Beeis for teorem 3.14:

$$DFT_N \vec{s} = \sqrt{N} F_N \vec{s} = \sqrt{N} F_N S \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \sqrt{N} \underbrace{F_N F_N^H}_{I} \underbrace{D F_N}_{S} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \sqrt{N} D F_N \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = D \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_{s,0} & & 0 \\ 0 & \lambda_{s,1} & \\ & & \ddots & \lambda_{s,N-1} \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_{s,0} \\ \lambda_{s,1} \\ \vdots \\ \lambda_{s,N-1} \end{pmatrix} = \vec{\lambda}_s$$

$\sqrt{N} F_N = DFT_N$

på komponentform: $\lambda_{s,n} = \sum_{k=0}^{N-1} s_k e^{-2\pi i k n/N}$

Eks 3.16: $S = \{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\}$ (t_{-1}, t_0, t_1)

$$\boxed{\begin{aligned} s_0 &= t_0 = \tfrac{1}{2} \\ s_1 &= t_1 = \tfrac{1}{4} \\ s_{N-1} &= t_{-1} = \tfrac{1}{4} \end{aligned}}$$

$$\vec{\lambda}_s = DFT_N \vec{s} = s_0 e^{-2\pi i 0 n/N} + s_1 e^{-2\pi i 1 \cdot n/N} + s_{N-1} e^{-2\pi i (N-1)n/N}$$

$$= \frac{1}{2} + \frac{1}{4} e^{-2\pi i n/N} + \frac{1}{4} e^{2\pi i n/N} = \frac{1}{2} + \frac{1}{2}\cos(2\pi n/N)$$

$\lambda_{s,0} = \frac{1}{2} + \frac{1}{2} = 1$ (bevarer laveste frekvens)

$\lambda_{s,\frac{N}{2}} = \frac{1}{2} + \frac{1}{2}\cos(\pi) = 0$ (nuller ut høyeste frekvens)

Eks. 3.17 Hva er egenverdiene/egenvektorene til $S = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$?

S er sirkulant/Toeplitz, så det er et filter, og egenvektorene er

$$\phi_0 = \frac{1}{\sqrt{2}}\left(e^{2\pi i 0 \cdot 0/2}, e^{2\pi i 0 \cdot 1/2}\right) = \underline{\underline{\frac{1}{\sqrt{2}}(1,1)}}$$

$$\phi_1 = \frac{1}{\sqrt{2}}\left(e^{2\pi i 1 \cdot 0/2}, e^{2\pi i 1 \cdot 1/2}\right) = \underline{\underline{\frac{1}{\sqrt{2}}(1,-1)}}$$

egenverdier: $\vec{\lambda}_s = DFT_2 \vec{s} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

$\lambda_{s,0} = 5$

$\underline{\underline{\lambda_{s,1} = 3}}$

eks 3.18

Anta S: $z_n = \frac{1}{6}(x_{n+2} + 4x_{n+1} + 6x_n + 4x_{n-1} + x_{n-2})$,

og $\vec{x} \in \mathbb{R}^N$ er vektoren der $x_k = \cos(2\pi 5k/N)$

Hva er $S\vec{x}$:

$$x_k = \frac{1}{2}\left(e^{2\pi i 5k/N} + e^{-2\pi i 5k/N}\right)$$

$$= \frac{\sqrt{N}}{2}\left(\frac{1}{\sqrt{N}} e^{2\pi i 5k/N} + \frac{1}{\sqrt{N}} e^{2\pi i (N-5)k/N}\right) = \frac{\sqrt{N}}{2}(\phi_5 + \phi_{N-5})$$

Da blir $S\vec{x} = \frac{\sqrt{N}}{2}(S\phi_5 + S\phi_{N-5}) = \frac{\sqrt{N}}{2}(\lambda_{S,5}\phi_5 + \lambda_{S,N-5}\phi_{N-5})$

Vi har her $t_{-2} = t_2 = \frac{1}{6}$, $t_{-1} = t_1 = \frac{2}{3}$, $t_0 = 1$

$$\Rightarrow s_2 = s_{N-2} = \frac{1}{6}, \quad s_1 = s_{N-1} = \frac{2}{3}, \quad s_0 = 1$$

$$\lambda_{S,5} = s_0 \cdot e^{-2\pi i 5\cdot 0/N} + s_1 e^{-2\pi i 5\cdot 1/N} + s_2 e^{-2\pi i 5\cdot 2/N} + s_{N-2} e^{-2\pi i 5(N-2)/N} + s_{N-1} e^{-2\pi i 5(N-1)/N}$$

$$= 1 + \underline{\frac{2}{3} e^{-2\pi i 5/N}} + \frac{1}{6} e^{-2\pi i 5\cdot 2/N} + \frac{1}{6} e^{2\pi i 5\cdot 2/N} + \underline{\frac{2}{3} e^{2\pi i 5/N}}$$

$$= 1 + \frac{4}{3}\cos(2\pi 5/N) + \frac{1}{3}\cos(2\pi \cdot 10/N)$$

$\lambda_{S,N-5} = \cdots = \lambda_{S,5}$

$$\Rightarrow S\vec{x} = \frac{\sqrt{N}}{2}\lambda_{S,5}(\phi_5 + \phi_{N-5}) = \frac{\sqrt{N}}{2}\lambda_{S,5}\left(\frac{1}{\sqrt{N}} e^{2\pi i 5k/N} + \frac{1}{\sqrt{N}} e^{-2\pi i 5k/N}\right)$$

$$= \lambda_{S,5}\cos(2\pi 5k/N) = \underline{\underline{\left(1 + \frac{4}{3}\cos(2\pi 5/N) + \frac{1}{3}\cos(2\pi \cdot 10/N)\right)\vec{x}}}$$

Oppg 3.7 - 3.9

Bevis for teorem 3.21

$$\lambda_{S,n} = \sum_{k=0}^{N-1} s_k e^{-2\pi i k n/N} = \sum_{0 \le k < \frac{N}{2}} s_k e^{-2\pi i k n/N} + \sum_{\frac{N}{2} \le k < N} s_k e^{-2\pi i k n/N}$$

$$= \sum_{0 \le k < \frac{N}{2}} t_k e^{-2\pi i k n/N} + \sum_{\frac{N}{2} \le k < N} t_{k-N} e^{-2\pi i k n/N}$$

$u = k - N$

$$= \sum_{0 \le k < \frac{N}{2}} t_k e^{-2\pi i k n/N} + \sum_{-\frac{N}{2} \le u < 0} t_u e^{-2\pi i (u+N) n/N}$$

$$= \sum_{0 \le k < \frac{N}{2}} t_k e^{-2\pi i k n/N} + \sum_{-\frac{N}{2} \le u < 0} t_u e^{-2\pi i u n/N}$$

$u \to k$

$$= \sum_{-\frac{N}{2} \le k < \frac{N}{2}} t_k e^{-2\pi i k n/N} = \sum_k t_k e^{-\overbrace{2\pi i k n/N}^{\omega = 2\pi n/N}} = \lambda_S(\omega)$$

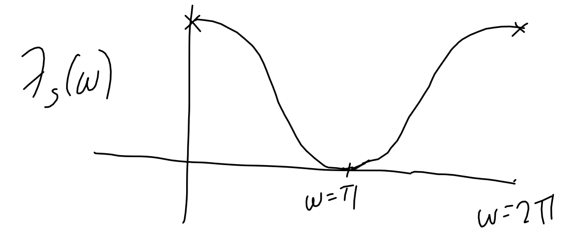
$$= \lambda_S(2\pi n/N)$$

Eks 3.25 Vi plotter kont. frekvensrespons for $S = \{\frac{1}{4}, \underline{\frac{1}{2}}, \frac{1}{4}\}$

Vi hadde $\lambda_{S,n} = \frac{1}{2} + \frac{1}{2}\cos(2\pi n/N)$ (fra eks. 3.16)

$= \lambda_S(2\pi n/N) \Rightarrow \lambda_S(\omega) = \frac{1}{2} + \frac{1}{2}\cos\omega.$

$(\lambda_S(\omega) = \frac{1}{4}e^{i\omega} + \frac{1}{2} + \frac{1}{4}e^{-i\omega} = \frac{1}{2} + \frac{1}{2}\cos\omega.$



$S = F_N^H D F_N \Rightarrow S^{-1} = F_N^H D^{-1} F_N$

$\lambda_{S,n}$ på diagonalen $\frac{1}{\lambda_{S,n}}$ på diagonalen.

$\Rightarrow \underline{\underline{\lambda_{S^{-1},n} = \frac{1}{\lambda_{S,n}}}}$ $\lambda_{S^{-1}}(\omega) = \frac{1}{\lambda_S(\omega)}$

eks 3.27 Anta at $\lambda_{S_1}(\omega) = \cos(2\omega)$

$\lambda_{S_2}(\omega) = 1 + 3\cos\omega$

Hva blir filterkoeffisientene til S_1S_2?

$$\lambda_{S_1S_2}(\omega) = \lambda_{S_1}(\omega)\lambda_{S_2}(\omega) = \cos(2\omega)(1+3\cos\omega)$$

$$= \frac{1}{2}(e^{2i\omega}+e^{-2i\omega})(1+\frac{3}{2}e^{i\omega}+\frac{3}{2}e^{-i\omega})$$

$$= \frac{3}{4}e^{3i\omega}+\frac{1}{2}e^{2i\omega}+\frac{3}{4}e^{i\omega}+\frac{3}{4}e^{-i\omega}\ \underbrace{\tfrac{1}{2}e^{-2i\omega}}_{t_2}+\underbrace{\tfrac{3}{4}e^{-3i\omega}}_{t_3}$$

t_1

$$= \sum_k t_k e^{-ik\omega}$$

t_0

$$\Rightarrow S_1S_2 = \{\frac{3}{4}, \frac{1}{2}, \frac{3}{4}, \underline{0}, \frac{3}{4}, \frac{1}{2}, \frac{3}{4}\}$$

Oppg 3.15: regn ut filterkoeffs til $S_1 \cdot S_2$ når $\lambda_{S_1}(\omega) = 2 + 4\cos\omega$, $\lambda_{S_2}(\omega) = 3\sin(2\omega)$