Fra sist: Rene toner

"Enhver lyd" kan skrives som en sum av rene toner ($\sin(2\pi\nu t)$)

(gjelder også trekantpuls/firkantpuls)

Kap 2.1: Digital lyd er en sekvens av målinger av lufttrykk:

$$x_k = f(k/f_s) = f(kT_s)$$

lydsampler

f_s: samplingsfrekvens $\quad f_s = \frac{1}{T_s}$

T_s: samplingsperiode

Oppg 1.2 Når er $f(t) = A_1 \sin(2\pi\nu_1 t) + A_2 \sin(2\pi\nu_2 t)$ periodisk?

perioder $\quad \frac{M}{\nu_1} = \frac{N}{\nu_2}$

$$\frac{\nu_1}{\nu_2} = \frac{M}{N}$$

1.2 Ide: Tilnærme funksjoner med sin/cos.

Spørsmål: 1. Hvilke funksjoner tilnærmer vi
2. Hvilke sin/cos-summer
3. Hvordan tilnærmer vi?

Anta at $\{\phi_i\}_{i=1}^{m}$ er en ortogonal basis for et rom W. Da er

$$\sum_{i=1}^{m} \frac{\langle f, \phi_i \rangle}{\langle \phi_i, \phi_i \rangle} \phi_i$$

en minste kvadraters tilnærming til f fra W. (Ortogonalt dekomposisjonsteorem)

Delvis bevis for teorem 1.15:

$$\langle \cos(2\pi m t/T), \cos(2\pi n t/T) \rangle$$

$$= \frac{1}{T}\int_0^T \cos(2\pi m t/T)\cos(2\pi n t/T)\,dt$$

$$= \frac{1}{2T}\int_0^T \Big(\cos(2\pi m t/T + 2\pi n t/T) + \cos(2\pi m t/T - 2\pi n t/T)\Big)\,dt$$

$m \neq n$

$$= \frac{1}{2T}\left[\frac{T}{2\pi(m+n)}\sin(2\pi(m+n)t/T) + \frac{T}{2\pi(m-n)}\sin(2\pi(m-n)t/T)\right]_0^T$$

$= 0$

Tilsvarende: $m = n$:

$$\langle \underset{\sin}{\cos}(2\pi m t/T), \underset{\sin}{\cos}(2\pi m t/T) \rangle = \frac{1}{2}$$

$$\langle 1, 1 \rangle = 1$$

$\cos(x+y)$
$= \cos x \cos y$
$- \sin x \sin y$

$\cos(x-y)$
$= \cos x \cos y$
$+ \sin x \sin y$

$2\cos x \cos y$
$= \cos(x+y) + \cos(x-y)$

Ortogonalt dekomp. teorem for f:

$$f_N(t) = \frac{\langle f, 1\rangle}{\underbrace{\langle 1, 1\rangle}_{\to 1}} 1 + \sum_{n=1}^{N} \frac{\langle f, \cos\frac{2\pi n t}{T}\rangle}{\underbrace{\langle \cos\frac{2\pi n t}{T}, \cos\frac{2\pi n t}{T}\rangle}_{\to \frac{1}{2}}} \cos\left(\frac{2\pi n t}{T}\right)$$

$$+ \sum_{n=1}^{N} \frac{\langle f, \sin\frac{2\pi n t}{T}\rangle}{\underbrace{\langle \sin\frac{2\pi n t}{T}, \sin\frac{2\pi n t}{T}\rangle}_{\to \frac{1}{2}}} \sin\left(\frac{2\pi n t}{T}\right)$$

$$= \underbrace{\frac{1}{T}\int_0^T f(t)\,dt}_{a_0} \cdot 1 + \sum_{n=1}^{N} \underbrace{\frac{2}{T}\int_0^T f(t)\cos\frac{2\pi n t}{T}\,dt}_{a_n} \cos\left(\frac{2\pi n t}{T}\right)$$

$$+ \sum_{n=1}^{N} \underbrace{\frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi n t}{T}\right)dt}_{b_n} \sin\left(\frac{2\pi n t}{T}\right)$$

$\Rightarrow$ har beviste formler for a_0, a_n, b_n

Når konvergerer Fourierrekka f_N mot f når $N \to \infty$?

Eks 1.17 Fourierrekka til firkantpulsen

$$f_s(t) = \begin{cases} 1 & 0 \le t < \frac{T}{2} \\ -1 & \frac{T}{2} \le t < T \end{cases}$$

$$a_0 = \frac{1}{T} \int_0^T f_s(t)\,dt = 0$$

$$a_n = \frac{2}{T} \int_0^T f_s(t) \cos(2\pi n t/T)\,dt$$

$$= \frac{2}{T}\left(\int_0^{\frac{T}{2}} \cos(2\pi n t/T)\,dt - \int_{\frac{T}{2}}^{T} \cos(2\pi n t/T)\,dt \right)$$

$$= \frac{2}{T}\frac{T}{2\pi n}\left(\left[\sin(2\pi n t/T)\right]_0^{\frac{T}{2}} - \left[\sin 2\pi n t/T\right]_{\frac{T}{2}}^{T} \right) = \underline{0}$$

$$\left(\frac{T}{2} \to \sin(\pi n)\right)$$

$$b_n = \frac{2}{T}\left(\int_0^{\frac{T}{2}} \sin(2\pi n t/T) - \int_{\frac{T}{2}}^{T} \sin(2\pi n t/T)\,dt \right)$$

$$= \frac{2}{T}\frac{T}{2\pi n}\left(\left[-\cos(2\pi n t/T)\right]_0^{\frac{T}{2}} + \left[\cos(2\pi n t/T)\right]_{\frac{T}{2}}^{T} \right)$$

$$= \frac{1}{\pi n}\left(-\cos(\pi n) + 1 + 1 - \cos(\pi n)\right) = \frac{2 - 2\cos \pi n}{\pi n}$$

$$b_n = \begin{cases} 0 & n \text{ partall} \\ \frac{4}{\pi n} & n \text{ oddetall} \end{cases}$$

$\Rightarrow$ Fourierrekka blir $\underline{\sum_{n \text{ odde}} \frac{4}{\pi n} \sin(2\pi n t/T)}$

Eks 1.18 Fourierrekka firkantpulsen:

$$\sum_{n \text{ odde}} -\frac{8}{\pi^2 n^2} \cos(2\pi n t/T)$$

Teorem 1.20: Hvis $f(t) = f(-t)$ (symmetrisk) så er $b_n = 0$ ($\Rightarrow$ cos-rekke)

<u>Bevis</u>: $b_n = \frac{2}{T} \int_0^T f(t) \sin(2\pi n t/T)\, dt$

$$= \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin(2\pi n t/T)\, dt$$

$$= \frac{2}{T}\left(\int_{-\frac{T}{2}}^{0} f(t) \sin(2\pi n t/T)\, dt + \int_0^{\frac{T}{2}} f(t) \sin(2\pi n t/T)\, dt \right) \qquad u = -t$$

$$= \frac{2}{T}\left(\int_{-\frac{T}{2}}^{0} f(t) \sin(2\pi n t/T)\, dt - \int_0^{-\frac{T}{2}} \underbrace{f(-u)}_{f(u)} \underbrace{\sin(-2\pi n u/T)}_{-\sin(2\pi n u/T)}\, du \right)$$

$$= \frac{2}{T}\left(\int_{-\frac{T}{2}}^{0} f(t) \sin(2\pi n t/T)\, dt - \int_{-\frac{T}{2}}^{0} f(u) \sin(2\pi n u/T)\, du \right)$$

$$= 0$$