Seksjon 2.2

Vil bygge opp tilsvarende verktøy for digital lyd, som i kap. 1.

Bevis for at ϕ_n er ortonormale:

$$\langle \phi_{n_1}, \phi_{n_2} \rangle = \sum_{k=0}^{N-1} \frac{1}{\sqrt{N}} e^{2\pi i k n_1/N} \overline{\frac{1}{\sqrt{N}} e^{2\pi i k n_2/N}}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i k(n_1-n_2)/N}$$

$n_1 = n_2$: $\frac{1}{N} \sum_{k=0}^{N-1} 1 = 1$

$n_1 \neq n_2$: $\frac{1}{N} \frac{1-(e^{2\pi i(n_1-n_2)/N})^N}{1-e^{2\pi i(n_1-n_2)/N}} = \frac{1}{N} \frac{1-e^{\overbrace{2\pi i(n_1-n_2)}^{1}}}{1-e^{2\pi i(n_1-n_2)/N}}$

$= 0$

Kap. 1: $f(t) \xrightarrow{\text{Fourierrekke}} y_n$

Kap 2: $\vec{x} \xrightarrow{F_N} \vec{y}$ (F_N: koordinatskifte til standard basis,

$$\vec{x} = y_0 \phi_0 + y_1 \phi_1 + \dots + y_{N-1} \phi_{N-1}$$

$F_N \phi_n = \vec{e}_n$

$\phi_n = 0\phi_0 + \dots + 1\phi_n + 0 \dots$

eks. 2,12

Vi regner ut $F_N(2\vec{x}+3\vec{y})$

der $\vec{x}$ er gitt ved $x_k = \cos(2\pi 5k/N)$

$\vec{y}$ er gitt ved $y_k = \sin(2\pi 7k/N)$

$$= F_N(2\cos(2\pi 5k/N) + 3\sin(2\pi 7k/N))$$

$$= F_N(\underline{2\cdot\frac{1}{2}}(e^{2\pi i5k/N} + e^{-2\pi i5k/N}) + 3\cdot\frac{1}{2i}(e^{2\pi i7k/N} - e^{-2\pi i7k/N}))$$

$$\xrightarrow{\qquad} \text{indeks } -5 \quad e^{2\pi i(-5)k/N} \swarrow 2\pi ik$$

$$= F_N(e^{2\pi i5k/N}) + F_N(e^{-2\pi i5k/N}) + \cdots\cdots$$

$$e^{2\pi i(-5)k/N + \underline{2\pi iNk/N}}$$

$$= \sqrt{N}F_N(\frac{1}{\sqrt{N}}e^{2\pi i5k/N}) + \sqrt{N}F_N(\frac{1}{\sqrt{N}}e^{2\pi i(N-5)k/N}) + \cdots$$

$$= e^{2\pi i(N-5)k/N}$$

$$= e$$

$$\sqrt{N}F_N\phi_5 + \sqrt{N}F_N\phi_{N-5} + \frac{3}{2i}\sqrt{N}(F_N\phi_7 - F_N\phi_{N-7})$$

$$\underline{\underline{= \sqrt{N}\vec{e}_5 + \sqrt{N}\vec{e}_{N-5} + \frac{3}{2i}\sqrt{N}\vec{e}_7 - \frac{3}{2i}\sqrt{N}\vec{e}_{N-7}}}$$

Uttrykk for F_N:

Siden $F_N \phi_n = \vec{e}_n$, så er

$$F_N [\phi_0\ \phi_1, \dots, \phi_{N-1}] = [\vec{e}_0\ \vec{e}_1, \dots, \vec{e}_{N-1}] = I$$

$$\Rightarrow F_N^{-1} = [\phi_0\ \phi_1 \cdots \phi_{N-1}]$$

Siden $\phi_0, \phi_1, \dots$ er ortonormale, så er

$$F_N = (F_N^{-1})^{-1} = \begin{bmatrix} \phi_0^T \\ \hline \phi_1^T \\ \vdots \\ \hline \phi_{N-1}^T \end{bmatrix}$$

siden ϕ_n har komponenter $\frac{1}{\sqrt{N}} e^{2\pi i kn/N}$, så

har F_N elementer $\frac{1}{\sqrt{N}} e^{-2\pi i kn/N}$

$F_N^{-1} = (\overline{F_N})^T$ $\quad \frac{1}{\sqrt{N}} e^{2\pi i kn/N}$

(skrives også F_N^H for)

$\vec{y} = \mathrm{DFT}\,\vec{x}$ betyr at

$$y_n = \sum_{k=0}^{N-1} x_k e^{-2\pi i kn/N}$$

$$x_k = \sum_{n=0}^{N-1} y_n e^{2\pi i kn/N}$$

Sammenheng mellom frekvens ν, og Fourierindeks n

Vi ser at ϕ_n er samplene til $f(t) = \frac{1}{\sqrt{N}} e^{2\pi i t n/T}$,

der $t = \frac{kT}{N}$

avstand mellom sampler: $\frac{T}{N} = T_s \Rightarrow f_s = \frac{1}{T_s} = \frac{N}{T}$

$\Rightarrow T = \frac{N}{f_s}$

har frekvens $\nu = \frac{n}{T}$

$= \frac{n}{N} f_s$

$$\Rightarrow \underline{\underline{\nu = \frac{n f_s}{N}}}$$

Eksempel 2.15

DFT av en "firkantpuls":
(2L+1 enere, resten nuller)

$$\underbrace{\underbrace{(\overset{x_0}{1},1,\dots,\overset{x_L}{1}}_{L+1},0,\dots,0,\underbrace{1,\dots,\overset{x_{N-1}}{1})}_{L}}_{N}$$

$$y_n = \sum_{k=0}^{N-1} x_k e^{-2\pi i kn/N}$$

$$= \sum_{k=0}^{L} e^{-2\pi i kn/N} + \sum_{k=N-L}^{N-1} e^{-2\pi i \overset{(k+N)}{k} n/N}$$

$$\boxed{u = k - N}$$

$$e^{-2\pi i Nn/N} = e^{-2\pi i n} = 1$$

$$= \sum_{k=0}^{L} e^{-2\pi i kn/N} + \sum_{k=-L}^{-1} e^{-2\pi i kn/N}$$

sin

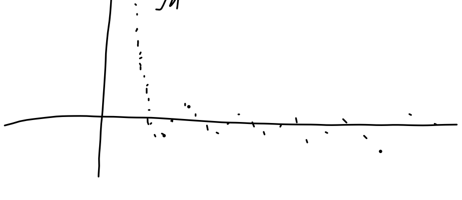
$$= \sum_{k=-L}^{L} e^{-2\pi i kn/N} = \frac{e^{-2\pi i(-L)n/N}\left(1-\left(e^{-2\pi i n/N}\right)^{2L+1}\right)}{1-e^{-2\pi i n/N}}$$

sin

$$= e^{2\pi i Ln/N}\frac{1-e^{-2\pi i n(2L+1)/N}}{1-e^{-2\pi i n/N}}$$

$$= \underbrace{e^{2\pi i Ln/N}\frac{e^{-\pi i n(2L+1)/N}}{e^{-\pi i n/N}}}_{1} \cdot \frac{\frac{1}{2i}\left(e^{\pi i n(2L+1)/N} - e^{-\pi i n(2L+1)/N}\right)}{\frac{1}{2i}\left(e^{\pi i n/N} - e^{-\pi i n/N}\right)}$$

$$= \frac{\sin(\pi n(2L+1)/N)}{\sin(\pi n/N)}$$



Bevis for teorem 2.16 i (2.18 python)

1. $$(\hat{x})_{N-n} = \sum_{k=0}^{N-1} x_k e^{-2\pi i k(N-n)/N}$$

$$= \sum_{k=0}^{N-1} x_k e^{2\pi i k n/N} = \overline{\sum_{k=0}^{N-1} x_k e^{-2\pi i k n/N}}$$

$$= (\hat{x})_n$$

4. Sett $z_k = x_{k-d}$

$$(\hat{z})_n = \sum_{k=0}^{N-1} z_k e^{-2\pi i k n/N} = \sum_{k=0}^{N-1} x_{k-d} e^{-2\pi i k n/N}$$

$$\overset{u=k-d}{=} \sum_{u=0}^{N-1} x_u e^{-2\pi i (u+d) n/N} = e^{-2\pi i d n/N} \sum_{u=0}^{N-1} x_u e^{-2\pi i u n/N}$$

$$= e^{-2\pi i d n/N} (\hat{x})_n$$