in/(2N)}S\left(e^{-2\pi ink/(2N)}\right)\right)\\
&=\frac{1}{2}\left(e^{\pi in/(2N)}\lambda_{S,n}e^{2\pi ink/(2N)}+e^{-\pi in/(2N)}\lambda_{S,2N-n}e^{-2\pi ink/(2N)}\right)\\
&=\frac{1}{2}\left(\lambda_{S,n}e^{2\pi i(n/(2N))(k+1/2)}+\lambda_{S,2N-n}e^{-2\pi i(n/(2N))(k+1/2)}\right)
\end{aligned}$$

where we have used that $e^{2\pi ink/(2N)}$ is an eigenvector of S with eigenvalue $\lambda_{S,n}$, and $e^{-2\pi ink/(2N)}=e^{2\pi i(2N-n)k/(2N)}$ is an eigenvector of S with eigenvalue $\lambda_{S,2N-n}$. If S preserves symmetric extensions, we see that we must have that $\lambda_{S,n}=\lambda_{S,2N-n}$, and also that the vectors listed in Equation (3.29) must be eigenvectors for S. This is reflected in that the entries in D in the diagonalization $S=(F_{2N})^H DF_{2N}$ are symmetric about the midpoint on the diagonal. From Exercise 3.3.8 we know that this occurs if and only if S is symmetric, which proves the first part of the theorem.

Since S preserves symmetric extensions it is clearly characterized by its restriction to the first N elements. With $S=\begin{pmatrix} S_1 & S_2\\ S_3 & S_4\end{pmatrix}$, we compute this restriction as

$$\begin{pmatrix} y_0\\ \vdots\\ y_{N-1}\end{pmatrix}=\begin{pmatrix} S_1 & S_2\end{pmatrix}\begin{pmatrix} x_0\\ \vdots\\ x_{N-1}\\ \hline x_N\\ \vdots\\ x_{2N-1}\end{pmatrix}=\begin{pmatrix} S_1+(S_2)^f\end{pmatrix}\begin{pmatrix} x_0\\ \vdots\\ x_{N-1}\end{pmatrix}.$$

S_2 contains the circulant part of the matrix, and forming $(S_2)^f$ means that the circulant parts switch corners. This shows that S is uniquely characterized by the matrix S_r as defined in the text of the theorem. Finally, since (3.29) are eigenvectors of S, the vectors in $\mathbb{R}^N$ restricted to their first N elements are eigenvectors for S_r. Since the scalar product of two symmetric extensions is the double of the scalar product of the first half of the vectors, we have that these vectors must also be orthogonal, and that

$$\left\{\frac{1}{\sqrt{N}}\cos\left(2\pi\frac{0}{2N}\left(k+\frac{1}{2}\right)\right),\left\{\sqrt{\frac{2}{N}}\cos\left(2\pi\frac{n}{2N}\left(k+\frac{1}{2}\right)\right)\right\}_{n=1}^{N-1}\right\}$$

is an orthonormal basis of eigenvectors for S_r. We see that we now can define $d_{n,N}$ and the vectors $\boldsymbol{d}_n$ as in the text of the theorem, and this completes the proof. □

From the proof we clearly see the analogy between symmetric functions and vectors: while the first can be written as a cosine-series, the second can be written as a sum of cosine-vectors:

Corollary 3.59.

$$\left\{\frac{1}{\sqrt{2N}}\cos\left(2\pi\frac{0}{2N}\left(k+\frac{1}{2}\right)\right), \left\{\frac{1}{\sqrt{N}}\cos\left(2\pi\frac{n}{2N}\left(k+\frac{1}{2}\right)\right)\right\}_{n=1}^{N-1}\right\}$$

form an orthonormal basis for the set of all symmetric vectors in $\mathbb{R}^{2N}$.

Note also that S_r is not a circulant matrix. Therefore, its eigenvectors are not pure tones. An example should clarify this:

Example 3.60. Consider the averaging filter $\boldsymbol{g} = \{\frac{1}{4}, \underline{\frac{1}{2}}, \frac{1}{4}\}$. Let us write down the matrix S_r for the case when $N = 4$. First we obtain the matrix S as

$$\left(\begin{array}{cccc|cccc}
\frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2}
\end{array}\right)$$

where we have drawn the boundaries between the blocks S_1, S_2, S_3, S_4. From this we see that

$$S_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \quad S_2 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 \end{pmatrix} \quad (S_2)^f = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

From this we get

$$S_r = S_1 + (S_2)^f = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

The orthogonal basis we have found is given its own name:

Definition 3.61 (DCT basis)**.** We denote by $\mathcal{D}_N$ the orthogonal basis $\{\boldsymbol{d}_0, \boldsymbol{d}_1, \ldots, \boldsymbol{d}_{N-1}\}$. We also call $\mathcal{D}_N$ the N-point *DCT* basis.

Using the DCT basis instead of the Fourier basis we can make the following definitions, which parallel those for the DFT:

Definition 3.62 (Discrete Cosine Transform)**.** The change of coordinates from the standard basis of $\mathbb{R}^N$ to the DCT basis $\mathcal{D}_N$ is called the *discrete cosine transform* (or DCT). The $N \times N$ matrix D_N that represents this change of basis is called the (N-point) DCT matrix. If $\boldsymbol{x}$ is a vector in R^N, its coordinates $\boldsymbol{y} = (y_0, y_1, \ldots, y_{N-1})$ relative to the DCT basis are called the DCT coefficients of $\boldsymbol{x}$ (in other words, $\boldsymbol{y} = D_N\boldsymbol{x}$).

As with the Fourier basis vectors, the DCT basis vectors are called synthesis vectors, since we can write

$$\boldsymbol{x} = y_0\boldsymbol{d}_0 + y_1\boldsymbol{d}_1 + \cdots + y_{N-1}\boldsymbol{d}_{N-1} \tag{3.30}$$

in the same way as for the DFT. Following the same reasoning as for the DFT, D_N^{-1} is the matrix where the $\boldsymbol{d}_n$ are columns. But since these vectors are real and orthonormal, D_N must be the matrix where the $\boldsymbol{d}_n$ are rows. Moreover, since Theorem 3.58 also states that the same vectors are eigenvectors for filters which preserve symmetric extensions, we can state the following:

Theorem 3.63. D_N is the orthogonal matrix where the rows are $\boldsymbol{d}_n$. Moreover, for any digital filter S which preserves symmetric extensions, $(D_N)^T$ diagonalizes S_r, i.e. $S_r = D_N^T D D_N$ where D is a diagonal matrix.

Let us also make the following definition:

Definition 3.64 (IDCT)**.** We will call $\boldsymbol{x} = (D_N)^T\boldsymbol{y}$ the inverse DCT or (IDCT) of $\boldsymbol{x}$.

Example 3.65. As with Example 3.9, exact expressions for the DCT can be written down just for a few specific cases. It turns out that the case $N = 4$ as considered in Example 3.9 does not give the same type of nice, exact values, so let us instead consider the case $N = 2$. We have that

$$D_4 = \begin{pmatrix} \frac{1}{\sqrt{2}}\cos(0) & \frac{1}{\sqrt{2}}\cos(0) \\ \cos\left(\frac{\pi}{2}\left(0+\frac{1}{2}\right)\right) & \cos\left(\frac{\pi}{2}\left(1+\frac{1}{2}\right)\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

The DCT of the same vector as in Example 3.9 can now be computed as:

$$D_2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Example 3.66. A direct implementation of the DCT could be made as follows:

```
function y=DCTImpl(x)
  N=length(x);
  DN=zeros(N);
  DN(1,:)=ones(1,N)/sqrt(N);
```

```
for n=1:N
  DN(n,:)=cos(2*pi*((n-1)/(2*N))*((0:N-1)+1/2))*sqrt(2/N);
end
y=DN*x;
```

In the next chapter we will see that one also can make a much more efficient implementation of the DCT than this.

With the DCT one constructs a vector twice as long. One might think due to this that one actually use matrices twice the size. This is, however, avoided since D_N has the same dimensions as F_N, and we will see shortly that the same algorithms as for the DFT also can be used for the DCT. By construction we also see that it is easy to express the N-point DCT in terms of the $2N$-point DFT. Let us write down this connection:

1. Write $e_0 = \sqrt{2}$ and $e_n = e^{\pi i n/(2N)}$ for $n \neq 0$, and define E as the diagonal matrix with the values $e_0, e_1, \ldots$ on the diagonal.
2. Let B be the $2N \times N$-matrix which is nonzero only when $i = j$ or $i + j = 2N - 1$, and 1 in all these places.
3. Let also A be the $N \times 2N$-matrix with 1 on the diagonal.

We can now write

$$D_N = E^{-1} A F_{2N} B. \tag{3.31}$$

A here extracts the first rows of the matrix, E^{-1} eliminates the complex coefficients, while B adds columns symmetrically. This factorization enables us to use the efficient FFT-implementation, since the matrices A, B, E all are sparse. We will, however, find an even more efficient implementation of the DCT, which will avoid computing a DFT of twice the size as here.

Similarly to Theorem 3.16 for the DFT, one can think of the DCT as a least squares approximation and the unique representation of a function having the same sample values, but this time in terms of sinusoids instead of complex exponentials:

Theorem 3.67 (Interpolation with the DCT basis)**.** Let f be a function defined on the interval $[0, T]$, and let $\boldsymbol{x}$ be the sampled vector given by

$$x_k = f((2k+1)T/(2N)) \quad \text{for } k = 0, 1, \ldots, N-1.$$

There is exactly one linear combination $g(t)$ on the form

$$\sum_{n=0}^{N-1} y_n d_{n,N} \cos(2\pi(n/2)t/T)$$

which satisfies the conditions

$$g((2k+1)T/(2N)) = f((2k+1)T/(2N)), \quad k = 0, 1, \ldots, N-1,$$

and its coefficients are determined by $\boldsymbol{y} = D_N \boldsymbol{x}$.

The proof for this follows by inserting $t = (2k+1)T/(2N)$ in the equation $g(t) = \sum_{n=0}^{N-1} y_n d_{n,N} \cos(2\pi(n/2)t/T)$ to arrive at the equations

$$f(kT/N) = \sum_{n=0}^{N-1} y_n d_{n,N} \cos\left(2\pi \frac{n}{2N}\left(k + \frac{1}{2}\right)\right) \qquad 0 \le k \le N-1.$$

This gives us an equation system for finding the y_n with the invertible DCT matrix as coefficient matrix, and the result follows.

Note the subtle change in the sample points of these cosine functions, from kT/N for the DFT, to $(2k+1)T/(2N)$ for the DCT. The sample points for the DCT are thus the midpoints on the intervals in a uniform partition of $[0, T]$ into N intervals, while they for the DFT are the start points on the intervals. Also, the frequencies are divided by 2. In Figure 3.9 we have plotted the sinusoids of Theorem 3.67 for $T = 1$, as well as the sample points used in that theorem. The sample points in (a) correspond to the first column in the DCT matrix, the sample points in (b) to the second column of the DCT matrix, and so on (up to normalization with $d_{n,N}$). As n increases, the functions oscillate more and more. As an example, y_5 says how much content of maximum oscillation there is. In other words, the DCT of an audio signal shows the proportion of the different frequencies in the signal, and the two formulas $\boldsymbol{y} = D_N\boldsymbol{x}$ and $\boldsymbol{x} = (D_N)^T\boldsymbol{y}$ allow us to switch back and forth between the time domain representation and the frequency domain representation of the sound. In other words, once we have computed $\boldsymbol{y} = D_N\boldsymbol{x}$, we can analyse the frequency content of $\boldsymbol{x}$. If we want to reduce the bass we can decrease the $\boldsymbol{y}$-values with small indices and if we want to increase the treble we can increase the $\boldsymbol{y}$-values with large indices.

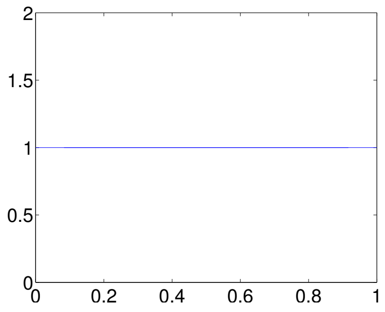
3.4.1 Other types of symmetric extensions

Note that our definition of symmetric extension duplicates the values x_0 and x_{N-1}: both are repeated when creating the symmetric extension. This is in fact unnecessary when we are creating a longer vector which has equal first and last values, and is primarily motivated from existing efficient implementations for the DCT when all vector lengths are powers of 2. When an efficient DCT implementation is not important, we can change the definition of the symmetric extension as follows (it is this type of symmetric extension we will use later):

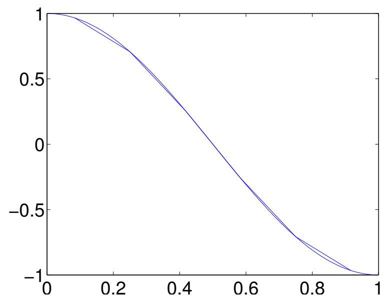
Definition 3.68 (Symmetric extension of a vector)**.** By the symmetric extension of $\boldsymbol{x} \in \mathbb{R}^N$, we mean $\breve{\boldsymbol{x}} \in \mathbb{R}^{2N-2}$ defined by

$$\breve{\boldsymbol{x}}_k = \begin{cases} x_k & 0 \le k < N \\ x_{2N-2-k} & N \le k < 2N-3 \end{cases} \tag{3.32}$$

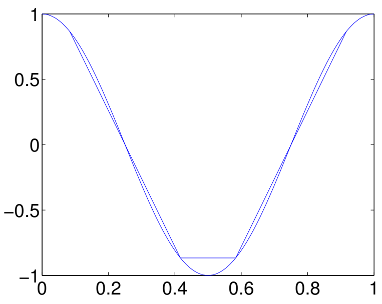
In other words, a vector in $\mathbb{R}^{2N}$ is a symmetric extension of a vector in $\mathbb{R}^N$ if and only if it is symmetric about $N-1$. Theorem 3.58 now instead takes the following form:



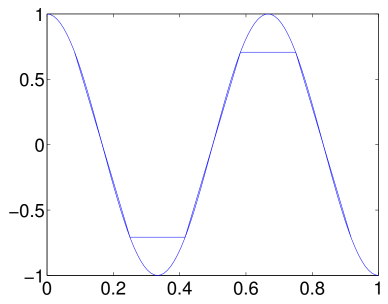
(a) $\cos(2\pi(0/2)t)$



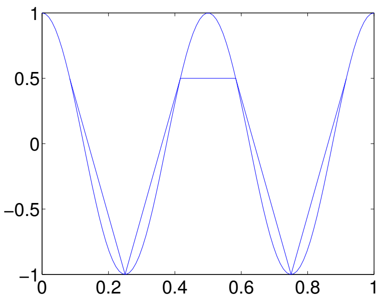
(b) $\cos(2\pi(1/2)t)$



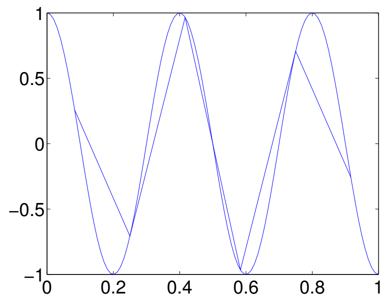
(c) $\cos(2\pi(2/2)t)$



(d) $\cos(2\pi(3/2)t)$



(e) $\cos(2\pi(4/2)t)$



(f) $\cos(2\pi(5/2)t)$

Figure 3.9: The 6 different sinusoids used in DCT for $N = 6$, i.e. $\cos(2\pi(n/2)t)$, $0 \leq n < 6$. The plots also show piecewise linear functions between the the sample points $\frac{2k+1}{2N}$ $0 \leq k < 6$, since only the values at these points are used in Theorem 3.67.

Theorem 3.69 (Characterization of filters which preserve symmetric extensions)**.** A real, circulant $(2N-2)\times(2N-2)$-Toeplitz matrix preverves symmetric extensions if and only if it is symmetric. For such S, S is uniquely characterized by its restriction to $\mathbb{R}^N$, denoted by S_r, which is given by $T_1 + \begin{pmatrix} 0 & (T_2)^f & 0 \end{pmatrix}$, where $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$, where T_1 is $N\times N$, T_2 is $N\times(N-2)$. Moreover, an orthogonal basis of eigenvectors for S_r are $\{\cos\left(2\pi\frac{n}{2N}n\right)\}_{n=0}^{N-1}$.

Proof. Let $\boldsymbol{z} = \hat{\breve{\boldsymbol{x}}}$. Since $\breve{\boldsymbol{x}}$ is symmetric about 0, by Theorem 3.11 it follows that $\boldsymbol{z}_n = \boldsymbol{z}_{2(N-1)-n)}$, so that the DFT of a symmetrix extension (as now defined) is in the span of the vectors

$$\{\boldsymbol{e}_0, \{\boldsymbol{e}_n + \boldsymbol{e}_{2(N-1)-n}\}_{n=1}^{N-1}\}.$$

It follows as before that

$$\cos\left(2\pi\frac{n}{2(N-1)}n\right)$$

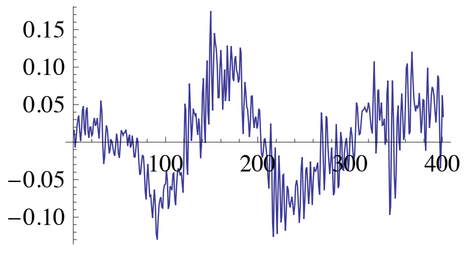
is a basis of eigenvectors. The same type of symmetry about the midpoint on the diagonal follows as before, which as before is equivalent to symmetry of the matrix. □

It is not customary to write down an orthonormal basis for the eigenvectors in this case, since we don't have the same type of efficient DCT implementation due to the lact of powers of 2.

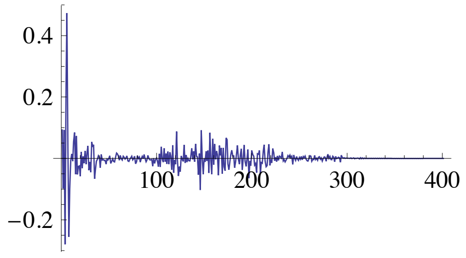
3.4.2 Use of DCT in lossy compression of sound

The DCT is particularly popular for processing the sound before compression. MP3 is based on applying a variant of the DCT (called the Modified Discrete Cosine Transform, MDCT) to groups of 576 (in special circumstances 192) samples. One does not actually apply the DCT directly. Rather one applies a much more complex transformation, which can be implemented in parts by using DCT in a clever way.

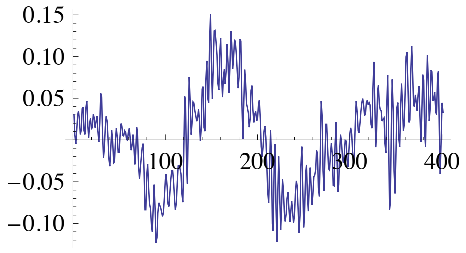
We mentioned previously that we could achieve compression by setting the Fourier coefficients which are small to zero. Translated to the DCT, we should set the DCT coefficients which are small to zero, and we apply the inverse DCT in order to reconstruct the signal in order to play it again. Let us test compression based on this idea. The plots in figure 3.10 illustrate the principle. A signal is shown in (a) and its DCT in (b). In (d) all values of the DCT with absolute value smaller than 0.02 have been set to zero. The signal can then be reconstructed with the inverse DCT of theorem **??**; the result of this is shown in (c). The two signals in (a) and (c) visually look almost the same even though the signal in (c) can be represented with less than 25 % of the information present in (a).



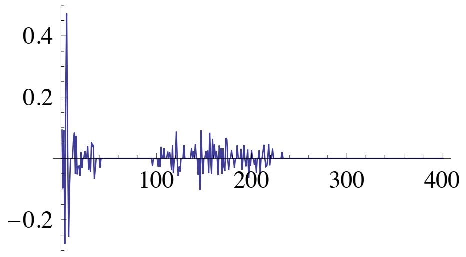
(a)



(b)



(c)



(d)

Figure 3.10: The signal in (a) are the sound samples from a small part of a song. The plot in (b) shows the DCT of the signal. In (d), all values of the DCT that are smaller than 0.02 in absolute value have been set to 0, a total of 309 values. In (c) the signal has been reconstructed from these perturbed values of the DCT. The values have been connected by straight lines to make it easier to interpret the plots.

We test this compression strategy on a data set that consists of 300 001 points. We compute the DCT and set all values smaller than a suitable tolerance to 0. With a tolerance of 0.04, a total of 142 541 values are set to zero. When we then reconstruct the sound with the inverse DCT, we obtain a signal that differs at most 0.019 from the original signal. To verify that the new file is not too different from the old file, we can take the read sound samples from `castanets.wav`, run the following function for different `eps`

```
function A=skipsmallvals(eps,A)
  B=dct(A);
  B=(B>=eps).*B;
  A=invdct(B);
```

and play the new samples. Finally we can store the signal by storing a `gzip`'ed version of the DCT-values (as 32-bit floating-point numbers) of the perturbed signal. This gives a file with 622 551 bytes, which is 88 % of the `gzip`'ed version of the original data.

The choice of the DCT in the MP3 standard has much to do with that the DCT, just as the DFT, has a very efficient implementation, as we will see next.

Exercises for Section 3.4

Ex. 1 — In Section 3.4.2 we implemented the function `skipsmallvals`, which ran a DCT on the entire vector. Explain why there are less computation involved in splitting the vector into many parts and performing a DCT for each part. Change the code accordingly.

Ex. 2 — As in Example 3.65, state the exact cartesian form of the DCT matrix for the case $N = 3$.

Ex. 3 — Assume that S is a symmetric digital filter with support $[-E, E]$. Let us, as in Exercise 3.3.10, see how we can make sure that the indices keep inside $[0, N-1]$. Show that $z_n = (T\boldsymbol{x})_n$ in this case can be split into the following different formulas, depending on n:

a. $0 \le n < E$:

$$z_n = T_{0,0}x_n + \sum_{k=1}^{n} T_{0,k}(x_{n+k} + x_{n-k}) + \sum_{k=n+1}^{E} T_{0,k}(x_{n+k} + x_{n-k+N}). \quad (3.33)$$

b. $E \le n < N - E$:

$$z_n = T_{0,0}x_n + \sum_{k=1}^{E} T_{0,k}(x_{n+k} + x_{n-k}). \quad (3.34)$$

c. $N - F \leq n < N$:

$$z_n = T_{0,0}x_n + \sum_{k=1}^{N-1-n} T_{0,k}(x_{n+k}+x_{n-k}) + \sum_{k=N-1-n+1}^{E} T_{0,k}(x_{n+k-N}+x_{n-k}). \tag{3.35}$$

Ex. 4 — Assume that $\{T_{0,-E}, \ldots, T_{0,0}, \ldots, T_{0,E}\}$ are the coefficicients of a symmetric, digital filter S, and let $\boldsymbol{t} = \{T_{0,1}, \ldots, T_{0,E}\}$. Write a function

```
function z=filterT(t,x)
```

which takes the vector $\boldsymbol{t}$ as input, and returns $\boldsymbol{z} = T\boldsymbol{x}$ using the formulas deduced in Exercise 3.

Ex. 5 — Repeat Exercise 1.4.9 by reimplementing the functions `reducetreble` and `reducesbass` using the function `filterT` from the previous exercise. The resulting sound files should sound the same, since the only difference is that we have modified the way we handle the beginning and end portion of the sound samples.

Ex. 6 — Using Python, define a class `Transform` with methods `transformImpl` and `ItransformImpl`. Define two subclasses of `Transform`, `DCTTransform`, `FFTTransform`), which implements these two functions by calling Python counterparts of `FFTImpl`, `IFFTImpl`, `DCTImpl`, and `IDCTImpl`.

3.5 Summary

We defined the Discrete Fourier transform, which could be thought of as the Fourier series of a vector. We exploited properties of the DFT, which corresponded nicely to the corresponding properties for Fourier series. We defined digital filters, which turned out to be linear transformations diagonalized by the DFT. Also we showed that the techniques from the last section we used to speed up the convergence of the Fourier series, could also be used for the DFT. In this way we arrived at the definition of the DCT.

Chapter 4

Implementation of the DFT and the DCT

The main application of the DFT and the DCT is as tools to compute frequency information in large datasets. It is therefore important that these operations can be performed by efficient algorithms. Straightforward implementation from the definition is not efficient if the data sets are large. However, it turns out that the underlying matrices may be factored in a way that leads to much more efficient algorithms, and this is the topic of the present chapter.

4.1 The Fast Fourier Transform (FFT)

In this section we will discuss the most widely used implementation of the DFT, which is usually referred to as the Fast Fourier Transform (FFT). For simplicity, we will assume that N, the length of the vector that is to be transformed by the DFT, is a power of 2. In this case it is relatively easy to simplify the DFT algorithm via a factorisation of the Fourier matrix. The foundation is provided by a simple reordering of the DFT.

Theorem 4.1 (FFT algorithm)**.** Let $\boldsymbol{y} = F_N\boldsymbol{x}$ be the N-point DFT of $\boldsymbol{x}$ with N an even number. Foran any integer n in the interval $[0, N/2-1]$ the DFT $\boldsymbol{y}$ of $\boldsymbol{x}$ is then given by

$$y_n = \frac{1}{\sqrt{2}}\left((F_{N/2}\boldsymbol{x}^{(e)})_n + e^{-2\pi in/N}(F_{N/2}\boldsymbol{x}^{(o)})_n\right), \tag{4.1}$$

$$y_{N/2+n} = \frac{1}{\sqrt{2}}\left((F_{N/2}\boldsymbol{x}^{(e)})_n - e^{-2\pi in/N}(F_{N/2}\boldsymbol{x}^{(o)})_n\right), \tag{4.2}$$

where $\boldsymbol{x}^{(e)}, \boldsymbol{x}^{(o)}$ are the sequences of length $N/2$ consisting of the even and odd samples of $\boldsymbol{x}$, respectively. In other words,

$$(\boldsymbol{x}^{(e)})_k = x_{2k} \text{ for } 0 \leq k \leq N/2-1,$$
$$(\boldsymbol{x}^{(o)})_k = x_{2k+1} \text{ for } 0 \leq k \leq N/2-1.$$

Put differently, the formulas (4.1)–(4.2) reduces the computation of an N-point DFT to 2 $N/2$-point DFT's. It turns out that this can speed up computations considerably, but let us first check that these formulas are correct.

Proof. Suppose first that $0 \leq n \leq N/2-1$. We start by splitting the sum in the expression for the DFT into even and odd indices,

$$\begin{aligned}
y_n &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k e^{-2\pi ink/N} \\
&= \frac{1}{\sqrt{N}} \sum_{k=0}^{N/2-1} x_{2k} e^{-2\pi in2k/N} + \frac{1}{\sqrt{N}} \sum_{k=0}^{N/2-1} x_{2k+1} e^{-2\pi in(2k+1)/N} \\
&= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{N/2}} \sum_{k=0}^{N/2-1} x_{2k} e^{-2\pi ink/(N/2)} \\
&\qquad\qquad + e^{-2\pi in/N} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{N/2}} \sum_{k=0}^{N/2-1} x_{2k+1} e^{-2\pi ink/(N/2)} \\
&= \frac{1}{\sqrt{2}} \left(F_{N/2} \boldsymbol{x}^{(e)} \right)_n + \frac{1}{\sqrt{2}} e^{-2\pi in/N} \left(F_{N/2} \boldsymbol{x}^{(o)} \right)_n,
\end{aligned}$$

where we have substituted $\boldsymbol{x}^{(e)}$ and $\boldsymbol{x}^{(o)}$ as in the text of the theorem, and recognized the $N/2$-point DFT in two places. For the second half of the DFT coefficients, i.e. $\{y_{N/2+n}\}_{0 \leq n \leq N/2-1}$, we similarly have

$$\begin{aligned}
y_{N/2+n} &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k e^{-2\pi i(N/2+n)k/N} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k e^{-\pi ik} e^{-2\pi ink/N} \\
&= \frac{1}{\sqrt{N}} \sum_{k=0}^{N/2-1} x_{2k} e^{-2\pi in2k/N} - \frac{1}{\sqrt{N}} \sum_{k=0}^{N/2-1} x_{2k+1} e^{-2\pi in(2k+1)/N} \\
&= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{N/2}} \sum_{k=0}^{N/2-1} x_{2k} e^{-2\pi ink/(N/2)} \\
&\qquad\qquad - e^{-2\pi in/N} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{N/2}} \sum_{k=0}^{N/2-1} x_{2k+1} e^{-2\pi ink/(N/2)} \\
&= \frac{1}{\sqrt{2}} \left(F_{N/2} \boldsymbol{x}^{(e)} \right)_n - \frac{1}{\sqrt{2}} e^{-2\pi in/N} \left(F_{N/2} \boldsymbol{x}^{(o)} \right)_n.
\end{aligned}$$

This concludes the proof. $\square$

It turns out that Theorem 4.1 can be interpreted as a matrix factorization. For this we need to define the concept of a block matrix.

Definition 4.2. Let $m_0, \ldots, m_{r-1}$ and $n_0, \ldots, n_{s-1}$ be integers, and let $A^{(i,j)}$ be an $m_i \times n_j$-matrix for $i = 0, \ldots, r-1$ and $j = 0, \ldots, s-1$. The notation

$$A = \left(\begin{array}{c|c|c|c} A^{(0,0)} & A^{(0,1)} & \cdots & A^{(0,s-1)} \\ \hline A^{(1,0)} & A^{(1,1)} & \cdots & A^{(1,s-1)} \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline A^{(r-1,0)} & A^{(r-1,1)} & \cdots & A^{(r-1,s-1)} \end{array}\right)$$

denotes the $(m_0 + m_1 + \ldots + m_{r-1}) \times (n_0 + n_1 + \ldots + n_{s-1})$-matrix where the matrix entries occur as in the $A^{(i,j)}$ matrices, in the way they are ordered, and with solid lines indicating borders between the blocks. When A is written in this way it is referred to as a block matrix.

We will express the Fourier matrix in factored form involving block matrices. The following observation is just a formal way to split a vector into its even and odd components.

Observation 4.3. Define the permutation matrix P_N by

$$\begin{aligned} (P_N)_{i,2i} &= 1, && \text{for } 0 \leq i \leq N/2-1; \\ (P_N)_{i,2i-N+1} &= 1, && \text{for } N/2 \leq i < N; \\ (P_N)_{i,j} &= 0, && \text{for all other } i \text{ and } j; \end{aligned}$$

and let $\boldsymbol{x}$ be a column vector. The mapping $\boldsymbol{x} \to P\boldsymbol{x}$ permutes the components of $\boldsymbol{x}$ so that the even components are placed first and the odd components last,

$$P_N\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}^{(e)} \\ \boldsymbol{x}^{(o)} \end{pmatrix},$$

with $\boldsymbol{x}^{(e)}$, $\boldsymbol{x}^{(o)}$ defined as in Theorem 4.1.

Let $D_{N/2}$ be the $(N/2) \times (N/2)$-diagonal matrix with entries $(D_{N/2})_{n,n} = e^{-2\pi in/N}$ for $n = 0, 1, \ldots, N/2-1$. It is clear from Equation (4.1) that the first half of $\boldsymbol{y}$ is then given by obtained as

$$\frac{1}{\sqrt{2}} \left(\begin{array}{c|c} F_{N/2} & D_{N/2}F_{N/2} \end{array} \right) P_N\boldsymbol{x},$$

and from Equation (4.2) that the second half of $\boldsymbol{y}$ can be obtained as

$$\frac{1}{\sqrt{2}} \left(\begin{array}{c|c} F_{N/2} & -D_{N/2}F_{N/2} \end{array} \right) P_N\boldsymbol{x}.$$

From these two formulas we can derive the promised factorisation of the Fourier matrix.

Theorem 4.4 (DFT matrix factorization)**.** The Fourier matrix may be factored as

$$F_N = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} F_{N/2} & D_{N/2}F_{N/2} \\ \hline F_{N/2} & -D_{N/2}F_{N/2} \end{array} \right) P_N. \tag{4.3}$$

This factorization in terms of block matrices is commonly referred to as the *FFT factorization* of the Fourier matrix. In implementations, this factorization is typically repeated, so that $F_{N/2}$ is replaced with a factorization in terms of $F_{N/4}$, this again with a factorization in terms of $F_{N/8}$, and so on.

The input vector $\boldsymbol{x}$ to the FFT algorithm is mostly assumed to be real. In this case, the second half of the FFT factorization can be simplified, since we have shown that the second half of the Fourier coefficients can be obtained by symmetry from the first half. In addition we need the formula

$$y_{N/2} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N/2-1} \left((\boldsymbol{x}^{(e)})_n - (\boldsymbol{x}^{(o)})_n \right)$$

to obtain coefficient $\frac{N}{2}$, since this is the only coefficient which can't be obtained from $y_0, y_1, \dots, y_{N/2-1}$ by symmetry.

In an implementation based on formula (4.3), we would first compute $P_N\boldsymbol{x}$, which corresponds to splitting $\boldsymbol{x}$ into the even-indexed and odd-indexed samples. The two leftmost blocks in the block matrix in (4.3) correspond to applying the $\frac{N}{2}$-point DFT to the even samples. The two rightmost blocks correspond to applying the $N/2$-point DFT to the odd samples, and multiplying the result with $D_{N/2}$. The results from these transforms are finally added together. By repeating the splitting we will eventually come to the case where $N = 1$. Then F_1 is just the scalar 1, so the DFT is the trivial assignment $y_0 = x_0$. The FFT can therefore be implemented by the following MATLAB code:

```
function y = FFTImpl(x)
N = length(x);
if N == 1
    y = x(1);
else
    xe = x(1:2:(N-1));
    xo = x(2:2:N);
    ye = FFTImpl(xe);
    yo = FFTImpl(xo);
    D=exp(-2*pi*1j*(0:N/2-1)'/N);
    y = [ ye + yo.*D; ye - yo.*D]/sqrt(2);
end
```

Note that this function is recursive; it calls itself. If this is you first encounter with a recursive program, it is worth running through the code for $N = 4$, say.

4.1.1 The Inverse Fast Fourier Transform (IFFT)

The IDFT is very similar to the DFT, and it is straightforward to prove the following analog to Theorem 4.1 and (4.3).

Theorem 4.5 (IDFT matrix factorization)**.** The inverse of the Fourier matrix can be factored as

$$(F_N)^H = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} (F_{N/2})^H & E_{N/2}(F_{N/2})^H \\ \hline (F_{N/2})^H & -E_{N/2}(F_{N/2})^H \end{array} \right) P_N, \tag{4.4}$$

where $E_{N/2}$ is the $(N/2) \times (N/2)$-diagonal matrix with entries given by $(E_{N/2})_{n,n} = e^{2\pi i n/N}$, for $n = 0, 1, \ldots, N/2 - 1$.

We note that the only difference between the factored forms of F_N and F_N^H is the positive exponent in $e^{2\pi i n/N}$. With this in mind it is straightforward to modify `FFTImpl.m` so that it performs the inverse DFT.

MATLAB has built-in functions for computing the DFT and the IDFT, called `fft` and `ifft`. Note, however, that these functions do not used the normalization $1/\sqrt{N}$ that we have adopted here. The MATLAB help pages give a short description of these algorithms. Note in particular that MATLAB makes no assumption about the length of the vector. MATLAB may however check if the length of the vector is 2^r, and in those cases a variant of the algorithm discussed here is used. In general, fast algorithms exist when the vector length N can be factored as a product of small integers.

Many audio and image formats make use of the FFT. To get optimal speed these algorithms typically split the signals into blocks of length 2^r with r some integer in the range 5–10 and utilise a suitable variant of the algorithms discussed above.

4.1.2 Reduction in the number of multiplications with the FFT

Before we continue we also need to explain why the FFT and IFFT factorizations lead to more efficient implementations than the direct DFT and IDFT implementations. We first need some terminology for how we count the number of operations of a given type in an algorithm. In particular we are interested in in the limiting behaviour when N becomes large, which is the motivation for the following definition.

Definition 4.6 (Order of an algorithm)**.** Let R_N be the number of operations of a given type (such as multiplication, addition) in an algorithm, where N describes the dimension of the data in the algorithm (such as the size of the matrix or length of the vector), and let f be a positive function. The algorithm is said to be of order N, which is written $O(f(N))$, if the number of operations grows as $f(N)$ for large N, or more precisely, if

$$\lim_{N\to\infty} \frac{R_N}{f(N)} = c > 0.$$

We will also use this notation for functions, and say that a real function g is $O(f(\boldsymbol{x}))$ if $\lim g(\boldsymbol{x})/f(\boldsymbol{x}) = 0$ where the limit mostly will be taken as $\boldsymbol{x} \to \boldsymbol{0}$ (this means that $g(\boldsymbol{x})$ is much smaller than $f(\boldsymbol{x})$ when $\boldsymbol{x}$ approaches the limit).

Let us see how we can use this terminology to describe the complexity of the FFT algorithm. Let M_N be the number of multiplications needed by the N-point FFT as defined by Theorem 4.1. It is clear from the algorithm that

$$M_N = 2M_{N/2} + N/2. \tag{4.5}$$

The factor 2 corresponds to the two matrix multiplications, while the term $N/2$ denotes the multiplications in the exponent of the exponentials that make up the matrix $D_{N/2}$ (or $E_{N/2}$) — the factor $2\pi i/N$ may be computed once and for all outside the loops. We have not counted the multiplications with `1/sqrt(2)`. The reason is that, in most implementations, this factor is absorbed in the definition of the DFT itself.

Note that all multiplications performed by the FFT are complex. It is normal to count the number of real multiplications instead, since any multiplication of two complex numbers can be performed as four multiplications of real numbers (and two additions), by writing the number in terms of its real and imaginary part, and myltiplying them together. Therefore, if we instead define M_N to be the number of real multiplications required by the FFT, we obtain the alternative recurrence relation

$$M_N = 2M_{N/2} + 2N. \tag{4.6}$$

In Exercise 1 you will be asked to derive the solution of this equation and show that the number of real multiplications required by this algorithm is $O(2N \log_2 N)$. In contrast, the direct implementation of the DFT requires N^2 complex multiplications, and thus $4N^2$ real multiplications. The exact same numbers are found for the IFFT.

Theorem 4.7 (Number of operations in the FFT and IFFT algorithms)**.** The N-point FFT and IFFT algorithms both require $O(2N \log_2 N)$ real multiplications. In comparison, the number of real multiplications required by direct implementations of the N-point DFT and IDFT is $4N^2$.

In other words, the FFT and IFFT significantly reduce the number of multiplications, and one can show in a similar way that the number of additions required by the algorithm is also roughly $O(N \log_2 N)$. This partially explains the efficiency of the FFT algorithm. Another reason is that since the FFT splits the calculation of the DFT into computing two DFT's of half the size, the FFT is well suited for parallel computing: the two smaller FFT's can be performed independently of one another, for instance in two different computing cores on the same computer.

Since filters are diagonalized by the DFT, it may be tempting to implement a filter by applying an FFT, multiplying with the frequency response, and then apply the IFFT. This is not usually done, however. The reason is that most filters have too few nonzero coefficients for this approach to be efficient — it is then better to use the direct algorithm for the DFT, since this may lead to fewer multiplications than the $O(N \log_2 N)$ required by the FFT.

Exercises for Section 4.1

Ex. 1 — In this exercise we will compute the number of real multiplications needed by the FFT algorithm given in the text. The starting point will be the difference equation (4.6) for the number of real multiplications for an N-point FFT.

a. Explain why $x_r = M_{2^r}$ is the solution to the difference equation $x_{r+1} - 2x_r = 4 \cdot 2^r$.

b. Show that the general solution to the difference equation is $x_r = 2r2^r + C2^r$.

c. Explain why $M_N = O(2N \log_2 N)$ (you do not need to write down the initial conditions for the difference equation in order to find the particular solution).

Ex. 2 — When we wrote down the difference equation $M_N = 2M_{N/2} + 2N$ for the number of multiplications in the FFT algorithm, you could argue that some multiplications were not counted. Which multiplications in the FFT algorithm were not counted when writng down this difference equation? Do you have a suggestion to why these multiplications were not counted?

Ex. 3 — Write down a difference equation for computing the number of real additions required by the FFT algorithm.

Ex. 4 — It is of course not always the case that the number of points in a DFT is $N = 2^n$. In this exercise we will see how we can attack the more general case.

a. Assume that N can be divided by 3, and consider the following splitting, which follows in the same way as the splitting used in the deduction of the FFT-algorithm:

$$\begin{aligned}
y_n &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k e^{-2\pi i n k/N} \\
&= \frac{1}{\sqrt{N}} \sum_{k=0}^{N/3-1} x_{3k} e^{-2\pi i n 3k/N} + \frac{1}{\sqrt{N}} \sum_{k=0}^{N/3-1} x_{3k+1} e^{-2\pi i n(3k+1)/N} \\
&+ \frac{1}{\sqrt{N}} \sum_{k=0}^{N/3-1} x_{3k+2} e^{-2\pi i n(3k+2)/N}
\end{aligned}$$

Find a formula which computes $y_0, y_1, \ldots, y_{N/3-1}$ by performing 3 DFT's of size $N/3$.

b. Find similar formulas for computing $y_{N/3}, y_{N/3+1}, \ldots, y_{2N/3-1}$, and $y_{2N/3}, y_{2N/3}+1, \ldots, y_{N-1}$. State a similar factorization of the DFT matrix as in Theorem 4.4, but this time where the matrix has 3×3 blocks.

c. Assume that $N = 3^n$, and that you implement the FFT using the formulas you have deduced in a. and b.. How many multiplications does this algorithm require?

d. Sketch a general procedure for speeding up the computation of the DFT, which uses the factorization of N into a product of prime numbers.

4.2 Efficient implementations of the DCT

In the preceding section we defined the DCT by expressing it in terms of the DFT. In particular, we can apply efficient implementations of the DFT, which we will shortly look at. However, the way we have defined the DCT, there is a penalty in that we need to compute a DFT of twice the length. We are also forced to use complex arithmetic (note that any complex multiplication corresponds to 4 real multiplications, and that any complex addition corresponds to 2 real additions). Is there a way to get around these penalties, so that we can get an implementation of the DCT which is more efficient, and uses less additions and multiplications than the one you made in Exercise 1? The following theorem states an expression of the DCT which achieves this. This expression is, together with a similar result for the DFT in the next section, much used in practical implementations:

Theorem 4.8 (DCT algorithm)**.** Let $\boldsymbol{y} = D_N \boldsymbol{x}$ be the N-point DCT of the vector $\boldsymbol{x}$. Then we have that

$$y_n = c_{n,N} \left(\cos\left(\pi \frac{n}{2N}\right) \Re((F_N \boldsymbol{x}^{(1)})_n) + \sin\left(\pi \frac{n}{2N}\right) \Im((F_N \boldsymbol{x}^{(1)})_n) \right), \quad (4.7)$$

where $c_{0,N} = 1$ and $c_{n,N} = \sqrt{2}$ for $n \geq 1$, and where $\boldsymbol{x}^{(1)} \in \mathbb{R}^N$ is defined by

$$(\boldsymbol{x}^{(1)})_k = x_{2k} \text{ for } 0 \leq k \leq N/2 - 1$$
$$(\boldsymbol{x}^{(1)})_{N-k-1} = x_{2k+1} \text{ for } 0 \leq k \leq N/2 - 1,$$

Proof. The N-point DCT of $\boldsymbol{x}$ is

$$y_n = d_{n,N} \sum_{k=0}^{N-1} x_k \cos\left(2\pi \frac{n}{2N}\left(k + \frac{1}{2}\right)\right).$$

Splitting this sum into two sums, where the indices are even and odd, we get

$$\begin{aligned} y_n = d_{n,N} &\sum_{k=0}^{N/2-1} x_{2k} \cos\left(2\pi \frac{n}{2N}\left(2k + \frac{1}{2}\right)\right) \\ + d_{n,N} &\sum_{k=0}^{N/2-1} x_{2k+1} \cos\left(2\pi \frac{n}{2N}\left(2k + 1 + \frac{1}{2}\right)\right). \end{aligned}$$

If we reverse the indices in the second sum, this sum becomes

$$d_{n,N} \sum_{k=0}^{N/2-1} x_{N-2k-1} \cos\left(2\pi \frac{n}{2N}\left(N - 2k - 1 + \frac{1}{2}\right)\right).$$

If we then also shift the indices with $N/2$ in this sum, we get

$$\begin{aligned} d_{n,N} &\sum_{k=N/2}^{N-1} x_{2N-2k-1} \cos\left(2\pi \frac{n}{2N}\left(2N - 2k - 1 + \frac{1}{2}\right)\right) \\ = d_{n,N} &\sum_{k=N/2}^{N-1} x_{2N-2k-1} \cos\left(2\pi \frac{n}{2N}\left(2k + \frac{1}{2}\right)\right), \end{aligned}$$

where we used that cos is symmetric and periodic with period 2π. We see that we now have the same cos-terms in the two sums. If we thus define the vector

$\boldsymbol{x}^{(1)}$ as in the text of the theorem, we see that we can write

$$\begin{aligned}
y_n &= d_{n,N} \sum_{k=0}^{N-1} (\boldsymbol{x}^{(1)})_k \cos\left(2\pi \frac{n}{2N}\left(2k+\frac{1}{2}\right)\right) \\
&= d_{n,N} \Re\left(\sum_{k=0}^{N-1} (\boldsymbol{x}^{(1)})_k e^{-2\pi i n\left(2k+\frac{1}{2}\right)/(2N)}\right) \\
&= \sqrt{N} d_{n,N} \Re\left(e^{-\pi i n/(2N)} \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} (\boldsymbol{x}^{(1)})_k e^{-2\pi i n k/N}\right) \\
&= c_{n,N} \Re\left(e^{-\pi i n/(2N)} (F_N \boldsymbol{x}^{(1)})_n\right) \\
&= c_{n,N}\left(\cos\left(\pi \frac{n}{2N}\right) \Re((F_N \boldsymbol{x}^{(1)})_n) + \sin\left(\pi \frac{n}{2N}\right) \Im((F_N \boldsymbol{x}^{(1)})_n)\right),
\end{aligned}$$

where we have recognized the N-point DFT, and where $c_{n,N} = \sqrt{N} d_{n,N}$. Inserting the values for $d_{n,N}$, we see that $c_{0,N} = 1$ and $c_{n,N} = \sqrt{2}$ for $n \geq 1$, which agrees with the definition of $c_{n,N}$ in the theorem. This completes the proof. □

With the result above we have avoided computing a DFT of double size. If we in the proof above define the $N \times N$-diagonal matrix Q_N by $Q_{n,n} = c_{n,N} e^{-\pi i n/(2N)}$, the result can also be written on the more compact form

$$\boldsymbol{y} = D_N \boldsymbol{x} = \Re\left(Q_N F_N \boldsymbol{x}^{(1)}\right).$$

We will, however, not use this form, since there is complex arithmetic involved, contrary to (4.7). Let us see how we can use (4.7) to implement the DCT, once we already have implemented the DFT in terms of the function `FFTImpl` as in Section 4.1:

```
function y = DCTImpl(x)
N = length(x);
if N == 1
    y = x;
else
    x1 = [x(1:2:(N-1)); x(N:(-2):2)];
    y = FFTImpl(x1);
    rp = real(y);
    ip = imag(y);
    y = cos(pi*((0:(N-1))')/(2*N)).*rp + sin(pi*((0:(N-1))')/(2*N)).*ip;
    y(2:N) = sqrt(2)*y(2:N);
end
```

In the code, the vector $\boldsymbol{x}^{(1)}$ is created first by rearranging the components, and it is sent as input to `FFTImpl`. After this we take real parts and imaginary parts, and multiply with the cos- and sin-terms in (4.7).

4.2.1 Efficient implementations of the IDCT

As with the FFT, it is straightforward to modify the DCT implementation so that it returns the IDCT. To see how we can do this, write from Theorem 4.8, for $n \geq 1$

$$\begin{aligned}
y_n &= c_{n,N}\left(\cos\left(\pi\frac{n}{2N}\right)\Re((F_N\boldsymbol{x}^{(1)})_n) + \sin\left(\pi\frac{n}{2N}\right)\Im((F_N\boldsymbol{x}^{(1)})_n)\right)\\
y_{N-n} &= c_{N-n,N}\left(\cos\left(\pi\frac{N-n}{2N}\right)\Re((F_N\boldsymbol{x}^{(1)})_{N-n}) + \sin\left(\pi\frac{N-n}{2N}\right)\Im((F_N\boldsymbol{x}^{(1)})_{N-n})\right)\\
&= c_{n,N}\left(\sin\left(\pi\frac{n}{2N}\right)\Re((F_N\boldsymbol{x}^{(1)})_n) - \cos\left(\pi\frac{n}{2N}\right)\Im((F_N\boldsymbol{x}^{(1)})_n)\right),
\end{aligned}$$

where we have used the symmetry of F_N for real signals. These two equations enable us to determine $\Re((F_N\boldsymbol{x}^{(1)})_n)$ and $\Im((F_N\boldsymbol{x}^{(1)})_n)$ from y_n and y_{N-n}. We get

$$\begin{aligned}
\cos\left(\pi\frac{n}{2N}\right)y_n + \sin\left(\pi\frac{n}{2N}\right)y_{N-n} &= c_{n,N}\Re((F_N\boldsymbol{x}^{(1)})_n)\\
\sin\left(\pi\frac{n}{2N}\right)y_n - \cos\left(\pi\frac{n}{2N}\right)y_{N-n} &= c_{n,N}\Im((F_N\boldsymbol{x}^{(1)})_n).
\end{aligned}$$

Adding we get

$$\begin{aligned}
c_{n,N}(F_N\boldsymbol{x}^{(1)})_n =& \cos\left(\pi\frac{n}{2N}\right)y_n + \sin\left(\pi\frac{n}{2N}\right)y_{N-n} + i(\sin\left(\pi\frac{n}{2N}\right)y_n - \cos\left(\pi\frac{n}{2N}\right)y_{N-n})\\
=&(\cos\left(\pi\frac{n}{2N}\right) + i\sin\left(\pi\frac{n}{2N}\right))(y_n - iy_{N-n}) = e^{\pi in/(2N)}(y_n - iy_{N-n}).
\end{aligned}$$

This means that $(F_N\boldsymbol{x}^{(1)})_n = \frac{1}{c_{n,N}}e^{\pi in/(2N)}(y_n - iy_{N-n})$ for $n \geq 1$. For $n = 0$, since $\Im((F_N\boldsymbol{x}^{(1)})_n) = 0$ we have that $(F_N\boldsymbol{x}^{(1)})_0 = \frac{1}{c_{0,N}}y_0$. This means that $\boldsymbol{x}^{(1)}$ can be recovered by taking the IDFT of the vector with component 0 being $\frac{1}{c_{0,N}}y_0 = y_0$, and the remaining components being $\frac{1}{c_{n,N}}e^{\pi in/(2N)}(y_n - iy_{N-n})$:

Theorem 4.9 (IDCT algorithm)**.** Let $\boldsymbol{x} = (D_N)^T\boldsymbol{y}$ be the IDCT of $\boldsymbol{y}$. and let $\boldsymbol{z}$ be the vector with component 0 being $\frac{1}{c_{0,N}}y_0$, and the remaining components being $\frac{1}{c_{n,N}}e^{\pi in/(2N)}(y_n - iy_{N-n})$. Then we have that

$$\boldsymbol{x}^{(1)} = (F_N)^H\boldsymbol{z},$$

where $\boldsymbol{x}^{(1)}$ is defined as in Theorem 4.8.

The implementation of IDCT can thus go as follows:

```
function x = IDCTImpl(y)
N = length(y);
if N == 1
    x = y(1);
```

```
else
  Q=exp(pi*1i*((0:(N-1))')/(2*N));
  Q(2:N)=Q(2:N)/sqrt(2);
  yrev=y(N:(-1):2);
  toapply=[ y(1); Q(2:N).*(y(2:N)-1i*yrev) ];
  x1=IFFTImpl(toapply);
  x=zeros(N,1);
  x(1:2:(N-1))=x1(1:(N/2));
  x(2:2:N)=x1(N:(-1):(N/2+1));
end
```

MATLAB also has a function for computing the DCT and IDCT, called `dct`, and `idct`. These functions are defined in MATLAB exactly as they are here, contrary to the case for the FFT.

4.2.2 Reduction in the number of multiplications with the DCT

Let us also state a result which confirms that the DCT and IDCT implementations we have described give the same type of reductions in the number multiplications as the FFT and IFFT:

Theorem 4.10 (Number of multiplications required by the DCT and IDCT algorithms)**.** Both the N-point DCT and IDCT factorizations given by Theorem 4.8 and Theorem 4.9 require $O(2(N+1)\log_2 N)$ real multiplications. In comparison, the number of real multiplications required by a direct implementation of the N-point DCT and IDCT is N^2.

Proof. By Theorem 4.7, the number of multiplications required by the FFT is $O(2N\log_2 N)$. By Theorem 4.8, two additional multiplications are needed for each index, giving additionally $2N$ multiplications in total, so that we end up with $O(2(N+1)\log_2 N)$ real multiplications. For the IDCT, note first that the vector $\boldsymbol{z} = \frac{1}{c_{n,N}} e^{\pi i n/(2N)}(y_n - i y_{N-n})$ seen in Theorem 4.9 should require $4N$ real multiplications to compute. But since the IDFT of $\boldsymbol{z}$ is real, $\boldsymbol{z}$ must have conjugate symmetry between the first half and the second half of the coefficients, so that we only need to perform $2N$ multiplications. Since the IFFT takes an additional $O(2N\log_2 N)$ real multiplications, we end up with a total of $O(2N + 2N\log_2 N) = O(2(N+1)\log_2 N)$ real multiplications also here. It is clear that the direct implementation of the DCT and IDCT needs N^2 multiplications, since only real arithmetic is involved. □

Since the DCT and IDCT can be implemented using the FFT and IFFT, it has the same advantages as the FFT when it comes to parallel computing. Much literature is devoted to reducing the number of multiplications in the DFT and the DCT even further than what we have done. In the next section we will show an example on how this can be achieved, with the help of extra work and some tedious math. Some more notes on computational complexity are in order. For instance, we have not counted the operations sin and cos in the DCT. The reason is that these values can be precomputed, since we take the sine and cosine of a specific set of values for each DCT or DFT of a given size. This is contrary to to multiplication and addition, since these include the input values, which are only known at runtime. We have, however, not written down that we use precomputed arrays for sine and cosine in our algorithms: This is an issue to include in more optimized algorithms. Another point has to do with multiplication of $\frac{1}{\sqrt{N}}$. As long as $N = 2^{2r}$, multiplication with N need not be considered as a multiplication, since it can be implemented using a bitshift.

4.2.3 *An efficient joint implementation of the DCT and the FFT

We will now present a more advanced FFT algorithm, which will turn out to decrease the number of multiplications and additions even further. It also has the advantage that it avoids complex number arithmetic altogether (contrary to Theorem 4.1), and that it factors the computation into smaller FFTs and DCTs so that we can also use our previous DCT implementation. This implementation of the DCT and the DFT is what is mostly used in practice. For simplicity we will drop this presentation for the inverse transforms, and concentrate only on the DFT and the DCT.

Theorem 4.11 (Revised FFT algorithm)**.** Let $\boldsymbol{y} = F_N\boldsymbol{x}$ be the N-point DFT of the real vector $\boldsymbol{x}$. Then we have that

$$\Re(y_n) = \begin{cases} \frac{1}{\sqrt{2}}\Re((F_{N/2}\boldsymbol{x}^{(e)})_n) + (ED_{N/4}\boldsymbol{z})_n & 0 \le n \le N/4-1 \\ \frac{1}{\sqrt{2}}\Re((F_{N/2}\boldsymbol{x}^{(e)})_n) & n = N/4 \\ \frac{1}{\sqrt{2}}\Re((F_{N/2}\boldsymbol{x}^{(e)})_n) - (ED_{N/4}\boldsymbol{z})_{N/2-n} & N/4+1 \le n \le N/2-1 \end{cases} \tag{4.8}$$

$$\Im(y_n) = \begin{cases} \frac{1}{\sqrt{2}}\Im((F_{N/2}\boldsymbol{x}^{(e)})_n) & q = 0 \\ \frac{1}{\sqrt{2}}\Im((F_{N/2}\boldsymbol{x}^{(e)})_n) + (ED_{N/4}\boldsymbol{w})_{N/4-n} & 1 \le n \le N/4-1 \\ \frac{1}{\sqrt{2}}\Im((F_{N/2}\boldsymbol{x}^{(e)})_n) + (ED_{N/4}\boldsymbol{w})_{n-N/4} & N/4 \le n \le N/2-1 \end{cases} \tag{4.9}$$

where $\boldsymbol{x}^{(e)}$ is as defined in Theorem 4.1, where $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{R}^{N/4}$ defined by

$$\begin{aligned} z_k &= x_{2k+1} + x_{N-2k-1} & 0 \le k \le N/4-1, \\ w_k &= (-1)^k(x_{N-2k-1} - x_{2k+1}) & 0 \le k \le N/4-1, \end{aligned}$$

and where E is a diagonal matrix with diagonal entries $E_{0,0} = \frac{1}{2}$ and $E_{n,n} = \frac{1}{2\sqrt{2}}$ for $n \ge 1$.

Proof. Taking real and imaginary parts in (4.1) we obtain

$$\Re(y_n) = \frac{1}{\sqrt{2}}\Re((F_{N/2}\boldsymbol{x}^{(e)})_n + \frac{1}{\sqrt{2}}\Re((D_{N/2}F_{N/2}\boldsymbol{x}^{(o)})_n)$$
$$\Im(y_n) = \frac{1}{\sqrt{2}}\Im((F_{N/2}\boldsymbol{x}^{(e)})_n + \frac{1}{\sqrt{2}}\Im((D_{N/2}F_{N/2}\boldsymbol{x}^{(o)})_n).$$

These equations explain the first parts on the right hand side in (4.8) and (4.9). Furthermore, for $0 \le n \le N/4 - 1$ we can write

$$\begin{aligned}
&\Re((D_{N/2}F_{N/2}\boldsymbol{x}^{(o)})_n) \\
&= \frac{1}{\sqrt{N/2}}\Re(e^{-2\pi i n/N}\sum_{k=0}^{N/2-1}(\boldsymbol{x}^{(o)})_k e^{-2\pi i n k/(N/2)}) \\
&= \frac{1}{\sqrt{N/2}}\Re(\sum_{k=0}^{N/2-1}(\boldsymbol{x}^{(o)})_k e^{-2\pi i n(k+\frac{1}{2})/(N/2)}) \\
&= \frac{1}{\sqrt{N/2}}\sum_{k=0}^{N/2-1}(\boldsymbol{x}^{(o)})_k \cos\left(2\pi\frac{n(k+\frac{1}{2})}{N/2}\right) \\
&= \frac{1}{\sqrt{N/2}}\sum_{k=0}^{N/4-1}(\boldsymbol{x}^{(o)})_k \cos\left(2\pi\frac{n(k+\frac{1}{2})}{N/2}\right) \\
&\quad + \frac{1}{\sqrt{N/2}}\sum_{k=0}^{N/4-1}(\boldsymbol{x}^{(o)})_{N/2-1-k} \cos\left(2\pi\frac{n(N/2-1-k+\frac{1}{2})}{N/2}\right) \\
&= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{N/4}}\sum_{k=0}^{N/4-1}((\boldsymbol{x}^{(o)})_k + (\boldsymbol{x}^{(o)})_{N/2-1-k})\cos\left(2\pi\frac{n\left(k+\frac{1}{2}\right)}{N/2}\right) \\
&= (E_0 D_{N/4}\boldsymbol{z})_n,
\end{aligned}$$

where we have used that cos is periodic with period 2π and symmetric, where z is the vector defined in the text of the theorem, where we have recognized the DCT matrix, and where E_0 is a diagonal matrix with diagonal entries $(E_0)_{0,0} = \frac{1}{\sqrt{2}}$ and $(E_0)_{n,n} = \frac{1}{2}$ for $n \ge 1$ (E_0 absorbs the factor $\frac{1}{\sqrt{N/2}}$, and the factor $d_{n,N}$ from the DCT). By absorbing the additional factor $\frac{1}{\sqrt{2}}$, we get a matrix E as stated in the theorem. For $N/4 + 1 \le n \le N/2 - 1$, everything above but the last statement is valid. We can now use that

$$\cos\left(2\pi\frac{n(k+\frac{1}{2})}{N/2}\right) = -\cos\left(2\pi\frac{\left(\frac{N}{2}-n\right)\left(k+\frac{1}{2}\right)}{N/2}\right)$$

to arrive at $-(E_0 D_{N/4}\boldsymbol{z})_{N/2-n}$ instead. For the case $n = \frac{N}{4}$ all the cosine entries are zero, and this completes (4.8). For the imaginary part, we obtain as above

$$\begin{aligned}
&\Im((D_{N/2}F_{N/2}\boldsymbol{x}^{(o)})_n) \\
&= \frac{1}{\sqrt{N/2}} \sum_{k=0}^{N/4-1} ((\boldsymbol{x}^{(o)})_{N/2-1-k} - (\boldsymbol{x}^{(o)})_k) \sin\left(2\pi \frac{n(k+\frac{1}{2})}{N/2}\right) \\
&= \frac{1}{\sqrt{N/2}} \sum_{k=0}^{N/4-1} ((\boldsymbol{x}^{(o)})_{N/2-1-k} - (\boldsymbol{x}^{(o)})_k)(-1)^k \cos\left(2\pi \frac{(N/4-n)(k+\frac{1}{2})}{N/2}\right).
\end{aligned}$$

where we have used that sin is periodic with period 2π and anti-symmetric, that

$$\begin{aligned}
\sin\left(2\pi\frac{n(k+\frac{1}{2})}{N/2}\right) &= \cos\left(\frac{\pi}{2} - 2\pi\frac{n(k+\frac{1}{2})}{N/2}\right) \\
&= \cos\left(2\pi\frac{(N/4-n)(k+\frac{1}{2})}{N/2} - k\pi\right) \\
&= (-1)^k \cos\left(2\pi\frac{(N/4-n)(k+\frac{1}{2})}{N/2}\right),
\end{aligned}$$

When $n = 0$ this is 0 since all the cosines entries are zero. When $1 \le n \le N/4$ this is $(E_0 D_{N/4}\boldsymbol{w})_{N/4-n}$, where $\boldsymbol{w}$ is the vector defined as in the text of the theorem. For $N/4 \le n \le N/2 - 1$ we arrive instead at $(E_0 D_{N/4}\boldsymbol{z})_{n-N/4}$, similarly to as above. This also proves (4.9), and the proof is done. □

As for Theorem 4.1, this theorem says nothing about the coefficients y_n for $n > \frac{N}{2}$. These are obtained in the same way as before through symmetry. The theorem also says nothing about $y_{N/2}$. This can be obtained with the same formula as in Theorem 4.1.

It is more difficult to obtain a matrix interpretation for Theorem 4.11, so we will only sketch an algorithm which implements it. The following code implements the recursive formulas for $\Re F_N$ and $\Im F_N$ in the theorem:

```
function y = FFTImpl2(x)
N = length(x);
if N == 1
    y = x;
elseif N==2
    y = 1/sqrt(2)*[x(1) + x(2); x(1) - x(2)];
else
    xe = x(1:2:(N-1));
    xo = x(2:2:N);
    yx = FFTImpl2(xe);

    z = x(N:(-2):(N/2+2))+x(2:2:(N/2));
    dctz = DCTImpl(z);
    dctz(1)=dctz(1)/2;
    dctz(2:length(dctz)) = dctz(2:length(dctz))/(2*sqrt(2));

    w = (-1).^((0:(N/4-1))').*(x(N:-2:(N/2+2))-x(2:2:(N/2)));
    dctw = DCTImpl(w);
    dctw(1)=dctw(1)/2;
    dctw(2:length(dctw)) = dctw(2:length(dctw))/(2*sqrt(2));

    y = yx/sqrt(2);
    y(1:(N/4))=y(1:(N/4))+dctz;
    if (N>4)
      y((N/4+2):(N/2))=y((N/4+2):(N/2))-dctz((N/4):(-1):2);
      y(2:(N/4))=y(2:(N/4))+1j*dctw((N/4):(-1):2);
    end
    y((N/4+1):(N/2))=y((N/4+1):(N/2))+1j*dctw;
    y = [y; ...
         sum(xe-xo)/sqrt(N); ...
         conj(y((N/2):(-1):2))];
end
```

In addition, we need to change the code for `DCTImpl` so that it calls `FFTImpl2` instead of `FFTImpl`. The following can now be shown:

Theorem 4.12 (Number of multiplications required by the revised FFT algorithm)**.** Let M_N be the number of real multiplications required by the revised algorithm of Theorem 4.11. Then we have that $M_N = O(\frac{2}{3}N \log_2 N)$.

This is a big reduction from the $O(2N \log_2 N)$ required by the FFT algorithm from Theorem 4.1. We will not prove Theorem 4.12. Instead we will go through the steps in a proof in Exercise 3. The revised FFT has yet a bigger advantage that the FFT when it comes to parallel computing: It splits the computation into, not two FFT computations, but three computations (one of which is an FFT, the other two DCT's). This makes it even easier to make use of many cores on computers which have support for this.

Exercises for Section 4.2

Ex. 1 — Write a function

```
function samples=DCTImpl(x)
```

which returns the DCT of the column vector $\boldsymbol{x} \in \mathbb{R}^{2N}$ as a column vector. The function should use the FFT-implementation from the previous section, and the factorization $C = E^{-1}AFB$ from above. The function should not construct the matrices A, B, E explicitly.

Ex. 2 — Explain why, if `FFTImpl` needs M_N multiplications A_N additions, then the number of multiplications and additions required by `DCTImpl` are $M_N + 2N$ and $A_N + N$, respectively.

Ex. 3 — In this exercise we will compute the number of real multiplications needed by the revised N-point FFT algorithm of Theorem 4.11, denoted M_N.

a. Explain from the algorithm of Theorem 4.11 that

$$M_N = 2(M_{N/4} + N/2) + M_{N/2} = N + M_{N/2} + 2M_{N/4}. \tag{4.10}$$

b. Explain why $x_r = M_{2^r}$ is the solution to the difference equation

$$x_{r+2} - x_{r+1} - 2x_r = 4 \times 2^r.$$

c. Show that the general solution to the difference equation is

$$x_r = \frac{2}{3}r2^r + C2^r + D(-1)^r.$$

d. Explain why $M_N = O(\frac{2}{3}N \log_2 N)$ (you do not need to write down the initial conditions for the difference equation in order to find the particular solution to it).

4.3 Summary

We obtained an implementation of the DFT which is more efficient in terms of the number of arithmetic operations than a direct implementation of the DFT. We also showed that this could be used for obtaining an efficient implementation of the DCT.