# AN INTRODUCTION TO CONVEXITY

GEIR DAHL *

NOVEMBER 2010

* University of Oslo, Centre of Mathematics for Applications, P.O.Box 1053, Blindern, 0316 Oslo, Norway (geird@math.uio.no)

# Contents

# List of Figures

# Preface.

This report is written in connection with a new course (in year 2000!) with the title *Convexity and optimization* at the University of Oslo. The course aims at upper undergraduate student in (applied) mathematics, statistics or mathematical economics. The main goal of the course is to give an introduction to the subjects of linear programming and convexity.

Many universities offer linear programming courses at an undergraduate level, and there are many books around written for that purpose. There are several interesting and important topics that one typically covers in such a course: modeling, the simplex algorithm, duality, sensitivity analysis, implementation issues, applications in network flows, game theory, approximation etc. However, for students in (applied) mathematics (or economics) I believe it is important to understand the "neighborhood" of linear programming, which is convexity (or convex analysis). Convexity is fundamental to the whole area of optimization, and it is also of great importance in mathematical statistics, economics, functional analysis, approximation theory etc.

The purpose of this report is to introduce the reader to convexity. The prerequisites are mainly linear algebra and linear programming (LP) including the duality theorem and the simplex algorithm. In our *Convexity and optimization* course we first teach LP, now based on the excellent book by Vanderbei, [15]. This book is extremely well written, and explains ideas and techniques elegantly without too many technicalities. The second, and final, part of the course is to go into convexity where this report may be used. There is plenty of material in convexity and the present text gradually became longer than originally planned. As a consequence, there is probably enough material in this report for a separate introductory course in convexity. In our *Convexity and optimization* course we therefore have to omit some of the material.

A classic book in convex analysis is Rockafellar's book [11]. A modern text which treats convex analysis in combination with optimization is [6]. Comprehensive treatments of convex analysis is [16] and [12]. The latter book is an advanced text which contains lots of recent results and historical notes. For a general treatment of convexity with application to theoretical statistics, see [14]. The book [17] also treats convexity in connection with a combinatorial study of polytopes.

In this text we restrict the attention to convexity in $\mathbb{R}^n$. However, the reader should know that the notion of convexity makes sense in vector spaces more generally. The whole theory can be directly translated to the case of finite-dimensional vector spaces (as e.g., the set of real $m \times n$-dimensional matrices). Many results, but not all of them, also hold in infinite-dimensional vector spaces; this is treated within the area of functional analysis.

**Acknowledgment.** I would like to thank Bjarne Johannessen for producing the figures and giving suggestions that improved the text. Moreover, I appreciate very much the stimulating environment here at the University of Oslo and I am grateful to my colleagues for support and interesting discussions.

*GD. Oslo, Jan. 7, 2000.*

In the present edition a number of misprints have been corrected and a few minor changes have been made.

*GD. Oslo, Dec. 15, 2000.*

Now a new chapter on convex optimization has been added and again some minor changes have been done.

*GD. Oslo, Oct. 13, 2004.*

Some minor corrections have been made.

*GD. Oslo, Aug. 25, 2009.*

# Chapter 1

# The basic concepts

This first chapter introduces convex sets and illustrates how convex sets arise in different contexts.

## 1.1 Is convexity useful?

Many people think that it is, even people not working with convexity! But this may not convince you, so maybe some of our examples below give you some motivation for working your way into the world of convexity. These examples are all presented in an informal style to increase readability.

**Example 1.1.1.** *(Optimization and convex functions)* Often one meets optimization problems where one wants to minimize a real-valued function of $n$ variables, say $f(x)$, where $x = (x_1, \ldots, x_n)$. This arises in e.g., economical applications (cost minimization or profit maximization), in statistical applications (estimation, regression, curve fitting), approximation problems, scheduling and planning problems, image analysis, medical imaging, engineering applications etc.

The ideal goal would be to find a point $x^*$ such that $f(x^*) \leq f(x)$ holds for all other points $x$; such a solution $x^*$ is called a globally optimal solution. The problem is that most (numerical) methods for minimizing a function can only find a locally optimal solution, i.e., a point $x_0$ with function value no greater than the function values of points "sufficiently near" $x_0$. Unfortunately, although a locally optimal solution is good locally, it may be very poor compared to some other solutions. Thus, for instance, in a cost minimization problem (where $x = (x_1, \ldots, x_n)$ is an activity vector) it would be very good news if we were able to prove (using our mathematical skills!) that our computed locally optimal solution is also a globally optimal solution. In that case we could say to our boss: "listen, here is my solution $x^*$ and no other person can come up with another solution having lower total cost".

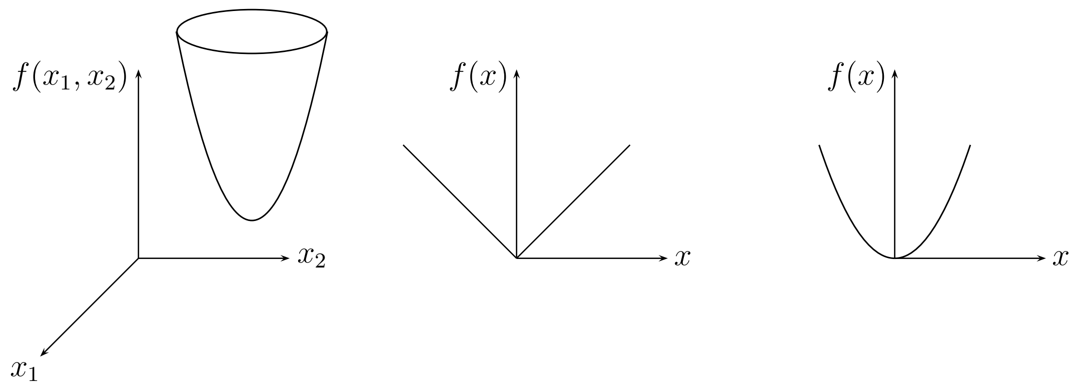


Figure 1.1: Some convex functions

*If the function $f$ is convex, then it is always true that a locally optimal solution is also globally optimal!*

We study convex functions in Chapter 5. Some convex functions are illustrated in Fig. 1.1. You will learn what a convex function is, how to decide if a given function is convex, and how to minimize a convex function.

At this point you might ask one of the following questions:

- I recall that a convex function $f : \mathbb{R} \to \mathbb{R}$ is convex whenever its second derivative is nonnegative, i.e., the "graph bends upwards". But what does it mean that a function of several variables is convex?
- Does the "local implies global property" above also hold for other functions than the convex ones?
- Will I meet convex functions in other areas of mathematics, statistics, numerical analysis etc?
- If the function $f$ is only defined on a subset $S$ of $\mathbb{R}^n$. Can $f$ still be convex? If so, how can we minimize it? And, does the "local implies global property" still hold?

You will get answers to these, and many more, questions. Concerning the last question, we shall see that the set $S$ of points should have a certain property in order to make an extended definition of convexity meaningful. This property is: $S$ is a convex set.

**Example 1.1.2.** *(Convex set)* Loosely speaking a convex set in $\mathbb{R}^2$ (or $\mathbb{R}^n$) is a set "with no holes". More accurately, a convex set $C$ has the following property: whenever we choose two points in the set, say $x, y \in C$, then all points on the line segment between $x$ and $y$ also lie in $C$. Some examples of convex sets in the plane are: a sphere (ball), an ellipsoid, a point, a line, a line segment, a rectangle, a triangle, see Fig. 1.2. But, for instance, a set with a finite number $p$ of points is only convex when $p = 1$. The union of two disjoint (closed) triangles is also nonconvex.

Why are convex sets important? They arise in lots of different situations where the convexity property is of importance. For instance, in optimization the set of feasible points is frequently convex. This is true for linear programming and many other important optimization problems. We can say more, the convexity of the feasible set plays a role for the existence of optimal solutions, the structure of the set of optimal solutions, and (very important!) how to solve optimization problems numerically.

But convex sets arise in other areas than optimization. For instance, an important area in statistics (both in theory and applications) is *estimation* where one uses statistical observations to "estimate" the value of one or more unknown parameters in a model. To measure quality of a solution one uses a "loss function" and, quite often, this loss function is convex. In statistical decision theory the concept of risk sets is central. Well, risk sets are convex sets. Moreover, under some additional assumption on the statistical setting, these risk sets are very special convex sets, so-called *polytopes*. We shall study polytopes in detail later.

Another example from statistics is the expectation operator. The expectation of a random variable relates to convexity. Assume that $X$ is a discrete variable taking values in some finite set of real numbers, say $\{x_1, \ldots, x_r\}$ with probabilities $p_i$ of the event $X = x_i$. Probabilities are all nonnegative and sum to one, so $p_j \geq 0$ and $\sum_{j=1}^r p_j = 1$. The *expectation* (or *mean*) of $X$ is the number

$$EX = \sum_{j=1}^{r} p_j x_j.$$

It should be regarded as a weighted average of the possible values that $X$ can attain, and the weights are simply the probabilities. Thus, a very likely event (meaning that $p_j$ is near one) gets large weight in this sum. Now, in the language of convexity, we say that $EX$ is a *convex combination* of the numbers $x_1, \ldots, x_r$. We shall work a lot with convex combinations. An extension is when the discrete random variable is a vector, so it attains values in a finite set $S = \{x_1, \ldots, x_r\}$ of points in $\mathbb{R}^n$. The expectation is now defined by $EX = \sum_{j=1}^r p_j x_j$ which, again, is a convex combination of the points in $S$.

A question: assume that $n = 2$ and $r = 4$ and choose some vectors $x_1, \ldots, x_4 \in \mathbb{R}^2$. Experiment with some different probabilities $p_1, \ldots, p_4$ and calculate $EX$ in each case. If you now vary the probabilities as much as possible (nonnegative and sum one), which set of possible expectations do you get?

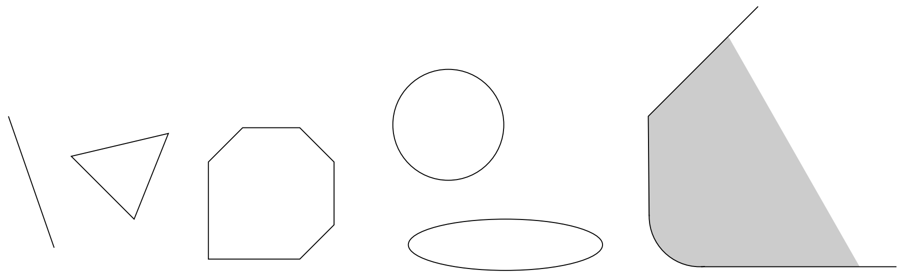

Figure 1.2: Some convex sets in the plane.

**Example 1.1.3.** *(Approximation)* In many applications of mathematics different approximation problems arise. Many such problems are of the following type: given some set $S \subset \mathbb{R}^n$ and a vector $a \notin S$, find a vector $x \in S$ which is as close to $a$ as possible among elements in $S$. The form of this problem depends on the set $S$ (and $a$) and how one measures the distance between vectors. In order to measure distance one may use the Euclidean norm (given by ($\|x\| = (\sum_{j=1}^n |x_j|^2)^{1/2}$) or some other norm. (We shall discuss different vector norms in Chapter 5).

Is there any connection to convexity here? First, norm functions, i.e., functions $x \to \|x\|$, are convex functions. This is so for all norms, not just the Euclidean norm. Second, a basic question is if a nearest point (to $a$ in $S$) exists. The answer is yes, provided that $S$ is a closed set. We discuss closed sets (and some topology) in Chapter 2. Next, we may be interested in knowing if there are more than one point that is nearest to $a$ in $S$. It turns out that if $S$ is a convex set (and the norm is the Euclidean norm), then the nearest point is *unique*. This may not be so for nonconvex sets. Even more can be said, as a theorem of Motzkin says that......

Well, we keep Motzkin's theorem a secret for the time being.

Hopefully, you now have an idea of what convexity is and where convexity questions arises. Let us start the work!

## 1.2 Nonnegative vectors

`nonnegative vector`

We are here concerned with the set $\mathbb{R}^n$ of real vectors $x = (x_1, \ldots, x_n)$. We use boldface symbols for vectors and matrices. The set (vector space) of all real matrices with $m$ rows and $n$ columns is denoted by $\mathbb{R}^{m,n}$. From linear algebra we know how to sum vectors and that we can multiply a vector by a scalar (a real number). Convexity deals with inequalities, and it is convenient to say that $x \in \mathbb{R}^n$ is *nonnegative* if each component $x_i$ is nonnegative. We let $\mathbb{R}^n_+$ denote the set of all nonnegative vectors. The zero vector is written $O$ (the dimension is suppressed, but should be clear from the context). We shall frequently use inequalities for vectors, so if $x, y \in \mathbb{R}^n$ we write

$$x \leq y \quad (\text{or } y \geq x)$$

and this means that $x_i \leq y_i$ for $i = 1, \ldots, n$. Note that this is equivalent to that $y - x \geq O$.

**Exercise 1.1.** *Let $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$ and assume that $x_1 \leq x_2$ and $y_1 \leq y_2$. Verify that the inequality $x_1 + y_1 \leq x_2 + y_2$ also holds. Let now $\lambda$ be a nonnegative real number. Explain why $\lambda x_1 \leq \lambda x_2$ holds. What happens if $\lambda$ is negative?*

**Example 1.2.1.** *(The nonnegative real vectors)* The sum of two nonnegative numbers is again a nonnegative number. Similarly, we see that the sum of two nonnegative vectors is a nonnegative vector. Moreover, if we multiply a nonnegative vector by a nonnegative number, we get another nonnegative vector. These two properties may be summarized by saying that $\mathbb{R}^n_+$ is *closed* under addition and multiplication by nonnegative scalars. We shall see that this means that $\mathbb{R}^n_+$ is a convex cone, a special type of convex set.

**Exercise 1.2.** *Think about the question in Exercise* 1.1 *again, now in light of the properties explained in Example* 1.2.1.

**Exercise 1.3.** *Let $a \in \mathbb{R}^n_+$ and assume that $x \leq y$. Show that $a^T x \leq a^T y$. What happens if we do not require $a$ to be nonnegative here?*

## 1.3 Linear programming

A *linear programming* problem (LP problem, for short) is an optimization problem where one wants to maximize or minimize some linear function $c^T x$ of the variable vector $x = (x_1, \ldots, x_n)$ over a certain set. This set is the solution set of a system of linear equations and inequalities in $x$. More specifically, an LP problem in standard form is

`linear programming problem`

$$\begin{array}{llllll}
\text{maximize} & c_1 x_1 & + & \ldots & + c_n x_n & \\
\text{subject to} & & & & & \\
& a_{11} x_1 & + & \ldots & + a_{1n} x_n & \leq b_1; \\
& & & \vdots & & \\
& a_{m1} x_1 & + & \ldots & + a_{mn} x_n & \leq b_m; \\
& & x_1, \ldots, x_n & & \geq 0. &
\end{array} \tag{1.1}$$

With our notion of nonnegativity of vectors this LP problem may be written nicely in matrix form as follows

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & \\ & \begin{array}{rl} Ax & \leq b; \\ x & \geq O. \end{array} \end{array} \tag{1.2}$$

Here $A = [a_{i,j}]$ is the $m \times n$ matrix with $(i,j)$th element being $a_{i,j}$ and $b$ is the column vector with $i$th component $b_i$. We recall that each vector $x$ is called *feasible* in the LP problem (1.2) if it satisfies $Ax \leq b$ and $x \geq O$. Let $P$ be the set of all feasible solutions in (1.2). The properties of this set depend, of course, on the coefficient matrix $A$ and the right-hand side $b$. But, is there some interesting property that is shared by all such sets $P$? Yes, it is described next.

**Example 1.3.1.** *(Linear programming)* Let $P$ be the feasible set of (1.2) and assume that $P$ is nonempty. Choose two distinct feasible points, say $x_1$ and $x_2$. Thus, $x_1, x_2 \in P$ and $x_1 \neq x_2$. What can be said about the vector $z = (1/2)x_1 + (1/2)x_2$? Geometrically, $z$ is the midpoint on the line segment $L$ in $\mathbb{R}^n$ between $x_1$ and $x_2$. But does $z$ lie in $P$? First, we see that $z \geq O$ (recall Example 1.2.1). Moreover, $Az = A[(1/2)x_1 + (1/2)x_2] = (1/2)Ax_1 + (1/2)Ax_2 \leq (1/2)b + (1/2)b = b$ again by our rules for calculating with nonnegative vectors. This shows that $z$ does lie in $P$, so it is also a feasible solution of the LP problem. Now, exactly the same thing happens if we consider another point, say $w$ on the line segment $L$. We know that $w$ may be written as $(1-\lambda)x_1 + \lambda x_2$ (or, if you prefer, $x_1 + \lambda(x_2 - x_1)$) for some scalar $\lambda$ satisfying $0 \leq \lambda \leq 1$. Thus, $P$ has the property that it contains all points on the line segment between two points in $P$. This is precisely the property that $P$ is convex.

An attempt to illustrate the geometry of linear programming is given in Fig. 1.3 (where the feasible region is the solution set of five linear inequalities).

## 1.4 Convex sets, cones and polyhedra

convex set

We now define our basic notion. A set $C \subseteq \mathbb{R}^n$ is called *convex* if $(1-\lambda)x_1 + \lambda x_2 \in C$ whenever $x_1, x_2 \in C$ and $0 \leq \lambda \leq 1$. Geometrically, this means that $C$ contains the line segment between each pair of points in $C$. In the previous example we showed that the set

$$P = \{x \in \mathbb{R}^n : Ax \leq b,\ x \geq O\} \tag{1.3}$$

is convex for all $A \in \mathbb{R}^{m,n}$ and $b \in \mathbb{R}^m$. In fact, this set is a very special convex set, called a polyhedron. Polyhedra is the subject of a later chapter. How can we

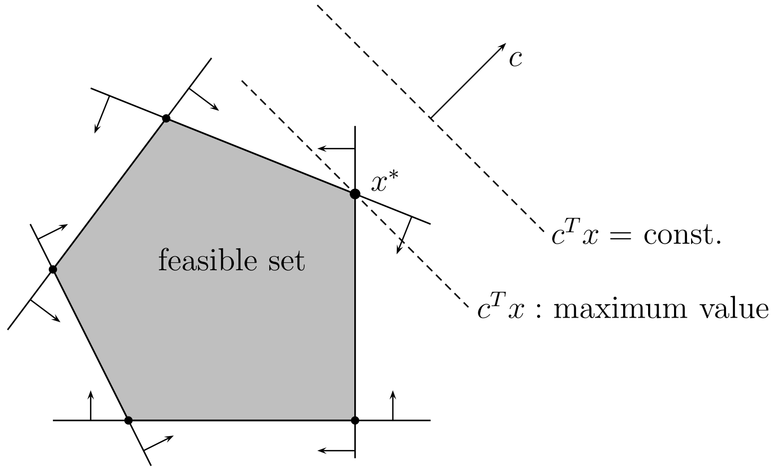


Figure 1.3: Linear programming

.

prove that a set is convex? The direct way is to use the definition as we did in Example 1.3.1. Later we learn some other useful techniques. How can we verify that a set $S$ is *not* convex? Well, it suffices to find two points $x_1$ and $x_2$ and $0 \leq \lambda \leq 1$ with the property that $(1-\lambda)x_1 + \lambda x_2 \notin S$ (you have then found a kind of "hole" in $S$).

**Example 1.4.1.** *(The unit ball)* The unit ball in $\mathbb{R}^n$ is the set $B = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, i.e., the set of points with Euclidean distance at most one to the origin. (So $\|x\| = (\sum_j |x_j|^2)^{1/2}$ is the Euclidean, or $l_2$-norm, of the vector $x \in \mathbb{R}^n$). We shall show that $B$ is convex. To do this we use the definition of convexity combined with the triangle inequality which says that

$$\|u+v\| \leq \|u\| + \|v\| \quad \text{for } u, v \in \mathbb{R}^n.$$

So let $x, y \in B$ and $\lambda \in [0,1]$. We want to show that $(1-\lambda)x + \lambda y \in B$, i.e., that $\|(1-\lambda)x + \lambda y\| \leq 1$. We use the triangle inequality (and norm properties) and calculate $\|(1-\lambda)x + \lambda y\| \leq \|(1-\lambda)x\| + \|\lambda y\| = (1-\lambda)\|x\| + \lambda\|y\| \leq (1-\lambda) + \lambda = 1$. Therefore $B$ is convex.

**Exercise 1.4.** *Show that every ball $B(a,r) := \{x \in \mathbb{R}^n : \|x - a\| \leq r\}$ is convex (where $a \in \mathbb{R}^n$ and $r \geq 0$).*

Some examples of convex sets in $\mathbb{R}^2$ are found in Fig. 1.2.

`linear system`

By a *linear system* we mean a finite set of linear equations and/or linear inequalities involving variables $x_1, \ldots, x_n$. For example, the set $P$ in (1.3) was defined as the solution set of a linear system. Consider the linear system $x_1 + x_2 = 3$, $x_1 \geq 0$,

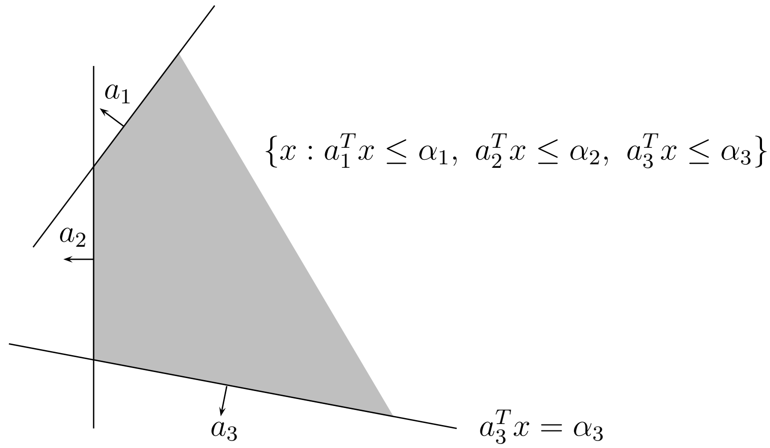


Figure 1.4: Linear system and polyhedron

$x_2 \ge 0$ in the variables $x_1, x_2$. The solution set is the set of points $(x_1, 3 - x_1)$ where $0 \le x_1 \le 3$. This linear system may be written differently. For instance, an equivalent form is $x_1 + x_2 \le 3$, $-x_1 - x_2 \le -3$, $-x_1 \le 0$, $-x_2 \le 0$. Here we only have $\le$-inequalities and these two systems clearly have the same solution set. From this small example, it should be clear that any linear system may easily be converted to a system (in the same variables) with only linear inequalities of $\le$-form, i.e., a linear system $ax \le b$. Motivated by these considerations, we define

`polyhedron` a *polyhedron* in $\mathbb{R}^n$ as a set of the form $\{x \in \mathbb{R}^n : Ax \le b\}$ where $A \in \mathbb{R}^{m,n}$ and $b \in \mathbb{R}^m$ ($m$ is arbitrary, but finite). Thus, a polyhedron is the solution set of a linear system $Ax \le b$, see Fig. 1.4. As we observed, this means that the solution set of *any* linear system is a polyhedron. Moreover, by repeating the argument of Exercise 1.3.1 we have the following result.

**Proposition 1.4.1** (Polyhedra)**.** *The solution set of any linear system in the variable $x \in \mathbb{R}^n$ is a polyhedron. Every polyhedron is a convex set.*

**Project 1.1** (Different LP forms)**.** Often LP problems are written in different forms than the one in (1.2). For instance, the feasible set may one of the following ones

$$\begin{aligned} P_0 &= \{x_0 \in \mathbb{R}^{n_0} : A_0 x_0 \le b_0,\ x_0 \ge O\}; \\ P_1 &= \{x_1 \in \mathbb{R}^{n_1} : A_1 x_1 = b_1,\ x_1 \ge O\}; \\ P_2 &= \{x_2 \in \mathbb{R}^{n_2} : A_2 x_2 \le b_2\}. \end{aligned} \tag{1.4}$$

All these sets are polyhedra as explained above. You are now asked to work out that these three sets are "equally general" in the sense that each $P_i$ may be written as a set $P_j$ for all $i$ and $j$. We have already mentioned how one can write $P_0$ and $P_1$ in the form $P_2$ (rewriting each equation as a pair of $\le$-inequalities). Note that, in this process, we could use the same number of variables (so, for instance, $n_2 = n_1$). However, this is not so when you write $P_2$ in the form $P_1$ (or $P_0$). Actually, we need two techniques for going from $P_2$ to (say) $P_1$.

The first technique is to introduce equations instead of inequalities. Recall that $A_2x_2 \leq b_2$ means that the vector $z$ defined by the equation $z = b_2 - A_2x_2$ is nonnegative. So, by introducing additional variables you may do the job. Explain the details.

The second technique is to make sure that *all* variables are required to be nonnegative. To see this, we observe that a variable $x_j$ with no sign constraint, may be replaced by two nonnegative variables $x'_j$ and $x''_j$ by introducing the equation $x_j = x'_j - x''_j$. The reason is simply that any real number may be written as a difference between two nonnegative numbers. Explain the details in the transformation.

Note that it is common to say simply that that "a linear system $Ax \leq b$ may be written in the form $Ax = b,\ x \geq O$" although this may require a different $x$, $A$ and $b$. Similar terminology is used for LP problems in different forms.

**Exercise 1.5.** *Explain how you can write the LP problem max* $\{c^Tx : Ax \leq b\}$ *in the form max* $\{c^Tx : Ax = b,\ x \geq O\}$.

**Example 1.4.2.** *(Optimal solutions in LP)* Consider an LP problem, for instance max $\{c^Tx : x \in P\}$ where $P$ is a polyhedron in $\mathbb{R}^n$. We assume that the problem has a finite optimal value $v := \max\{c^Tx : x \in P\}$. Recall that the set of optimal solutions is the set

$$F = \{x \in P : c^Tx = v\}.$$

Give an example with two variables and illustrate $P$, $c$ and $F$.

Next, show that $F$ is a convex set. In fact, $F$ is a polyhedron. Why? We mention that $F$ is a special subpolyhedron of $P$, contained in the boundary of $P$. Later we shall study such sets $F$ closer, they are so-called faces of $P$. For instance, we shall see that there are only finitely many faces of $P$. Thus, there are only finitely many possible sets of optimal solutions of LP problems with $P$ as the feasible set.

**Example 1.4.3.** *(Probabilities)* Let $T = \{t_1, \ldots, t_n\}$ be a set of $n$ real numbers. Consider a discrete *stochastic variable* $X$ with values in $T$ and let the probability of the event that $X = t_j$ be equal to $p_j$ for $j = 1, \ldots, n$. Probabilities are nonnegative and sum to 1, so the vector of probabilities $p = (p_1, \ldots, p_n)$ lies in the set $S_n = \{x \in \mathbb{R}^n : x \geq O,\ \sum_{j=1}^n x_j = 1\}$. This set is a polyhedron. It is called the standard simplex in $\mathbb{R}^n$ for reasons we explain later.

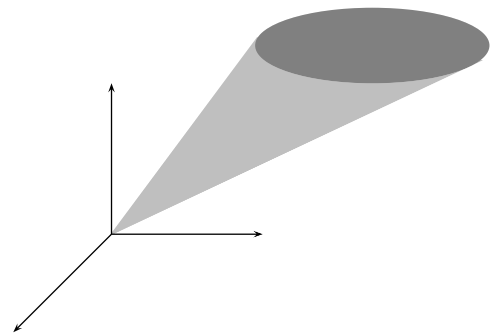

Figure 1.5: A convex cone in $\mathbb{R}^3$

**Exercise 1.6.** *Make a drawing of the standard simplices $S_1$, $S_2$ and $S_3$. Verify that each unit vector $e_j$ lies in $S_n$ ($e_j$ has a one in position $j$, all other components are zero). Each $x \in S_n$ may be written as a linear combination $x = \sum_{j=1}^n \lambda_j e_j$ where each $\lambda_j$ is nonnegative and $\sum_{j=1}^n \lambda_j = 1$. How? Can this be done in several ways?*

`convex cone` A set $C \subseteq \mathbb{R}^n$ is called a *convex cone* if $\lambda_1 x_1 + \lambda_2 x_2 \in C$ whenever $x_1, x_2 \in C$ and $\lambda_1, \lambda_2 \geq 0$. An example is $\mathbb{R}^n_+$, the set of nonnegative vectors in $\mathbb{R}^n$. A convex cone in $\mathbb{R}^3$ is shown in Fig. 1.5. Note that every (nonempty) convex cone contains $O$ (just let $\lambda_1 = \lambda_2 = 0$ in the definition). Moreover, a convex cone is closed under multiplication by a nonnegative scalar: if $x \in C$ and $\lambda \in \mathbb{R}_+$, then $\lambda x \in C$. The reader should verify this property based on the definition.

**Exercise 1.7.** *Show that each convex cone is indeed a convex set.*

There are two examples of convex cones that are important for linear programming.

**Exercise 1.8.** *Let $A \in \mathbb{R}^{m,n}$ and consider the set $C = \{x \in \mathbb{R}^n : Ax \leq O\}$. Prove that $C$ is a convex cone.*

`polyhedral cone` A convex cone of the form $\{x \in \mathbb{R}^n : Ax \leq O\}$ where $A \in \mathbb{R}^{m,n}$ is called a *polyhedral cone* . Let $x_1, \ldots, x_t \in \mathbb{R}^n$ and let $C(x_1, \ldots, x_t)$ be the set of vectors of the form

$$\sum_{j=1}^{t} \lambda_j x_j$$

where $\lambda_j \geq 0$ for each $j = 1, \ldots, t$.

**Exercise 1.9.** *Prove that $C(x_1, \ldots, x_t)$ is a convex cone.*

`finitely generated cone` `ray` A convex cone of the form $C(x_1, \ldots, x_t)$ is called a *finitely generated cone*, and we say that it is *generated* by the vectors $x_1, \ldots, x_t$. If $t = 1$ so $C = \{\lambda x_1 : \lambda \geq 0\}$, $C$ is called a *ray*. More generally, the set $R = \{x_0 + \lambda x_1 : \lambda \geq 0\}$ is called a *halfline* and we say that $x_1$ is a *direction vector* for $R$. Thus, a ray is a halfline starting in the origin. halfline direction vector

Later we shall see (and prove) the interesting fact that these two classes of cones coincide: a convex cone is polyhedral if and only if it is finitely generated.

**Exercise 1.10.** *Let $S = \{(x, y, z) : z \geq x^2 + y^2\} \subset \mathbb{R}^3$. Sketch the set and verify that it is a convex set. Is $S$ a finitely generated cone?*

## 1.5 Linear algebra and affine sets

Although we assume that the reader is familiar with linear algebra, it is useful to have a quick look at some important linear algebra notions at this point. Here is a small linear algebra project.

**Project 1.2** (A linear algebra reminder)**.** Linear algebra is the foundation of convex analysis. We should recall two important notions: linear independence and linear subspace. linear algebra concepts

Let $x_1, \ldots, x_t$ be vectors in $\mathbb{R}^n$. We say that $x_1, \ldots, x_t$ are *linearly independent* if $\sum_{j=1}^t \lambda_j x_j = O$ implies that $\lambda_1 = \ldots = \lambda_t = 0$. Thus, the only way to write the zero vector $O$ as a linear combination $\sum_{j=1}^t \lambda_j x_j$ of the given vectors $x_1, \ldots, x_t$ is the trivial way with all coefficients $\lambda_j$ being zero. This condition may be expressed in matrix notation when we introduce a matrix $x$ with $j$th column being the vector $x_j$. Thus, $x \in \mathbb{R}^{n,t}$ and linear independence of $x_1, \ldots, x_t$ means that $x\lambda = O$ implies that $\lambda = O$ (we then say that $x$ has full *column rank*). As a small example, consider the vectors $x_1 = (1, 0, -1)$ and $x_2 = (1, 2, 3)$ in $\mathbb{R}^3$. These are linearly independent as $\lambda_1 x_1 + \lambda_2 x_2$ implies that $\lambda_2 = 0$ (consider the second component) and therefore also $\lambda_1 = 0$. Note that any set of vectors containing the zero vector is linearly dependent (i.e., not linearly independent). linearly independent

Show the following: if $x = \sum_{j=1}^t \lambda_j x_j = \sum_{j=1}^t \mu_j x_j$, then $\lambda_j = \mu_j$ for each $j = 1, \ldots, n$. Thus, the vector $x$ can only be written as a linear combination of the vectors $x_1, \ldots, x_t$ in a *unique* way. Give an example illustrating that such a uniqueness result does not hold for linearly dependent vectors.

We proceed to linear subspaces. Recall that a set $L \subseteq \mathbb{R}^n$ is called a *(linear) subspace* if it is closed under addition and multiplication with scalars. This means that $\lambda_1 x_1 + \lambda_2 x_2 \in L$ whenever $x_1, x_2 \in L$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. A very important fact is that every linear subspace $L$ may be represented in two different ways. First, consider a maximal set of linearly independent vectors, say $x_1, \ldots, x_t$, in $L$. This means that if we add a vector in $L$ to this set we obtain a linearly dependent set of vectors. Then $x_1, \ldots, x_t$ *spans* $L$ in the sense that $L$ is precisely the set of linear combinations of the vectors $x_1, \ldots, x_t$. Moreover, as explained above due to the linear independence, each vector in $L$ may be written uniquely as a linear combination of $x_1, \ldots, x_t$. This set of $t$ vectors is called a *basis* of $L$. A crucial fact is that $L$ may have many bases, but they all have the same number of elements. This number $t$ is called the *dimension* of $L$. The second representation of a linear subspace $L$ is as the kernel of some matrix. We recall that the *kernel* (or *nullspace*) of a matrix $A \in \mathbb{R}^{m,n}$ is the set of vectors $x$ satisfying $Ax = O$. This set is denoted by $\mathrm{Ker}(A)$.

`basis` `kernel (nullspace)`

Check that the kernel of any $m \times n$ matrix is a linear subspace. Next, try to show the opposite, that every linear subspace is the kernel of some matrix. Confer with some linear algebra textbook (hint: orthogonal complements).

Why bother with linear subspaces in a text on convexity? One reason is that every linear subspace is a (very special) polyhedron; this is seen from the kernel representation $L = \{x \in \mathbb{R}^n : Ax = O\}$. It follows that every linear subspace is a convex set.

Prove, using the definitions, that every linear subspace is a convex set.

Our final point here is that the two different representations of linear spaces may be generalized to hold for large classes of convex sets. This will be important to us later, but we need to do some more work before these results can be discussed.

Linear algebra, of course, is much more than a study of linear subspaces. For instance, one of the central problems is to solve linear systems of equations. Thus, given a matrix $A \in \mathbb{R}^{m,n}$ and a vector $b \in \mathbb{R}^m$ we want to solve the linear equation $Ax = b$. Often, we have that $m = n$ and that $A$ is nonsingular (invertible). This means that the columns of $a$ are linearly independent and therefore $Ax = b$ has a unique solution. However, there are many interesting situations where one is concerned with rectangular linear systems, i.e., where the number of equations may not be equal to the number of variables. Examples here are optimization and regression analysis and approximation problems.

Now, any linear system of equations $Ax = b$ is also a linear system as we have defined it. Thus, the solution set of $Ax = b$ must be a polyhedron. But, this polyhedron is very special as we shall see next.

`affine set`

**Project 1.3** (Affine sets)**.** We say that a set $C \subseteq \mathbb{R}^n$ is *affine* provided that it contains the line through any pair of its points. This means that whenever $x_1, x_2 \in C$ and $\lambda \in \mathbb{R}$ the vector $(1-\lambda)x_1 + \lambda x_2$ also lies in $C$. Note that this vector equals $x_1 + \lambda(x_2 - x_1)$ and that, when $x_1$ and $x_2$ are distinct, the vector $x_2 - x_1$ is a direction vector for the line through $x_1$ and $x_2$.

For instance, a line in $\mathbb{R}^n$ is an affine set. Another example is the set $C = \{x_0 + \lambda_1 r_1 + \lambda_2 r_2 : \lambda_1, \lambda_2 \in \mathbb{R}\}$ which is a two-dimensional "plane" going through $x_0$ and spanned by the nonzero vectors $r_1$ and $r_2$. See Fig. 1.6 for an example.

Show that every affine set is a convex set!

Here is the connection between affine sets and linear systems of equations. Let $C$ be the solution set of $Ax = b$ where $A \in \mathbb{R}^{m,n}$ and $b \in \mathbb{R}^m$. Show that $C$ is an affine set! In particular, the solution set $H$ of a single equation $a^T x = \alpha$, where $a \neq O$, is an affine set. Such a set $H$ is called a hyperplane, and the vector $a$ is called a *normal vector* of the hyperplane. Give some examples of hyperplanes in $\mathbb{R}^2$, and in $\mathbb{R}^3$! We say that two hyperplanes $H$ and $H'$ are *parallel* if they have parallel normal vectors. Show that two hyperplanes in $\mathbb{R}^n$ that are not parallel must intersect! What kind of set is the intersection?

Are there any affine sets that are not the solution set of some system of equations? The answer is no, so we have

**Proposition 1.5.1** (Affine sets)**.** *Let $C$ be a nonempty subset of $\mathbb{R}^n$. Then $C$ is an affine set if and only if there is a matrix $A \in \mathbb{R}^{m,n}$ and a vector $b \in \mathbb{R}^m$ for some $m$ such that*

$$C = \{x \in \mathbb{R}^n : Ax = b\}.$$

*Moreover, $C$ may be written as $C = L + x_0 = \{x + x_0 : x \in L\}$ for some linear subspace $L$ of $\mathbb{R}^n$. The subspace $L$ is unique.*

We leave the proof as an exercise.

**Project 1.4** (Preservation of convexity)**.** Convexity is preserved under several operations, and the next result describes a few of these. We here use som set notation. When $A, B \subseteq \mathbb{R}^n$ their sum is the set $A + B = \{x + y : x \in A, y \in B\}$. Similarly, when $\lambda \in \mathbb{R}$ we let $\lambda A := \{\lambda x : x \in A\}$. In each situation below you should give an example, and try to prove the statement.

1. Let $C_1, C_2$ be convex sets in $\mathbb{R}^n$ and let $\lambda_1, \lambda_2$ be real numbers. Then $\lambda_1 C_1 + \lambda_2 C_2$ is convex.

2. The intersection of any (even infinite) family of convex sets is a convex set (you may have shown this already!).

3. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an affine transformation, i.e., a function of the form $T(x) = Ax + b$, for some $A \in \mathbb{R}^{m,n}$ and $b \in \mathbb{R}^m$. Then $T$ maps convex sets to convex sets, i.e, if $C$ is a convex set in $\mathbb{R}^n$, then $T(C) = \{T(x) : x \in C\}$ is a convex set in $\mathbb{R}^m$.

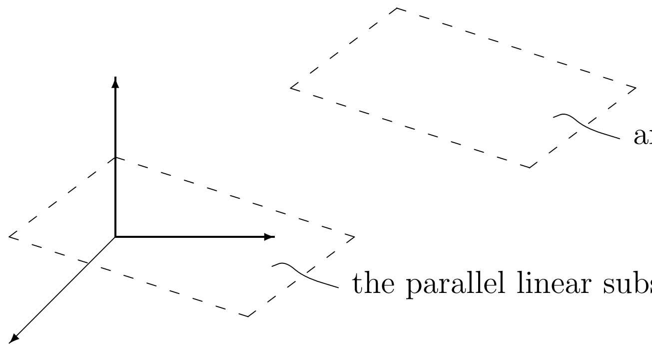


Figure 1.6: Affine set

## 1.6 Exercises

**Exercise 1.11.** *Consider the linear system* $0 \leq x_i \leq 1$ *for* $i = 1, \ldots, n$ *and let* $P$ *denote the solution set. Explain how to solve a linear programming problem*

$$\max\{c^T x : x \in P\}.$$

*What if the linear system was* $a_i \leq x_i \leq b_i$ *for* $i = 1, \ldots, n$*. Here we assume* $a_i \leq b_i$ *for each* $i$*.*

**Exercise 1.12.** *Is the union of two convex sets again convex?*

**Exercise 1.13.** *Determine the sum* $A + B$ *in each of the following cases:*

$$\begin{array}{lll}
\text{(i)} & A = \{(x,y) : x^2 + y^2 \leq 1\}, & B = \{(3,4)\}; \\
\text{(ii)} & A = \{(x,y) : x^2 + y^2 \leq 1\}, & B = [0,1] \times \{0\}; \\
\text{(iii)} & A = \{(x,y) : x + 2y = 5\}, & B = \{(x,y) : x = y,\ 0 \leq x \leq 1\}; \\
\text{(iv)} & A = [0,1] \times [1,2], & B = [0,2] \times [0,2].
\end{array}$$

**Exercise 1.14.** *(i) Prove that, for every* $\lambda \in \mathbb{R}$ *and* $A, B \subseteq \mathbb{R}^n$*, it holds that* $\lambda(A+B) = \lambda A + \lambda B$*. (ii) Is it true that* $(\lambda + \mu)A = \lambda A + \mu A$ *for every* $\lambda, \mu \in \mathbb{R}$ *and* $A \subseteq \mathbb{R}^n$*? If not, find a counterexample. (iii) Show that, if* $\lambda, \mu \geq 0$ *and* $A \subseteq \mathbb{R}^n$ *is convex, then* $(\lambda + \mu)A = \lambda A + \mu A$*.*

**Exercise 1.15.** *Show that if* $C_1, \ldots, C_t \subseteq \mathbb{R}^n$ *are all convex sets, then* $C_1 \cap \ldots \cap C_t$ *is convex. Do the same when all sets are affine (or linear subspaces, or convex cones). In fact, a similar result for the intersection of any family of convex sets. Explain this.*

**Exercise 1.16.** *Consider a family (possibly infinite) of linear inequalities* $a_i^T x \leq b_i$, $i \in I$*, and* $C$ *be its solution set, i.e.,* $C$ *is the set of points satisfying all the inequalities. Prove that* $C$ *is a convex set.*

**Exercise 1.17.** *Consider the unit disc* $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$ *in* $\mathbb{R}^2$*. Find a family of linear inequalities as in the previous problem with solution set* $S$*.*

**Exercise 1.18.** *Is the unit ball* $B = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$ *a polyhedron?*

**Exercise 1.19.** *Consider the unit ball* $B_\infty = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$ *is convex. Here* $\|x\|_\infty = \max_j |x_j|$ *is the max norm of* $x$*. Show that* $B_\infty$ *is a polyhedron. Illustrate when* $n = 2$*.*

**Exercise 1.20.** *Consider the unit ball* $B_1 = \{x \in \mathbb{R}^n : \|x\|_1 \leq 1\}$ *is convex. Here* $\|x\|_1 = \sum_{j=1}^n |x_j|$ *is the absolute norm of* $x$*. Show that* $B_1$ *is a polyhedron. Illustrate when* $n = 2$*.*

**Exercise 1.21.** *Prove Proposition* 1.5.1*.*

**Exercise 1.22.** *Let* $C$ *be a nonempty affine set in* $\mathbb{R}^n$*. Define* $L = C - C$*. Show that* $L$ *is a subspace and that* $C = L + x_0$ *for some vector* $x_0$*.*

## SUMMARY OF NEW CONCEPTS AND RESULTS:

- convex set
- convex cone (finitely generated, polyhedral)
- polyhedron
- linear system
- linear programming
- linear algebra: linear independence, linear subspace, representations
- affine set
- the norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$

# Chapter 2

# Convex hulls and Carathéodory's theorem

We have now introduced our main objects: convex sets and special convex sets (convex cones, polyhedra). In this chapter we investigate these objects further, and a central notion is that of convex combinations of points. We shall define the dimension of a set and study the topological properties of convex sets.

## 2.1 Convex and nonnegative combinations

In convex analysis one is interested in certain special linear combinations of vectors that represent "mixtures" of points. Consider vectors $x_1, \ldots, x_t \in \mathbb{R}^n$ and nonnegative numbers (coefficients) $\lambda_j \geq 0$ for $j = 1, \ldots, t$ such that $\sum_{j=1}^{t} \lambda_j = 1$. Then the vector $x = \sum_{j=1}^{t} \lambda_j x_j$ is called a *convex combination* of $x_1, \ldots, x_t \in \mathbb{R}^n$, see Fig. 2.1. Thus, a convex combinations is a special linear combination where the coefficients are nonnegative and sum to one. A special case is when $t = 2$ and we have a convex combination of two points: $\lambda_1 x_1 + \lambda_2 x_2 = (1 - \lambda_2)x_2 + \lambda_2 x_2$. Note that we may reformulate our definition of a convex set by saying that it is closed under convex combinations of each pair of its points.

`convex combination`

We give a remark on the terminology here. If $S \subseteq \mathbb{R}^n$ is any set, we say that $x$ is a convex combination of points in $S$ if $x$ may be written as a convex combination of a *finite* number of points in $S$. Thus, there are no infinite series or convergence questions we need to worry about.

**Example 2.1.1.** *(Convex combinations)* Consider the following four vectors in $\mathbb{R}^2$: $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. The point $(1/2, 1/2)$ is a convex combination of $(1, 0)$ and $(0, 1)$ as we have $(1/2, 1/2) = (1/2) \cdot (1, 0) + (1/2) \cdot (0, 1)$. We also see that $(1/2, 1/2)$ is a convex combination of the vectors $(0, 0)$ and $(1, 1)$. Thus, a point may have different representations as convex combinations.

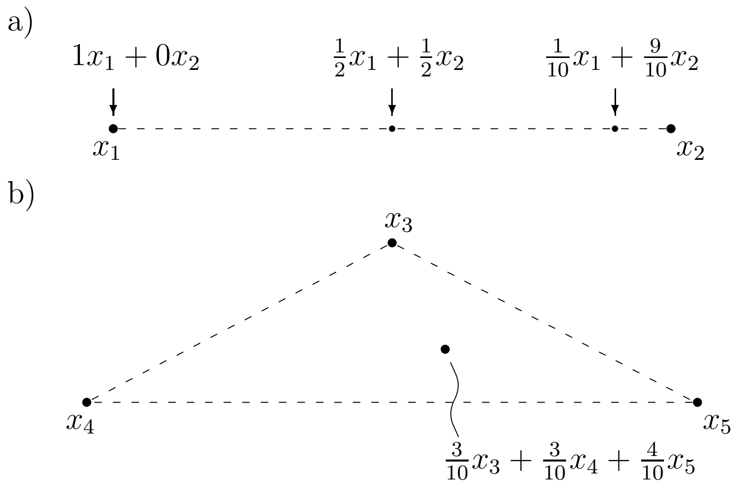


Figure 2.1: Convex combinations

Similarly, we call a vector $\sum_{j=1}^{t} \lambda_j x_j$ a *nonnegative combination* of the vectors $x_1, \ldots, x_t$ when $\lambda_1, \ldots, \lambda_t \geq 0$. It is clear that every convex combination is also a nonnegative combination, and that every nonnegative combination is a linear combination. `nonnegative combination`

**Exercise 2.1.** Illustrate some combinations (linear, convex, nonnegative) of two vectors in $\mathbb{R}^2$.

The following result says that a convex set is closed under the operation of taking convex combinations. This is similar to a known fact for linear subspaces: they are closed under linear combinations.

**Proposition 2.1.1** (Convex sets)**.** *A set $C \subseteq \mathbb{R}^n$ is convex if and only if it contains all convex combinations of its points. A set $C \subseteq \mathbb{R}^n$ is a convex cone if and only if it contains all nonnegative combinations of its points.*

*Proof.* If $C$ contains all convex combinations of its points, then this also holds for combinations of two points, and then $C$ must be convex. Conversely, assume that $C$ is convex. We prove that $C$ contains every convex combination of $t$ of its elements using induction on $t$. When $t = 2$ this is clearly true as $C$ is convex. Assume next that $C$ contains any convex combination of $t-1$ elements (where $t \geq 3$). Let $x_1, \ldots, x_t \in C$ and $\lambda_j > 0$ for $j = 1, \ldots, t$ where $\sum_{j=1}^{t} \lambda_j = 1$. Thus, $0 < \lambda_1 < 1$ (if $\lambda_1 = 1$ we would get $t = 1$). We have that

$$(*) \;\; x = \lambda_1 x_1 + (1 - \lambda_1) \sum_{j=2}^{t} (\lambda_j / (1 - \lambda_1)) x_j.$$

Note that $\sum_{j=2}^{t} \lambda_j/(1-\lambda_1) = 1$, and each element is nonnegative. Therefore the vector $y = \sum_{j=2}^{t}(\lambda_j/(1-\lambda_1))x_j$ is a convex combination of $t-1$ elements in $C$ so $y \in C$, by the induction hypothesis. Moreover, $x$ is a convex combination of $x_1$ and $y$, both in $C$, and therefore $x \in C$ as desired. The result concerning conical combinations is proved similarly. □

**Exercise 2.2.** *Choose your favorite three points $x_1, x_2, x_3$ in $\mathbb{R}^2$, but make sure that they do not all lie on the same line. Thus, the three points form the corners of a triangle $C$. Describe those points that are convex combinations of two of the three points. What about the interior of the triangle $C$, i.e., those points that lie in $C$ but not on the boundary (the three sides): can these points be written as convex combinations of $x_1, x_2$ and $x_3$? If so, how?*

## 2.2 The convex hull

Consider two distinct points $x_1, x_2 \in \mathbb{R}^n$ (let $n = 2$ if you like). There are many convex sets that contain both these points. But, is there a smallest convex set that contains $x_1$ and $x_2$? It is not difficult to answer this positively. The line segment $L$ between $x_1$ and $x_2$ has these properties: it is convex, it contains both points and any other convex set containing $x_1$ and $x_2$ must also contain $L$. Note here that $L$ is precisely the set of convex combinations of the two points $x_1$ and $x_2$. Similarly, if $x_1, x_2, x_3$ are three points in $\mathbb{R}^2$ (or $\mathbb{R}^n$) not all on the same line, then the triangle $T$ that they define must be the smallest convex set containing $x_1, x_2$ and $x_3$. And again we note that $T$ is also the set of convex combinations of $x_1, x_2, x_3$ (confer Exercise 2.2).

More generally, let $S \subseteq \mathbb{R}^n$ be any set. Define the *convex hull* of $S$, denoted by $\text{conv}(S)$ as the set of all convex combinations of points in $S$ (see Fig. 2.2). The convex hull of two points $x_1$ and $x_2$, i.e., the line segment between the two points, is often denoted by $[x_1, x_2]$. An important fact is that $\text{conv}(S)$ is a convex set, whatever the set $S$ might be. Thus, taking the convex hull becomes a way of producing new convex sets. `convex hull`

**Exercise 2.3.** *Show that* $\text{conv}(S)$ *is convex for all $S \subseteq \mathbb{R}^n$. (Hint: look at two convex combinations $\sum_j \lambda_j x_j$ and $\sum_j \mu_j y_j$, and note that both these points may be written as a convex combination of the same set of vectors.)*

**Exercise 2.4.** *Give an example of two distinct sets $S$ and $T$ having the same convex hull. It makes sense to look for a smallest possible subset $S_0$ of a set $S$ such that $S = \text{conv}(S_0)$. We study this question later.*

**Exercise 2.5.** *Prove that if $S \subseteq T$, then $\text{conv}(S) \subseteq \text{conv}(T)$.*

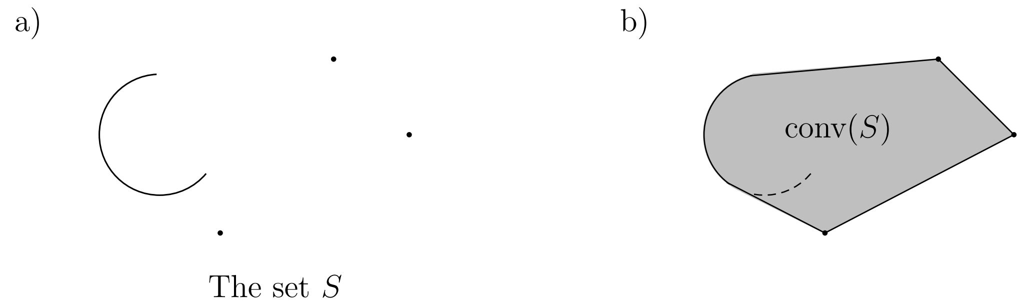


Figure 2.2: Convex hull

The following proposition tells us that the convex hull of a set $S$ is the smallest convex set containing $S$. Recall that the intersection of an arbitrary family of sets consists of the points that lie in all of these sets.

**Proposition 2.2.1** (Convex hull)**.** *Let $S \subseteq \mathbb{R}^n$. Then $\mathrm{conv}(S)$ is equal to the intersection of all convex sets containing $S$. Thus, $\mathrm{conv}(S)$ is the smallest convex set containing $S$.*

*Proof.* From Exercise 2.3 we have that $\mathrm{conv}(S)$ is convex. Moreover, $S \subseteq \mathrm{conv}(S)$; just look at a convex combination of one point! Therefore $W \subseteq \mathrm{conv}(S)$ where $W$ is defined as the intersection of all convex sets containing $S$. Now, consider a convex set $C$ containing $S$. Then $C$ must contain all convex combinations of points in $S$, this follows from Proposition 2.1.1. But then $\mathrm{conv}(S) \subseteq C$ and we conclude that $W$ (the intersection of such sets $C$) must contain $\mathrm{conv}(S)$. This concludes the proof. ☐

conical hull

What we have just done concerning convex combinations may be repeated for nonnegative combinations. Thus, when $S \subseteq \mathbb{R}^n$ we define the *conical hull* of $S$, denoted by $\mathrm{cone}(S)$ as the set of all nonnegative combinations of points in $S$. This set is always a convex cone. Moreover, we have that

**Proposition 2.2.2** (Conical hull)**.** *Let $S \subseteq \mathbb{R}^n$. Then $\mathrm{cone}(S)$ is equal to the intersection of all convex cones containing $S$. Thus, $\mathrm{cone}(S)$ is the smallest convex cone containing $S$.*

The proof is left as an exercise.

**Exercise 2.6.** *If $S$ is convex, then $\mathrm{conv}(S) = S$. Show this!*

**Exercise 2.7.** *Let $S = \{x \in \mathbb{R}^2 : \|x\|_2 = 1\}$, this is the unit circle in $\mathbb{R}^2$. Determine $\mathrm{conv}(S)$ and $\mathrm{cone}(S)$.*

**Example 2.2.1.** *(LP and convex cones)* Consider a linear programming problem $\max\{c^T x : x \in P\}$ where $P = \{x \in \mathbb{R}^n : Ax = b,\ x \geq O\}$. We have that $Ax = \sum_{j=1}^n x_j a_j$ where $a_j$ is the $j$th column of $a$. Thus, $P$ is nonempty if and only if $b \in \text{cone}(\{a_1, \ldots, a_n\})$. Moreover, a point $x$ is feasible in the LP problem (i.e., $x \in P$) if its components $x_j$ are the coefficients of $a_j$ when $b$ is represented as a nonnegative combination of $a_1, \ldots, a_n$.

We have seen that by taking the convex hull we produce a convex set whatever set we might start with. If we start with a finite set, a very interesting class of convex sets arise. A set $P \subset \mathbb{R}^n$ is called a *polytope* if it is the convex hull of a finite set of points in $\mathbb{R}^n$. Polytopes have been studied a lot during the history of mathematics. Some polytopes are illustrated in Fig. 2.4. The convex set in Fig. 2.2 b) is not a polytope. Today polytope theory is still a fascinating subject with a lot of activity. One of the reasons is its relation to linear programming, because most LP problems have a feasible set which is a polytope. In fact, we shall later prove an important result in polytope theory saying that a set is a polytope if and only if it is a bounded polyhedron. Thus, in LP problems with bounded feasible set, this set is really a polytope. `polytope`

**Example 2.2.2.** *(LP and polytopes)* Consider a polytope $P = \text{conv}(\{x_1, \ldots, x_t\})$. We want to solve the optimization problem

$$(*) \quad \max\{c^T x : x \in P\}$$

where $c \in \mathbb{R}^n$. As mentioned above, this problem is an LP problem, but we do not worry too much about this now. The interesting thing is the combination of a linear objective function and the fact that the feasible set is a convex hull of finitely many points. To see this, consider an arbitrary feasible point $x \in P$. Then $x$ may be written as a convex combination of the points $x_1, \ldots, x_t$, say $x = \sum_{j=1}^t \lambda_j x_j$ for some $\lambda_j \geq 0$, $j = 1, \ldots, t$ where $\sum_j \lambda_j = 1$. Define now $v = \max_j c^T x_j$. We then calculate

$$c^T x = c^T \sum_j \lambda_j x_j = \sum_{j=1}^t \lambda_j c^T x_j \leq \sum_{j=1}^t \lambda_j v = v \sum_{j=1}^t \lambda_j = v.$$

Thus, $v$ is an upper bound for the optimal value in the optimization problem $(*)$. We also see that this bound is attained whenever $\lambda_j$ is positive only for those indices $j$ satisfying $c^T x_j = v$. Let $J$ be the set of such indices. We conclude that the optimal solutions of the problem $(*)$ is the set

$$\text{conv}(\{x_j : j \in J\})$$

which is another polytope (contained in $P$). The procedure just described may be useful computationally if the number $t$ of points defining $P$ is not too large. In some cases, $t$ *is* too large, and then we may still be able to solve the problem $(*)$ by different methods, typically linear programming related methods.

## 2.3 Affine independence and dimension

We know what we mean by the dimension $\dim(L)$ of a linear subspace $L$ of $\mathbb{R}^n$: $\dim(L)$ is the cardinality of a basis in $L$, or equivalently, the maximal number of linearly independent vectors lying in $L$. This provides a starting point for defining the dimension of more general sets, in fact any set, in $\mathbb{R}^n$.

The forthcoming definition of dimension may be loosely explained as follows. Let $S$ be a set and pick a point $x_1$ in $S$. We want (the undefined) dimension of $S$ to tell how many (linearly) independent directions we can move in, starting from $x$ and still hit some point in $S$. For instance, consider the case when $S$ is convex (which is of main interest here). Say that we have a point $x \in S$ and can find other points $x_1, \ldots, x_t$ that also lie in $S$. Thus, by convexity we can "move" from $x$ in each of the directions $x_j - x$ for $j = 1, \ldots, t$ and still be in $S$ (if we do not go too far). If the vectors $x_1 - x, \ldots, x_t - x$ are linearly independent, and $t$ is largest possible, we say that $S$ has dimension $t$. We now make these ideas more precise.

`affinely independent vectors`

First, we introduce the notion of affine independence. A set of vectors $x_1, \ldots, x_t \in \mathbb{R}^n$ are called *affinely independent* if $\sum_{j=1}^t \lambda_j x_j = O$ and $\sum_{j=1}^t \lambda_j = 0$, imply that $\lambda_1 = \ldots = \lambda_t = 0$. This definition resembles the definition of linear independence except for the extra condition that the sum of the $\lambda$'s is zero. Note that if a set of vectors is linearly independent, then it is also affinely independent. In fact, there is a useful relationship between these two notions as the next proposition tells us.

**Proposition 2.3.1** (Affine independence)**.** *The vectors $x_1, \ldots, x_t \in \mathbb{R}^n$ are affinely independent if and only if the $t-1$ vectors $x_2 - x_1, \ldots, x_t - x_1$ are linearly independent.*

*Proof.* Let $x_1, \ldots, x_t \in \mathbb{R}^n$ be affinely independent and assume that $\lambda_2, \ldots, \lambda_t \in \mathbb{R}$ and $\sum_{j=2}^t \lambda_j (x_j - x_1) = O$. Then $(-\sum_{j=2}^t \lambda_j) x_1) + \sum_{j=2}^t \lambda_j x_j = O$. Note here that the sum of all the coefficients is zero, so by affine independence of $x_1, \ldots, x_t$ we get that $\lambda_2 = \ldots = \lambda_t = 0$. This proves that $x_2 - x_1, \ldots, x_t - x_1$ are linearly independent. Conversely, let $x_2 - x_1, \ldots, x_t - x_1$ be linearly independent and assume $\sum_{j=1}^t \lambda_j x_j = O$ and $\sum_{j=1}^t \lambda_j = 0$. Then $\lambda_1 = -\sum_{j=2}^t \lambda_j$ and therefore $O = (-\sum_{j=2}^t \lambda_j) x_1 + \sum_{j=2}^t \lambda_j x_j = \sum_{j=2}^t \lambda_j (x_j - x_1)$. But, as $x_2 - x_1, \ldots, x_t - x_1$

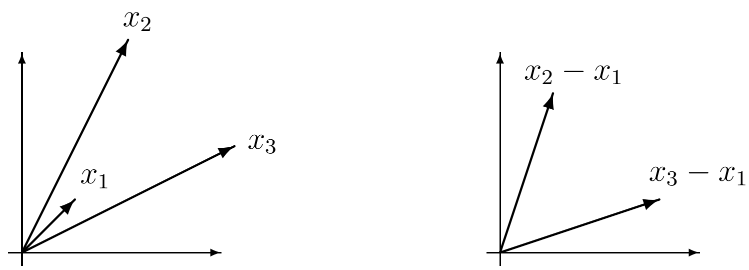


Figure 2.3: Affine independence

are linearly independent, we must have $\lambda_2 = \ldots = \lambda_t = 0$ and therefore also $\lambda_1 = -\sum_{j=2}^{t} \lambda_j = 0$. □

In the example shown in Fig. 2.3 the vectors $x_1, x_2, x_3$ are affinely independent and the vectors $x_2 - x_1$ and $x_3 - x_1$ are linearly independent.

**Exercise 2.8.** *Does affine independence imply linear independence? Does linear independence imply affine independence? Prove or disprove!*

A useful property of affine independence is that this property still holds whenever all our vectors are translated with a fixed vector as discussed in the next exercise.

**Exercise 2.9.** *Let $x_1, \ldots, x_t \in \mathbb{R}^n$ be affinely independent and let $w \in \mathbb{R}^n$. Show that $x_1 + w, \ldots, x_t + w$ are also affinely independent.*

We can now, finally, define the dimension of a set. The *dimension* of a set $S \subseteq \mathbb{R}^n$, denoted by $\dim(S)$, is the maximal number of affinely independent points in $S$ minus 1. So, for example in $\mathbb{R}^3$, the dimension of a point and a line is 0 and 1 respectively, and the dimension of the plane $x_3 = 0$ is 2. See Fig. 2.4 for some examples. `dimension`

**Exercise 2.10.** *Let $L$ be a linear subspace of dimension (in the usual linear algebra sense) $t$. Check that this coincides with our new definition of dimension above. (Hint: add $O$ to a "suitable" set of vectors).*

Consider a convex set $C$ of dimension $d$. Then there are (and no more than) $d+1$ affinely independent points in $C$. Let $S = \{x_1, \ldots, x_{d+1}\}$ denote a set of such points. Then the set of all convex combinations of these vectors, i.e., $\mathrm{conv}(S)$, is a polytope contained in $C$ and $\dim(S) = \dim(C)$. Moreover, let $A$ be the set of all vectors of the form $\sum_{j=1}^{t} \lambda_j x_j$ where $\sum_{j=1}^{t} \lambda_j = 1$ (no sign restriction of the $\lambda$'s). Then $A$ is an affine set containing $C$, and it is the smallest affine set with this property. $A$ is called the *affine hull* of $C$. `affine hull`

**Exercise 2.11.** *Prove the last statements in the previous paragraph.*

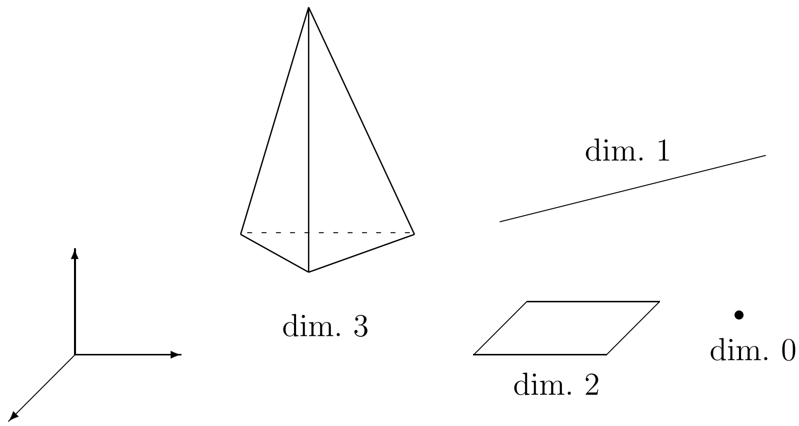


Figure 2.4: Dimensions

simplex

Some special polytopes and finitely generated cones are of particular interest. A *simplex* $P$ in $\mathbb{R}^n$ is the convex hull of a set $S$ of affinely independent vectors in $\mathbb{R}^n$. We recall from Example 1.4.3 the set $S_n = \{x \in \mathbb{R}^n : x \geq O,\ \sum_{j=1}^n x_j = 1\}$ which is the standard simplex in $\mathbb{R}^n$. It is indeed a simplex as $S_n = \mathrm{conv}(\{e_1, \ldots, e_n\})$ and the unit vectors $e_1, \ldots, e_n$ are affinely (even linearly) independent. Thus, $\dim(S_n) = n - 1$.

simplex cone

A *simplex cone* in $\mathbb{R}^n$ is a finitely generated convex cone $K$ spanned by linearly independent vectors. Then, clearly, $\dim(K)$ equals the number of these generating vectors.

**Proposition 2.3.2** (Unique representation)**.** *(i) Let the vectors $x_1, \ldots, x_t \in \mathbb{R}^n$ be affinely independent and consider the simplex $P = \mathrm{conv}(\{x_1, \ldots, x_t\})$ generated by these vectors. Then each point in $P$ has a unique representation as a convex combination of $x_1, \ldots, x_t$.*

*(ii) Let $x_1, \ldots, x_t \in \mathbb{R}^n$ be linearly independent and consider the simplex cone $C = \mathrm{cone}(\{x_1, \ldots, x_t\})$ generated by these vectors. Then each point in $C$ has a unique representation as a nonnegative combination of $x_1, \ldots, x_t$.*

## 2.4 Convex sets and topology

To study convex sets it is useful with some basic knowledge to topology. For instance, we want to discuss the boundary of a convex set. If this set is full-dimensional, like the unit ball $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$ in $\mathbb{R}^n$, then the boundary is $\{x \in \mathbb{R}^n : \|x\| = 1\}$ which may not be so surprising. But what is the "boundary" of the set $C = \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2+x_2^2 \leq 1\}$? We now give the proper definitions of these concepts.

A *closed ball* is a set of the form $B(a, r) = \{x \in \mathbb{R}^n : \|x-a\| \leq r\}$ where $a \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$, i.e. this set consists of all points with distance not larger than $r$ from $a$. The corresponding *open ball*, defined whenever $r > 0$, is $B^\circ(a, r) = \{x \in \mathbb{R}^n : \|x-a\| < r\}$. We shall now define the notions of open and closed sets. First, we should say that every closed ball is indeed a closed set, and every open ball is an open set. A set $S \subseteq \mathbb{R}^n$ is called *open* if it contains an open ball around each of its points, that is, for each $x \in S$ there is an $\epsilon > 0$ such that $B^\circ(a, \epsilon) \subseteq S$. For instance, in $\mathbb{R}$ each open interval $\{x \in \mathbb{R} : a < x < b\}$ is an open set. A set $S \subseteq \mathbb{R}^n$ is called *closed* if its (set) complement $\bar{S} = \{x \in \mathbb{R}^n : x \notin S\}$ is open. Every closed interval $\{x \in \mathbb{R} : a \leq x \leq b\}$ is a closed set.

`closed ball` `open ball` `open set` `closed set`

**Exercise 2.12.** *Construct a set which is neither open nor closed.*

A very useful fact is that closed sets may be characterized in terms of convergent sequences. We say that a point sequence $\{x^k\}_{k=1}^\infty \subset \mathbb{R}^n$ *converges* to $x$ if $\|x^k - x\| \to 0$ as $k \to \infty$. Each such sequence is called *convergent* and $x$ is called the *limit point* of the sequence. We also write $x^k \to x$ in this case.

`convergent sequence`

**Exercise 2.13.** *Show that $x^k \to x$ if and only if $x_j^k \to x_j$ for $j = 1, \ldots, n$. Thus, convergence of a point sequence simply means that all the component sequences are convergent.*

Consider now a set $S$ and a sequence $\{x^k\}$ in $S$ (meaning that each point $x^k$ lies in $S$). Assume that the sequence converges to the point $x$. Does this limit point lie in $S$? No, not in general, but this is true if $S$ is closed, see below.

A set $S$ in $\mathbb{R}^n$ is called *bounded* if there is a number $M$ such that $\|x\| \leq M$ for all $x \in S$. This means that $S$ does not contain points with arbitrary large norm, or equivalently, it does not contain points with arbitrary large components. A set is called *compact* if it is both closed and bounded. In optimization problems the feasible set is (almost always) a closed set, often it is even compact.

`bounded` `compact set`

A function $f : \mathbb{R}^n \to \mathbb{R}^k$ is *continuous* at the point $x \in \mathbb{R}^n$ if for each $\epsilon > 0$ there is an $\delta > 0$ such that if $y$ satisfies $\|y - x\| < \delta$ then $\|f(y) - f(x)\| < \epsilon$. If $f$ is continuous at every point, $f$ is simply said to be *continuous*.

`continuous function`

Some other useful basic results in topology are summarized next. We refer to any introductory book in analysis or topology for proofs and more on the subject. A rather naive attempt to illustrate topology is seen in Fig. 2.5.

**Theorem 2.4.1** (Topology, continuity etc)**.** *(i) A set $S$ is closed if and only if it contains the limit point of each convergent point sequence in $S$.*

*(ii) A set $S$ is compact if and only if each point sequence in $S$ has a convergent subsequence (with limit point in $S$).*

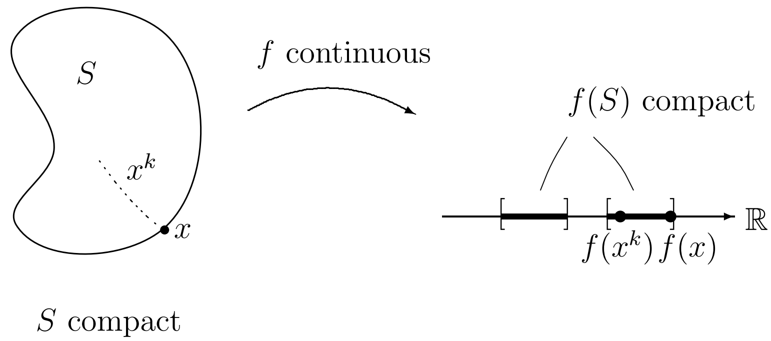


Figure 2.5: Compactness and continuity

*(iii) The union of any family of open sets is an open set, and the intersection of a finite number of open sets is an open set.*

*(iv) The union of a finite number of closed sets is a closed set, and the intersection of any family of closed sets is a closed set.*

*(v) If $f : \mathbb{R}^n \to \mathbb{R}^k$ is continuous and $S \subseteq \mathbb{R}^n$ is compact, then the image $f(S) = \{f(x) : x \in S\}$ is also compact.*

*(vi) A function $f : \mathbb{R}^n \to \mathbb{R}^k$ is continuous if and only if the inverse image $f^{-1}(S) = \{x \in \mathbb{R}^n : f(x) \in S\}$ is closed for every closed set $S$ in $\mathbb{R}^k$.*

*(vii) If $f : \mathbb{R}^n \to \mathbb{R}^k$ is continuous and $x^k \to x$, then $f(x^k) \to f(x)$.*

*(viii) (Weierstrass' theorem) A continuous mapping $f$ of a compact set into $\mathbb{R}$ attains its maximum and minimum in $S$, i.e., there are points $x_1, x_2 \in S$ such that*

$$f(x_1) \leq f(x) \leq f(x_2) \text{ for all } x \in S.$$

**Example 2.4.1.** *(Polyhedra are closed) Consider a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ in $\mathbb{R}^n$, where $A \in \mathbb{R}^{m,n}$ and $b \in \mathbb{R}^m$. Then $P$ is closed. We can see this from statement (vi) of Theorem 2.4 in the following way. The function $f : \mathbb{R}^n \to \mathbb{R}^m$ defined by $f(x) = Ax$ is continuous (every linear transformation is continuous). Moreover, the set $S = \{y \in \mathbb{R}^m : y \leq b\}$ is clearly closed. Therefore $P = f^{-1}(S)$ must be closed. Note that a polyhedron may not be compact (there are unbounded polyhedra). The class of compact polyhedra consists precisely of all polytopes, we shall return to this later.*

**Exercise 2.14.** *Show that every simplex cone is closed.*

`interior` The *interior* $\text{int}(S)$ of a set $S$ is defined as the union of all open sets contained in $S$. This set must be open due to property (iii) of Theorem 2.4.1.

In fact, $\mathrm{int}(S)$ is the unique largest open set contained in $S$. For instance, we have $\mathrm{int}(B(a,r)) = B^\circ(a,r)$. The *closure* $\mathrm{cl}(S)$ of a set $S$ is the intersection of all closed sets containing $S$. This set must be closed due to property (iv) of Theorem 2.4.1 and it is the unique smallest closed set containing $S$. Note that $\mathrm{int}(S) \subseteq S \subseteq \mathrm{cl}(S)$. We have that $S$ is open if and only if $\mathrm{int}(S) = S$, and that $S$ is closed if and only if $\mathrm{cl}(S) = S$. closure

The *boundary* $\mathrm{bd}(S)$ of $S$ is defined by $\mathrm{bd}(S) = \mathrm{cl}(S) \setminus \mathrm{int}(S)$. For instance, we have that $\mathrm{bd}(B(a,r)) = \{x \in \mathbb{R}^n : \|x - a\| = r\}$. boundary

**Exercise 2.15.** *Prove that $x \in \mathrm{bd}(S)$ if and only if each ball with center $x$ intersects both $S$ and the complement of $S$.*

Thus, if $x \in \mathrm{bd}(S)$, then we can find a sequence of points in $S$ that converges to $x$ (as $x \in \mathrm{cl}(S)$), and we can also find a sequence of points outside $S$ that converges to $x$.

As mentioned above, in convex analysis, we also need the concept of *relative topology.* Since a convex set $C$ in $\mathbb{R}^n$ may have dimension smaller than $n$, it is of interest to study $C$ as a subset of "the smallest space" it lies in. Here the proper space is the affine hull of $C$, $\mathrm{aff}(C)$. We recall that this is the smallest affine set that contains $C$. Our next definition of relative interior point also makes sense for arbitrary sets. Let $S \subseteq \mathbb{R}^n$ and $x \in S$. We say that $x$ is a *relative interior point* of $S$ if there is $r > 0$ such that relative topology

$$B^\circ(x,r) \cap \mathrm{aff}(S) \subseteq S.$$

This means that $x$ is the center of some open ball whose intersection with $\mathrm{aff}(S)$ is contained in $S$. We let $\mathrm{rint}(S)$ denote the *relative interior* of $S$, this is the set of all relative interior points of $S$. Finally, we define the *relative boundary* of $S$, denoted by $\mathrm{rbd}(S)$, as those points in the closure of $S$ not lying in the relative interior of $S$, i.e., relative interior relative boundary

$$\mathrm{rbd}(S) = \mathrm{cl}(S) \setminus \mathrm{rint}(S).$$

Each point in $\mathrm{rbd}(S)$ is called a *relative boundary point* of $S$, see Fig. 2.6.

**Exercise 2.16.** *Consider again the set $C = \{(x_1,x_2,0) \in \mathbb{R}^3 : x_1^2 + x_2^2 \le 1\}$. Verify that (i) $C$ is closed, (ii) $\dim(C) = 2$, (iii) $\mathrm{int}(C) = \emptyset$, (iv) $\mathrm{bd}(C) = C$, (v) $\mathrm{rint}(C) = \{(x_1,x_2,0) \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1\}$ and (vi) $\mathrm{rbd}(C) = \{(x_1,x_2,0) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1\}$.*

With all the machinery introduced above, let us look at convex sets from a topological perspective. A basic result in this area is the following theorem. It says that when $C$ is convex then (except for one of the end points) every line segment between a point in $\mathrm{rint}(C)$ and a point in the closure of $C$ also lies in $\mathrm{rint}(C)$.

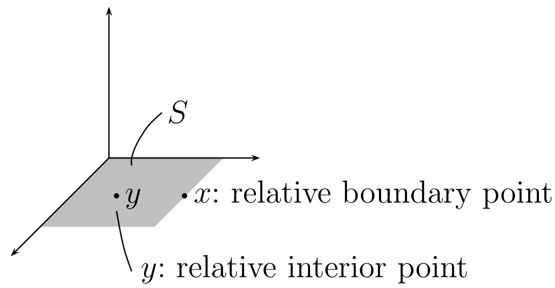


Figure 2.6: Relative topology

Several other results on the topology of convex sets may be derived from this theorem.

**Theorem 2.4.2** (A convex set has "thin boundary")**.** *Let* $C \subseteq \mathbb{R}^n$ *be a nonempty convex set. Let* $x_1 \in \mathrm{rint}(C)$ *and* $x_2 \in \mathrm{cl}(C)$*. Then* $(1-\lambda)x_1 + \lambda x_2 \in \mathrm{rint}(C)$ *for all* $0 \leq \lambda < 1$*.*

*Proof.* We first prove this result for the case when $x_1 \in \mathrm{rint}(C)$ and $x_2 \in C$. Since $x_1 \in \mathrm{rint}(C)$ there is a $r > 0$ such that $B(x_1, r) \cap \mathrm{aff}(C) \subseteq C$. Define $w = (1-\lambda)x_1 + \lambda x_2$ where $0 \leq \lambda < 1$. We shall show that

$$(*) \quad B(w, (1-\lambda)r) \cap \mathrm{aff}(C) \subseteq C.$$

To do this, let $x \in B(w, (1-\lambda)r) \cap \mathrm{aff}(C)$ and let

$$y := x_1 + (1/(1-\lambda))(x - w) = (1/(1-\lambda))x + (1 - 1/(1-\lambda))x_2.$$

Since $y$ is an affine combination of $x$ and $x_2$, both lying in $\mathrm{aff}(C)$, we see that $y \in \mathrm{aff}(C)$. Moreover, $\|y - x_1\| = (1/(1-\lambda))\|x - w\| \leq r$. This shows that $y \in B(x_1, r) \cap \mathrm{aff}(C)$ and therefore $y \in C$. We also have that $x = (1-\lambda)y + \lambda x_2$ and because $C$ is convex, this implies that $x \in C$. This proves the relation in $(*)$ and it follows that $w \in \mathrm{rint}(C)$.

We have finished the proof of the special case when $x_2 \in C$. Consider now the general case when $x_1 \in \mathrm{rint}(C)$ and $x_2 \in \mathrm{cl}(C)$. Again we define $w = (1-\lambda)x_1 + \lambda x_2$ where $0 \leq \lambda < 1$. Since $x_2 \in \mathrm{cl}(C)$, there is a point $x_2' \in C$ which is sufficiently near $x_2$, namely $\lambda\|x_2' - x_2\| < (1-\lambda)r$. Define next

$$y := x_1 + (\lambda/(1-\lambda))(x_2 - x_2') = (1/(1-\lambda))w + (1 - 1/(1-\lambda))x_2'.$$

Then $y \in \mathrm{aff}(C)$ and $\|y - x_1\| = (\lambda/(1-\lambda))\|x_2 - x_2'\| < r$. Therefore $y \in B(x_1, r) \cap \mathrm{aff}(C)$ so $y \in \mathrm{rint}(C)$. Finally, we have $w = (1-\lambda)y + \lambda x_2'$ so $w \in \mathrm{rint}(C)$ and the proof is complete. $\square$

We now give a main result on how convexity is preserved under several topological operations on a convex set. The proof of this result is left as an exercise; it is an immediate consequence of Theorem 2.4.2.

**Theorem 2.4.3** (Convexity under topological operations)**.** *If $C \subseteq \mathbb{R}^n$ is a convex set, then all the sets* $\mathrm{rint}(C)$, $\mathrm{int}(C)$ *and* $\mathrm{cl}(C)$ *are convex.*

Another result that also follows from Theorem 2.4.2 is the following one. It says that the three sets $\mathrm{rint}(C)$, $C$ (assumed to be convex) and $\mathrm{cl}(C)$ are very close. For instance, they all have the same relative boundary.

**Theorem 2.4.4** (Sets that are "close")**.** *Let $C \subseteq \mathbb{R}^n$ be a convex set. Then each object in the following list coincide for these three sets* $\mathrm{rint}(C)$, $C$ *and* $\mathrm{cl}(C)$*: (i) the affine hull, (ii) the dimension, (iii) the relative interior, (iv) the closure, and (v) the relative boundary.*

We omit the proof, but a reader who likes topology might try to prove it. Again, the main tool is Theorem 2.4.2.

## 2.5 Carathéodory's theorem and some consequences

If a point $x$ lies in the convex hull of some set $S \subseteq \mathbb{R}^n$ we know that $x$ may be written as convex combination of points in $S$. But how many points do we need for this? Maybe there is some way of reducing a convex combination $x$ of "many" points to a convex combination of fewer points, still producing the same point $x$. This may seem reasonable, especially when we compare to what we know from linear algebra: a point in a linear subspace of dimension $t$ may be written as a linear combination of a set of $t$ basis vectors of $L$. We shall see that a related result holds for convex hulls, although things are a little bit more complicated.

The following result, due to Carathéodory, says that any point in the convex hull of some points may be written as a convex combination of "few" of these points, see Fig. 2.7.

**Theorem 2.5.1** (Carathéodory's theorem)**.** *Let $S \subseteq \mathbb{R}^n$. Then each $x \in \mathrm{conv}(S)$ may be written as a convex combination of (say) $m$ affinely independent points in $S$. In particular, $m \leq n+1$.*

*Proof.* Since $x \in \mathrm{conv}(S)$ there are nonnegative numbers $\lambda_1, \ldots, \lambda_t$ and vectors $x_1, \ldots, x_t \in S$ such that $\sum_j \lambda_j = 1$ and $x = \sum_j \lambda_j x_j$. In fact, we may assume that each $\lambda_j$ is positive, otherwise we could omit some $x_j$ from the representation. If $x_1, \ldots, x_t$ are affinely independent, we are done, so assume that they are not. Then there are numbers $\mu_1, \ldots, \mu_t$ not all zero such that $\sum_{j=1}^t \mu_j x_j = O$ and $\sum_{j=1}^t \mu_j = 0$. Since the $\mu_j$s are not all zero and sum to zero, at least one of

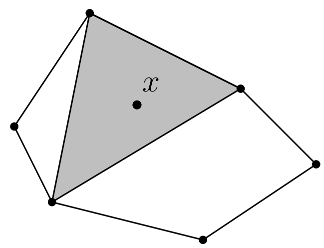


Figure 2.7: Carathéodory's theorem

these numbers must be positive, say that $\mu_1 > 0$. We now multiply the equation $\sum_j \mu_j x_j = O$ by a nonnegative number $\alpha$ and subtract the resulting equation from the equation $x = \sum_j \lambda_j x_j$. This gives

$$x = \sum_j (\lambda_j - \alpha\mu_j) x_j.$$

Note that $\sum_j (\lambda_j - \alpha\mu_j) = \sum_j \lambda_j - \alpha \sum_j \mu_j = 1$. When $\alpha = 0$ this is just our original representation of $x$. But now we gradually increase $\alpha$ from zero until one of the coefficients $\lambda_j - \alpha\mu_j$ becomes zero, say this happens for $\alpha = \alpha_0$. Recall here that each $\lambda_j$ is positive and so is $\mu_1$. Then each coefficient $\lambda_j - \alpha_0\mu_j$ is nonnegative and at least one of them is zero. But this means that we have found a new representation of $x$ as a convex combination of $t-1$ vectors from $S$. Clearly, this reduction process may be continued until we have $x$ written as a convex combination of, say, $m$ affinely independent points in $S$. Finally, there are at most $n+1$ affinely independent points in $\mathbb{R}^n$ so $m \leq n+1$. $\square$

There is a similar result for conical hulls which may be proved similarly (another exercise!).

**Theorem 2.5.2** (Carathéodory for cones)**.** *Let $S \subseteq \mathbb{R}^n$. Then each $x \in \text{cone}(S)$ may be written as a nonnegative combination of (say) $m$ linearly independent points in $S$. In particular, $m \leq n$.*

Carathéodory's theorem says that, for a given point $x \in \mathbb{R}^n$ in the convex hull of a set $S$ of points, we can write $x$ as a convex combination of at most $n+1$ affinely independent points from $S$. This, however, does not mean, in general, that there is a "convex basis" in the sense that the *same set* of $n+1$ points may be used to generate any point $x$. Thus, the "generators" have to be chosen specifically for each $x$. This is in contrast to the existence of a basis for linear subspaces. It should be noted that a certain class of convex sets, simplices, discussed below, has a "convex basis"; this is seen directly from the definitions below.

We discuss some interesting consequences of Carathéodory's theorem.

**Example 2.5.1.** *(Carathéodory and LP)* Consider the LP problem $\max\{c^T x : x \in P\}$ where $P = \{x \in \mathbb{R}^n : Ax = b,\ x \geq O\}$. We assume that the rows of $A$ are linearly independent, and therefore $m \leq n$, and that $P$ is nonempty. As discussed before this means that $b$ lies in the finitely generated convex cone $\text{cone}(\{a_1, \ldots, a_n\})$ where $a_1, \ldots, a_n$ are the columns of the matrix $A$. So $b$ may be written as a nonnegative combination of $a_1, \ldots, a_n \in \mathbb{R}^m$. Moreover, due to Carathéodory's theorem for cones (Theorem 2.5.2), it is possible to write $b$ as a nonnegative combination of $t$ linearly independent $a_j$'s. Since $a_j \in \mathbb{R}^m$, we must have $t \leq m$. In other words, there is a nonnegative $x \in \mathbb{R}^n$ with at least $n - t$ components being zero such that $Ax = b$, and the nonzeros correspond to linearly independent columns of $A$. The basic feasible solutions of our LP problem are all of this form. Thus, we have a new proof of a fundamental fact in linear programming: if an LP problem (of the form above) is feasible, it contains a basic feasible solution.

Recall that a simplex is a very special polytope, it is the convex hull of affinely independent points. It turns out that any polytope may be written as a union of simplices. To get the idea, consider a (convex) pentagon in the plane. For each subset of three of these five vertices ("corners") we get a triangle (a simplex), and the union of all these triangles is the pentagon. Here is the general result, obtained using Carathéodory's theorem.

**Theorem 2.5.3** (Simplex decomposition of polytopes)**.** *Every polytope in $\mathbb{R}^n$ can be written as the union of a finite number of simplices. Each finitely generated cone can be written as the union of a finite number of simplex cones.*

*Proof.* Consider a polytope $P = \text{conv}(\{x_1, \ldots, x_t\})$ in $\mathbb{R}^n$. For each subset of $x_1, \ldots, x_t$ consisting of affinely independent vectors we take the convex hull and thereby obtain a simplex contained in $P$. Now, the union of all these simplices must be equal to $P$ for, Carathéodory's theorem, each point $x$ may be written as a convex combination of affinely independent points selected from $x_1, \ldots, x_t$, so $x$ lies in the corresponding simplex. Things are similar for cones! □

Recall that a set $S$ in $\mathbb{R}^n$ is bounded if there is a number $M$ such that $\|x\| \leq M$ for all $x \in S$. This means that $S$ does not contain points with arbitrary large norm, or equivalently, it does not contain points with arbitrary large components.

**Exercise 2.17.** *Show that every polytope in $\mathbb{R}^n$ is bounded. (Hint: use the properties of the norm: $\|x + y\| \leq \|x\| + \|y\|$ and $\|\lambda x\| = \lambda\|x\|$ when $\lambda \geq 0$).*

**Exercise 2.18.** *Consider the standard simplex $S_t$. Show that it is compact, i.e., closed and bounded.*

What about polytopes more generally, are they compact as well? The answer is "yes"!

**Proposition 2.5.4** (Compactness of polytopes). *Every polytope in $\mathbb{R}^n$ is compact, i.e., closed and bounded.*

*Proof.* Let $P = \text{conv}(\{x_1, \ldots, x_t\}) \subset \mathbb{R}^n$. Then $P$ is the image of the function $f$ from the standard simplex $S_t$ to $\mathbb{R}^n$ which maps the point $(\lambda_1, \ldots, \lambda_t)$ (nonnegative and with sum one) to the point $\sum_{j=1}^t \lambda_j x_j$. The function $f$ is continuous and $S_t$ is compact, so from Theorem 2.4.1 (v) we may then conclude that the image $P$ is compact. □

This proof was interesting! It also showed that every polytope $P$ is the image of a standard simplex under a certain mapping. This mapping is of the form $x = A\lambda$ ($\lambda \in \mathbb{R}^t$ and $x \in \mathbb{R}^n$ is the point in the polytope) so it is linear. At first, this may sound strange as simplices are very simple objects while polytopes are not. But when $t$ is large compared to $n$ (see notation in the proof) our operation may be viewed as a projection from a higher dimensional space to a lower dimensional one, and this operation can be quite complicated.

**Proposition 2.5.5.** *Every finitely generated convex cone in $\mathbb{R}^n$ is closed.*

*Proof.* From Theorem 2.5.3 we know that each finitely generated cone $C$ can be written as the union of a finite number of simplex cones. But every simplex cone is closed (see Exercise 2.14), so then $C$ must be closed. □

**Exercise 2.19.** *Give an example of a convex cone which is not closed.*

Finally, we mention that it also holds that the convex hull of a compact set is compact. The proof is similar to the one above.

## 2.6 Exercises

**Exercise 2.20.** *Let $S \subseteq \mathbb{R}^n$ and let $W$ be the set of all convex combinations of points in $S$. Prove that $W$ is convex.*

**Exercise 2.21.** *Prove the second statement of Proposition* 2.1.1.

**Exercise 2.22.** *Give a geometrical interpretation of the induction step in the proof of Proposition* 2.1.1.

**Exercise 2.23.** *Let $S = \{(0,0),(1,0),(0,1)\}$. Show that* $\text{conv}(S) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq 1\}$.

**Exercise 2.24.** *Let $S$ consist of the points* $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ *and* $(1,1,1)$. *Show that* $\mathrm{conv}(S) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leq x_i \leq 1 \text{ for } i = 1,2,3\}$. *Also determine* $\mathrm{conv}(S \setminus \{(1,1,1)\}$ *as the solution set of a system of linear inequalities. Illustrate all these cases geometrically.*

**Exercise 2.25.** *Let* $A, B \subseteq \mathbb{R}^n$. *Prove that* $\mathrm{conv}(A+B) = \mathrm{conv}(A) + \mathrm{conv}(B)$. *Hint: it is useful to consider the sum* $\sum_{j,k} \lambda_j \mu_k (a_j + b_k)$ *where* $a_j \in A$, $b_k \in B$ *and* $\lambda_j \geq 0$, $\mu_k \geq 0$ *and* $\sum_j \lambda_j = 1$ *and* $\sum_k \mu_k = 1$.

**Exercise 2.26.** *When* $S \subset \mathbb{R}^n$ *is a finite set, say* $S = \{x_1, \ldots, x_t\}$, *then we have*

$$\mathrm{conv}(S) = \{\sum_{j=1}^{t} \lambda_j x_j : \lambda_j \geq 0 \text{ for each } j, \ \sum_j \lambda_j = 1\}.$$

*Thus, every point in* $\mathrm{conv}(S)$ *is a convex combination of the points* $x_1, \ldots, x_t$. *What happens if, instead, $S$ has an infinite number of elements? Then it may not be possible to give a fixed, finite subset $S_0$ of $S$ such that every point in* $\mathrm{conv}(S)$ *is a convex combination of elements in $S_0$. Give an example which illustrates this.*

**Exercise 2.27.** *Let* $x_0 \in \mathbb{R}^n$ *and let* $C \subseteq \mathbb{R}^n$ *be a convex set. Show that* $\mathrm{conv}(C \cup \{x_0\}) = \{(1-\lambda)x_0 + \lambda x : x \in C, \lambda \in [0,1]\}$.

**Exercise 2.28.** *Prove Proposition* 2.2.2.

**Exercise 2.29.** *Confer Exercise* 2.9. *Give an example showing that a similar property for linear independence does not hold. Hint: consider the vectors* $(1,0)$ *and* $(0,1)$ *and choose some* $w$.

**Exercise 2.30.** *If* $x = \sum_{j=1}^{t} \lambda_j x_j$ *and* $\sum_{j=1}^{t} \lambda_j = 1$ *we say that $x$ is an affine combination of* $x_1, \ldots, x_t$. *Show that* $x_1, \ldots, x_t$ *are affinely independent if and only if none of these vectors may be written as an affine combination of the remaining ones.*

**Exercise 2.31.** *Prove that* $x_1, \ldots, x_t \in \mathbb{R}^n$ *are affinely independent if and only if the vectors* $(x_1, 1), \ldots, (x_t, 1) \in \mathbb{R}^{n+1}$ *are linearly independent.*

**Exercise 2.32.** *Prove Proposition* 2.3.2.

**Exercise 2.33.** *Prove that* $\mathrm{cl}(A_1 \cup \ldots \cup A_t) = \mathrm{cl}(A_1) \cup \ldots \cup \mathrm{cl}(A_t)$ *holds whenever* $A_1, \ldots, A_t \subseteq \mathbb{R}^n$.

**Exercise 2.34.** *Prove that every bounded point sequence in* $\mathbb{R}^n$ *has a convergent subsequence.*

**Exercise 2.35.** *Find an infinite set of closed intervals whose union is the open interval* $\langle 0, 1 \rangle$. *This proves that the union of an infinite set of closed intervals may not be a closed set.*

**Exercise 2.36.** *Let $S$ be a bounded set in* $\mathbb{R}^n$. *Prove that* $\mathrm{cl}(S)$ *is compact.*

**Exercise 2.37.** *Let* $S \subseteq \mathbb{R}^n$. *Show that either* $\mathrm{int}(S) = \mathrm{rint}(S)$ *or* $\mathrm{int}(S) = \emptyset$.

**Exercise 2.38.** *Prove Theorem* 2.4.3 *Hint: To prove that* $\mathrm{rint}(C)$ *is convex, use Theorem* 2.4.2. *Concerning* $\mathrm{int}(C)$, *use Exercise* 2.37. *Finally, to show that* $\mathrm{cl}(C)$ *is convex, let* $x, y \in \mathrm{cl}(C)$ *and consider two point sequences that converge to* $x$ *and* $y$, *respectively. Then look at a convex combination of* $x$ *and* $y$ *and construct a suitable sequence!*

**Exercise 2.39.** *Prove Theorem* 2.5.2.

**SUMMARY OF NEW CONCEPTS AND RESULTS:**

- convex combination, nonnegative combination
- convex hull, conical hull, affine hull
- affinely independent vectors
- Carathéodory's theorem
- polytope
- simplex, simplex cone
- dimension of a set
- topology: open, closed, closure, interior, boundary, relative boundary, relative boundary

# Chapter 3

# Projection and separation

This chapter deals with some properties of convex sets that are useful in linear programming, mathematical economics, statistics etc. The two central concepts are projection and separation. Projection deals with the problem of finding a nearest point in a set to a given point outside the set. Geometrically, in $\mathbb{R}^3$, separation is to distinguish two sets by a (two-dimensional) plane so the two sets are on different sides of the plane. This notion is treated in $\mathbb{R}^n$ as well.

## 3.1 The projection operator

A problem that often arises in applications is to find a nearest point in some set $S \subset \mathbb{R}^n$ to a given point $x$. This is a problem of best approximation of $x$ from the set $S$. We here consider such problems and the role of convexity. In this section some basic knowledge of topology is useful.

Let $S$ be a closed subset of $\mathbb{R}^n$. From topology we recall that $S$ is closed if and only if $S$ contains the limit point of each convergent sequence of points in $S$. Thus, if $\{x^k\}_{k=1}^{\infty}$ is a convergent sequence of points where $x^k \in S$, then the limit point $x = \lim_{k\to\infty} x^k$ also lies in $S$.

Closedness is important for the mentioned approximation problem, as it assures that a nearest point in $S$ exists. For $S \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$ we define

$$d_S(x) = \inf\{\|x - s\| : s \in S\} \tag{3.1}$$

where $\|\cdot\|$ denotes the Euclidean norm ($\|x\| = (\sum_{j=1}^n |x_j|^2)^{1/2}$). The function $d_S(\cdot)$ is called the *distance function* of $S$ as it simply measures the distance from $x$ to the set $S$. Recall that "infimum" means the "the largest lower bound". A point $s \in S$ such that $\|x - s\| = d_S(x)$ is called a *nearest point* to $x$ in $S$. We now give a basic existence result.

distance function

nearest point

**Proposition 3.1.1** (Nearest point). *Let $S \subseteq \mathbb{R}^n$ be a nonempty closed set and let $x \in \mathbb{R}^n$. Then there is a nearest point $s \in S$ to $x$, i.e., $\|x - s\| = d_S(x)$.*

*Proof.* From the definition of $d_S(x)$ we see that there is a sequence $\{s^k\}_{k=1}^{\infty}$ of points in $S$ such that $\lim_{k\to\infty} \|x - s^k\| = d_S(x)$. In particular we must have that $\|x - s^k\| \leq M$ for some number $M$ (the sequence $\{\|x - s^k\|\}$ of real numbers is convergent an therefore bounded). This implies that our point sequence $\{s^k\}_{k=1}^{\infty}$ is bounded as well: $\|s^k\| = \|s^k - x + x\| \leq \|s^k - x\| + \|x\| \leq M + \|x\|$ by the triangle inequality for norms. Thus, by Exercise 2.34, there is a subsequence $s^{i_1}, s^{i_2}, \ldots$ which converges to some point $s$. Since $S$ is closed $s$ must lie in $S$. Moreover, $d_S(x) = \lim_{j\to\infty} \|x - s^{i_j}\| = \|x - s\|$. To get the last equality we used the continuity of the norm function. □

Thus, closedness of $S$ assures that a nearest point exists. But such a point may not be unique.

**Exercise 3.1.** *Give an example where the nearest point is unique, and one where it is not. Find a point $x$ and a set $S$ such that every point of $S$ is a nearest point to $x$!*

Here is a nice property of closed convex sets: the nearest point is unique!

**Theorem 3.1.2** (Unique nearest point for convex sets). *Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Then, for every $x \in \mathbb{R}^n$, the nearest point $x_0$ to $x$ in $C$ is unique. Moreover, $x_0$ is the unique solution of the inequalities*

$$(x - x_0)^T (y - x_0) \leq 0 \quad \text{for all } y \in C. \tag{3.2}$$

*Proof.* Let $x_0$ be a nearest point to $x$ in $C$. Such a point exists according to Proposition 3.1.1. We shall first establish a useful inequality related to $x_0$.

Let $y \in C$ and let $0 < \lambda < 1$. Since $C$ is convex, $(1 - \lambda)x_0 + \lambda y \in C$ and since $x_0$ is a nearest point we have that $\|(1 - \lambda)x_0 + \lambda y - x\| \geq \|x_0 - x\|$, i.e., $\|(x_0 - x) + \lambda(y - x_0)\| \geq \|x_0 - x\|$. By squaring both sides and calculating the inner products we obtain $\|x_0 - x\|^2 + 2\lambda(x_0 - x)^T(y - x_0) + \lambda^2\|y - x_0\|^2 \geq \|x_0 - x\|^2$. We now subtract $\|x_0 - x\|^2$ on both sides, divide by $\lambda$, let $\lambda \to 0^+$ and finally multiply by $-1$. This proves that the inequality (3.2) holds for every $y \in C$. Let now $x_1$ be another nearest point to $x$ in $C$; we want to show that $x_1 = x_0$. By letting $y = x_1$ in (3.2) we get

$$(*_1) \;\; (x - x_0)^T (x_1 - x_0) \leq 0.$$

By symmetry (or repeating the arguments that lead to (3.2) and now letting $y = x_0$) we also get that

$$(*_2) \;\; (x - x_1)^T (x_0 - x_1) \leq 0.$$

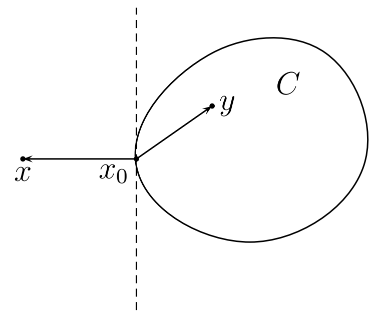


Figure 3.1: Projection onto a convex set

By adding the inequalities $(*_1)$ and $(*_2)$ we obtain $\|x_1 - x_0\|^2 = (x_1 - x_0)^T(x_1 - x_0) \leq 0$ which implies that $x_1 = x_0$. Thus, the nearest point is unique. $\square$

The inequality (3.2) in the previous proof has a nice geometrical interpretation (see Fig. 3.1): the angle between the vectors $x - x_0$ and $y - x_0$ (both starting in the point $x_0$) is obtuse, i.e., larger than $90°$. Note that, if one finds a point $x_0$ that satisfies (3.2) for any $y \in C$, then $x_0$ is the projection of $x$ in $C$.

projection

For every nonempty closed convex set $C$ and $x \in \mathbb{R}^n$ let $p_C(x)$ denote the unique nearest point to $x$ in $S$. This defines a function $p_C : \mathbb{R}^n \to C$ which is called the *projection* or *nearest-point map* of $C$. Note that $\|x - p_C(x)\| = d_C(x)$.

least squares

**Example 3.1.1.** *(The least squares problem)* Let $A \in \mathbb{R}^{m,n}$ and $b \in \mathbb{R}^m$. The *linear least squares approximation problem* , or simply the *least squares problem*, is to find a vector $x \in \mathbb{R}^n$ which minimizes $\|Ax - b\|$. The problem arises, for instance, in many statistical applications. We want to relate this problem to projection. To do so, define $L = \{Ax : x \in \mathbb{R}^n\}$ so this is the linear subspace of $\mathbb{R}^m$ spanned by the columns of the matrix $a$. The least squares problem is equivalent to finding a vector $z \in L$ (so $z = Ax$ for some $x$) such that $\|z - b\|$ is minimized. Thus, we want to find the nearest point to $b$ in the convex set $L$. From above we therefore know that the optimal solution $z$ is unique (but $x$ may not be unique, see below) and that $z$ is characterized by the inequality in (3.2):

$$(b - z)^T(w - z) \leq 0 \quad \text{for all } w \in L.$$

By setting $z = Ax$ in this inequality and varying $w$ we obtain the following system of so-called *normal equations*

$$A^T Ax = A^T b.$$

This system has a unique solution $x$ if the columns of $a$ are linearly independent (then the coefficent matrix is nonsingular; it is even positive definite). If the columns of $a$ are linearly dependent, there are many solutions $x$: the set of solutions is a nontrivial affine set. Note, however, that in any case the projection $z = Ax$ is unique.

We also mention (without proof) an interesting result which says that the property of having unique projection is characteristic for closed convex sets. The result is due to T.S. Motzkin (1935).

**Theorem 3.1.3** (Motzkin's characterization of convex sets)**.** *Let $S \subseteq \mathbb{R}^n$ be a nonempty closed set. Assume that each $x \in \mathbb{R}^n$ has a unique nearest point in $S$. Then $S$ is convex.*

## 3.2 Separation of convex sets

Consider the sphere $S = \{x \in \mathbb{R}^3 : \|x\| \leq 1\}$ in $\mathbb{R}^3$. Through each boundary point $a$ of $S$, i.e., a point with $\|a\| = 1$ we can place a tangential plane $H$. It consists of the points $x$ satisfying the equation $a^T x = 1$. We say that $H$ supports $S$ at $x$, meaning that all points of $S$ lie on the same side of the plane, as they satisfy $a^T x \leq 1$, and, moreover, $H$ intersects $S$ (in $a$). In this section we discuss supporting hyperplanes more generally. Our presentation is influenced by [12].

We call $H \subset \mathbb{R}^n$ a *hyperplane* if it is of the form $H = \{x \in \mathbb{R}^n : a^T x = \alpha\}$ for some nonzero vector $a$ and a real number $\alpha$. The vector $a$ is called the *normal vector* of the hyperplane. We see that every hyperplane is an affine set of dimension $n-1$. Each hyperplane divides the space into two sets $H^+ = \{x \in \mathbb{R}^n : a^T x \geq \alpha\}$ and $H^- = \{x \in \mathbb{R}^n : a^T x \leq \alpha\}$. These sets $H^+$ and $H^-$ are called *halfspaces* and they intersect in $H$. Let $S \subset \mathbb{R}^n$ and let $H$ be a hyperplane in $\mathbb{R}^n$. If $S$ is contained in one of the halfspaces $H^+$ or $H^-$ and $H \cap S$ is nonempty, we say that $H$ is a *supporting hyperplane* of $S$. Moreover, in this situation we say that $H$ *supports* $S$ at $x$, for each $x \in H \cap S$.

hyperplane

halfspace

supporting hyperplane

**Exercise 3.2.** *Let $a \in \mathbb{R}^n \setminus \{O\}$ and $x_0 \in \mathbb{R}^n$. Then there is a unique hyperplane $H$ that contains $x_0$ and has normal vector $a$. Verify this and find the value of the constant $\alpha$ (see above).*

To simplify the presentation we now restrict the attention to closed convex sets. Then each point outside our set $C$ gives rise to a supporting hyperplane as the following lemma tells us. Recall that $p_C(x)$ is the (unique) nearest point to $x$ in $C$.

**Proposition 3.2.1** (Supporting hyperplane)**.** *Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set and let $x \in \mathbb{R}^n \setminus C$. Consider the hyperplane $H$ containing $p_C(x)$ and having normal vector $a = x - p_C(x)$. Then $H$ supports $C$ at $p_C(x)$ and $C$ is contained in the halfspace $H^- = \{y : a^T y \leq \alpha\}$ where $\alpha = a^T p_C(x)$.*

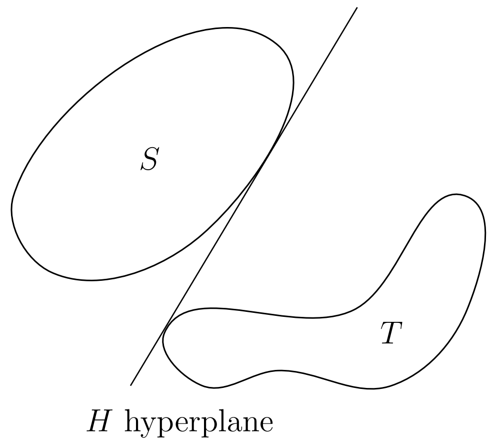


Figure 3.2: Separation

*Proof.* Note that $a$ is nonzero as $x \notin C$ while $p_C(x) \in C$. Then $H$ is the hyperplane with normal vector $a$ and given by $a^T y = \alpha = a^T p_C(x)$. We shall show that $C$ is contained in the halfspace $H^-$. So, let $y \in C$. Then, by (3.2) we have $(x - p_C(x))^T(y - p_C(x)) \leq 0$, i.e., $a^T y \leq a^T p_C(x) = \alpha$ as desired. $\square$

We now explain the concept of separation of sets. It is useful to extend our notation for hyperplanes and halfspaces as follows:

$$\begin{aligned} H_{a,\alpha} &:= \{x \in \mathbb{R}^n : a^T x = \alpha\}; \\ H^-_{a,\alpha} &:= \{x \in \mathbb{R}^n : a^T x \leq \alpha\}; \\ H^+_{a,\alpha} &:= \{x \in \mathbb{R}^n : a^T x \geq \alpha\}. \end{aligned}$$

`separating hyperplane`

Here the normal vector $a$ is nonzero as usual. Consider two sets $S$ and $T$ in $\mathbb{R}^n$. We say that the hyperplane $H_{a,\alpha}$ *separates* $S$ and $T$ if $S \subseteq H^-_{a,\alpha}$ and $T \subseteq H^+_{a,\alpha}$ or vice versa. See Fig. 3.2 for an example in $\mathbb{R}^2$. Note that both $S$ and $T$ may intersect the hyperplane $H_{a,\alpha}$ in this definition, in fact, they may even be contained in the hyperplane. The special case where one of the sets $S$ and $T$ has a single point is important and will be discussed soon.

`strong separation`

There are some stronger notions of separation, and we are here concerned with one of these. We say that the hyperplane $H_{a,\alpha}$ *strongly separates* $S$ and $T$ if there is an $\epsilon > 0$ such that $S \subseteq H^-_{a,\alpha-\epsilon}$ and $T \subseteq H^+_{a,\alpha+\epsilon}$ or vice versa. This means that

$$\begin{aligned} a^T x &\leq \alpha - \epsilon \quad \text{for all } x \in S; \\ a^T x &\geq \alpha + \epsilon \quad \text{for all } x \in T. \end{aligned}$$

**Exercise 3.3.** *Give an example of two disjoint sets $S$ and $T$ that cannot be separated by a hyperplane.*

**Remark.** Consider a set $S$ and a point $p$ (so $T = \{p\}$) lying in $\mathbb{R}^n$. There may be many hyperplanes separating $S$ and $p$, and we make an observation which will be useful later. Assume that $p$ lies in the affine hull $\mathrm{aff}(S)$. Let $L$ be the unique linear subspace which is parallel to $\mathrm{aff}(S)$, so $\mathrm{aff}(S) = L + x_0$ for some $x_0 \in S$. Assume that it is possible to separate $S$ and $p$ and consider a separating hyperplane with normal vector $a$. Thus, we have $a^T x \leq a^T p$ for each $x \in S$, i.e., $a^T(x-p) \leq 0$ for all $x \in S$. Consider any point $x \in S$. Since both $x$ and $p$ lie in $\mathrm{aff}(S)$, we have that $x = x_0 + l_1$ and $p = x_0 + l_2$ for suitable $l_1, l_2 \in L$. Therefore, $x - p = l_1 - l_2 \in L$. Now, the normal vector $a$ may be decomposed as $a = a_1 + a_2$ where $a_1 \in L$ and $a_2 \in L^\perp$ (the orthogonal complement of $L$). It follows that $a^T(x-p) = a_1^T(x-p)$. Thus, the component $a_2$ of $a$ plays no role for the separation property. The lesson is: *in order to separate $S$ and $p \in \mathrm{aff}(S)$ we may choose a normal vector lying in the unique linear subspace that is parallel to* $\mathrm{aff}(S)$.

**Exercise 3.4.** *In view of the previous remark, what about the separation of $S$ and a point $p \notin \mathrm{aff}(S)$? Is there an easy way to find a separating hyperplane?*

We may now present one of the basic results on separation of convex sets.

**Theorem 3.2.2** (Strong separation)**.** *Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set and assume that $x \in \mathbb{R}^n \setminus C$. Then $C$ and $x$ can be strongly separated.*

*Proof.* Let $H$ be the hyperplane containing $p_C(x)$ and having normal vector $x - p_C(x)$. From Proposition 3.2.1 we know that $H$ supports $C$ at $p_C(x)$. Moreover $x \neq p_C(x)$ (as $x \notin C$). Consider the hyperplane $H^*$ which is parallel to $H$ (i.e., having the same normal vector) and contains the point $(1/2)(x + p_C(x))$. Then $H^*$ strongly separates $x$ and $C$. $\square$

Our next result is along the same lines as Proposition 3.2.1, except that it treats a more general situation where the set may not be closed. To motivate the result consider a convex set $C \subseteq \mathbb{R}^n$ with $d = \dim(C) < n$, i.e., $C$ is not fulldimensional. Then any hyperplane contained in $\mathrm{aff}(C)$ will be a supporting hyperplane to $C$ at every point of $C$. These supporting hyperplanes are clearly not of much interest, so we need another concept here. Consider a hyperplane $H$ which supports a convex set $C$ at the point $x$ where we now allow $x$ to be a relative boundary point of $C$. We say that $H$ is a *nontrivial* supporting hyperplane at $x$ provided that $C$ is not contained in $H$. For instance, in the example of Fig. 3.3 the set $C$ lies in the $xy$-plane while the (nontrivial) supporting hyperplane does not.

`nontrivial supporting hyperplane`

**Theorem 3.2.3** (Supporting hyperplane, more generally)**.** *Let $C \subseteq \mathbb{R}^n$ be a nonempty convex set and let $x \in \mathrm{rbd}(C)$. Then $C$ has a nontrivial supporting hyperplane at $x$.*

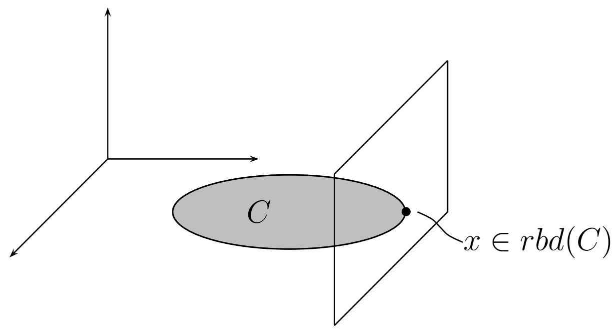


Figure 3.3: Nontrivial supporting hyperplane

*Proof.* From Theorem 2.4.4 we know that $C$ and its closure $\mathrm{cl}(C)$ have the same relative boundary, so $x$ is also a relative boundary point of $\mathrm{cl}(C)$. Therefore there is a sequence of points $x^k$ lying in $\mathrm{aff}(C) \setminus \mathrm{cl}(C)$ that converges to $x$. Now, for each $k$ we may find a hyperplane that separates $\mathrm{cl}(C)$ (which is closed) and $x^k$; this follows from Proposition 3.2.1. More precisely, there is a vector $a^k \in \mathbb{R}^n$ such that

$$(*) \;\; (a^k)^T y \le (a^k)^T x^k \quad \text{for all } y \in \mathrm{cl}(C).$$

We may here choose the normal vector $a^k$ to have length 1, i.e., $\|a^k\| = 1$ and with $a^k \in L$ where $L$ is the unique linear subspace parallel to $\mathrm{aff}(C)$ (confer the remark given before Theorem 3.2.2). Since the set $K = \{z \in L : \|z\| = 1\}$ is compact (closed and bounded), it follows from Theorem 2.4.1 that we can find a convergent subsequence $\{a^{k_j}\}$ of $\{a^k\}$ which converges to, say, $a \in K$. Note that $a$ has norm 1, so it is nonzero. Passing to the limit for this subsequence in $(*)$, we obtain the inequality

$$a^T y \le a^T x \quad \text{for all } y \in \mathrm{cl}(C).$$

Thus, the corresponding hyperplane $H$ supports $C$ at $x$ and it only remains to prove that $H$ is a nontrivial supporting hyperplane. Assume that $C \subseteq H$. Then $C \subseteq H \cap \mathrm{aff}(C)$. As $a$ lies in $L$ (which is parallel to $\mathrm{aff}(C)$) the set $H \cap \mathrm{aff}(C)$ has strictly lower dimension than $\mathrm{aff}(C)$. But this would imply that $C$ has lower dimension than $\mathrm{aff}(C)$ which is impossible due to Theorem 2.4.4. Thus, $C$ is not contained in $H$ and therefore $H$ is a nontrivial supporting hyperplane. $\square$

Let us now consider an important application of these separation results. It concerns an exterior (or outer) representation of closed convex sets. This result may be seen as a generalization of the useful facts in linear algebra: (i) every linear subspace is the intersection of linear hyperplanes (i.e., hyperplanes containing the

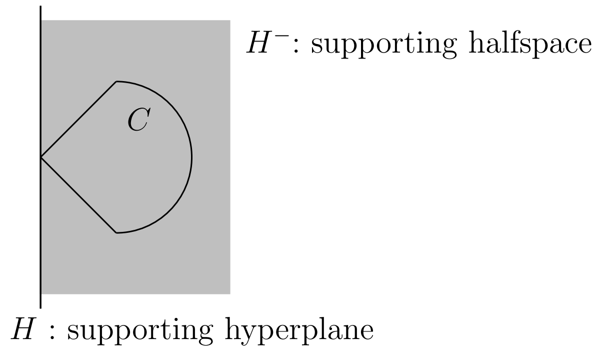


Figure 3.4: Supporting hyperplane and halfspace

origin), and (ii) every affine set is the intersection of hyperplanes. For convex sets we just have to intersect halfspaces instead. If $H$ is a supporting hyperplane of the set $C$ and $C \subseteq H^+$ (or $C \subseteq H^-$) we call $H^+$ (or $H^-$) a *supporting halfspace* of $C$, see Fig. 3.4. `supporting halfspace`

**Corollary 3.2.4** (Outer description of closed convex sets)**.** *Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Then $C$ is the intersection of all its supporting halfspaces.*

*Proof.* Since $C$ is contained in each of its supporting halfspaces, $C$ must also be contained in the intersection of these sets. Assume that this inclusion is strict. Then there is a point $x \notin C$ so that $x$ lies in every supporting halfspace of $C$. Since $C$ is a closed convex set we may apply Proposition 3.2.1 and find a supporting hyperplane $H$ which supports $C$ at the projection point $p_C(x)$. But, as pointed out in the proof of Proposition 3.2.1 (or Theorem 3.2.2), one of the halfspaces associated with $H$ supports $C$ but it does not contain $x$. This contradicts our assumption, and we conclude that $C$ is the intersection of all its supporting halfspaces. $\square$

In Fig. 3.5 some of the supporting hyperplanes that define the convex set $C$ are shown.

Let us consider an important application of separation. Farkas' lemma for linear inequalities tells us under which conditions a linear system of inequalities has a solution. It is closely related to the strong duality theorem for linear programming. In fact, Farkas' lemma may be derived from the duality theorem and vice versa. In many books on LP one proves the duality theorem algorithmically by showing that the simplex algorithm terminates. Here we want to illustrate how Farkas' lemma is obtained from our convexity theory. We consider the following version of Farkas' lemma. We let the $j$th column of a matrix $A$ be denoted by $a^j$.

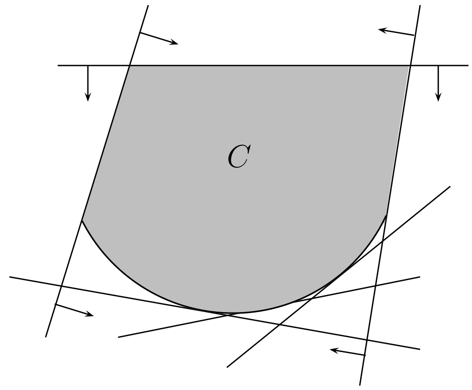


Figure 3.5: Outer description

**Theorem 3.2.5** (Farkas' lemma)**.** *Let $A \in \mathbb{R}^{m,n}$ and $b \in \mathbb{R}^m$. Then there exists an $x \geq O$ satisfying $Ax = b$ if and only if for each $y \in \mathbb{R}^m$ with $y^T A \geq O$ it also holds that $y^T b \geq 0$.*

*Proof.* Consider the finitely generated convex cone $C = \text{cone}(\{a^1, \ldots, a^n\}) \subseteq \mathbb{R}^m$. Then $C$ is closed (recall Proposition 2.5.5). We observe that $Ax = b$ has a nonnegative solution simply means (geometrically) that $b \in C$. Assume now that $Ax = b$ and $x \geq O$. If $y^T A \geq O$, then $y^T b = y^T(Ax) = (y^T A)x \geq 0$ as the inner product of two nonnegative vectors. Conversely, if $Ax = b$ has no nonnegative solution, then $b \notin C$. But then, by Theorem 3.2.2, $C$ and $b$ can be strongly separated, so there is a nonzero vector $y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ with $y^T x \geq \alpha$ for each $x \in C$ and $y^T b < \alpha$. As $O \in C$, we have $\alpha \leq 0$. We claim that $y^T x \geq 0$ for each $x \in C$: for if $y^T x < 0$ for some $x \in C$, there would be a point $\lambda x \in C$ with $\lambda > 0$ ($C$ is a cone) such that $y^T(\lambda x) < \alpha$, a contradiction. Therefore (as $a^j \in C$) $y^T a^j \geq 0$ so $y^T A \geq O$. Since $y^T b < 0$ we have proved the other direction of Farkas' lemma. □

It is important to understand the geometrical content of Farkas' lemma: $b$ lies in the finitely generated cone $C = \text{cone}(\{a^1, \ldots, a^n\})$ if and only if there is no hyperplane $H = \{x \in \mathbb{R}^n : y^T x = 0\}$ (having normal vector $y$) that (strongly) separates $b$ and $C$, i.e., $y^T b < 0$ and $y^T a^j > 0$ for each $j$. See the illustration in Fig. 3.6.

We conclude this section with another separation theorem. It may be derived from Theorem 3.2.2, but we omit the proof here.

**Theorem 3.2.6** (More on separation)**.** *Let $S$ and $T$ be disjoint nonempty closed convex sets in $\mathbb{R}^n$. Then $S$ and $T$ can be separated. If, moreover, $S$ is compact and $T$ is closed, then $S$ and $T$ can be strongly separated.*

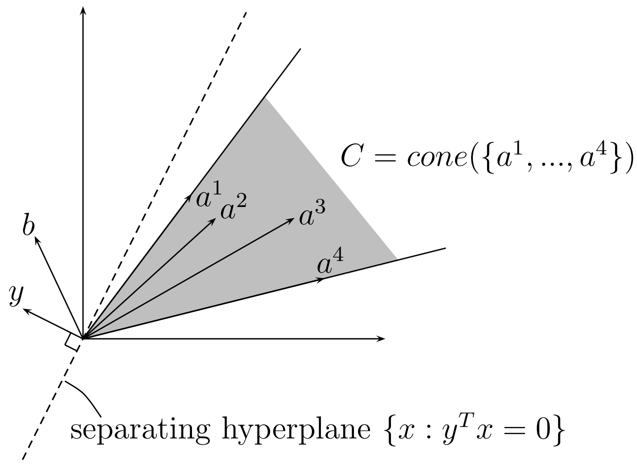


Figure 3.6: Geometry of Farkas' lemma

## 3.3 Exercises

**Exercise 3.5.** *Let $C \subseteq \mathbb{R}^n$ be convex. Recall that if a point $x_0 \in C$ that satisfies (3.2) for any $y \in C$, then $x_0$ is the (unique) nearest point to $x$ in $C$. Now, let $C$ be the unit ball in $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$ satisfy $\|x\| > 1$. Find the nearest point to $x$ in $C$. What if $\|x\| \leq 1$?*

**Exercise 3.6.** *Let $L$ be a line in $\mathbb{R}^n$. Find the nearest point in $L$ to a point $x \in \mathbb{R}^n$. Use your result to find the nearest point on the line $L = \{(x, y) : x+3y = 5\}$ to the point $(1, 2)$.*

**Exercise 3.7.** *Let $H$ be a hyperplane in $\mathbb{R}^n$. Find the nearest point in $H$ to a point $x \in \mathbb{R}^n$. In particular, find the nearest point to each of the points $(0, 0, 0)$ and $(1, 2, 2)$ in the hyperplane $H = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1\}$.*

**Exercise 3.8.** *Let $L$ be a linear subspace in $\mathbb{R}^n$ and let $q_1, \ldots, q_t$ be an orthonormal basis for $L$. Thus, $q_1, \ldots, q_t$ span $L$, $q_i^T q_j = 0$ when $i \neq j$ and $\|q_j\| = 1$ for each $j$. Let $q$ be the $n \times t$-matrix whose $j$th column is $q_j$, for $j = 1, \ldots, t$. Define the associated matrix $p = qq^T$. Show that $px$ is the nearest point in $L$ to $x$. (The matrix $P$ is called an orthogonal projector (or projection matrix)). Thus, performing the projection is simply to apply the linear transformation given by $p$. Let $L^\perp$ be the orthogonal complement of $L$. Explain why $(I - P)x$ is the nearest point in $L^\perp$ to $x$.*

**Exercise 3.9.** *Let $L \subset \mathbb{R}^3$ be the subspace spanned by the vectors $(1, 0, 1)$ and $(0, 1, 0)$. Find the nearest point to $(1, 2, 3)$ in $L$ using the results of the previous exercise.*

**Exercise 3.10.** *Show that the nearest point in $\mathbb{R}^n_+$ to $x \in \mathbb{R}^n$ is the point $x^+$ defined by $x^+_j = x^+_j = \max\{x_j, 0\}$ for each $j$.*

**Exercise 3.11.** *Find a set $S \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ with the property that* every *point of $S$ is nearest to $x$ in $S$!*

**Exercise 3.12.** *Verify that every hyperplane in $\mathbb{R}^n$ has dimension $n-1$.*

**Exercise 3.13.** *Let $C = [0,1]\times[0,1] \subset \mathbb{R}^2$ and let $a = (2,2)$. Find* all *hyperplanes that separates $C$ and $a$.*

**Exercise 3.14.** *Let $C$ be the unit ball in $\mathbb{R}^n$ and let $a \notin C$. Find a hyperplane that separates $C$ and $a$.*

**Exercise 3.15.** *Find an example in $\mathbb{R}^2$ of two sets that have a unique separating hyperplane.*

**Exercise 3.16.** *Let $S, T \subseteq \mathbb{R}^n$. Explain the following fact: there exists a hyperplane that separates $S$ and $T$ if and only if there is a linear function $l : \mathbb{R}^n \to \mathbb{R}$ such that $l(s) \leq l(t)$ for all $s \in S$ and $t \in T$. Is there a similar equivalence for the notion of strong separation?*

**Exercise 3.17.** *Show that a convex set $C$ is bounded if and only if* $\mathrm{rec}(C) = \{O\}$.

**Exercise 3.18.** *Let $C$ be a nonempty closed convex set in $\mathbb{R}^n$. Then the associated projection operator $p_C$ is Lipschitz continuous with Lipschitz constant 1, i.e.,*

$$\|p_C(x) - p_C(y)\| \leq \|x - y\| \quad \text{for all } x, y \in \mathbb{R}^n.$$

*(Such an operator is called nonexpansive). You are asked to prove this using the following procedure. Define $a = x - p_C(x)$ and $b = y - p_C(y)$. Verify that $(a-b)^T(p_C(x)-p_C(y)) \geq 0$. (Show first that $a^T(p_C(y)-p_C(x) \leq 0$ and $b^T(p_C(x)-p_C(y) \leq 0$ using* (3.2)*. Then consider $\|x-y\|^2 = \|(a-b)+(p_C(x)-p_C(y))\|^2$ and do some calculations.)*

**Exercise 3.19.** *Consider the outer description of closed convex sets given in Corollary* 3.2.4*. What is this description for each of the following sets: (i) $C_1 = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, (ii) $C_2 = \mathrm{conv}(\{0,1\}^n)$, (iii) $C_3$ is the convex hull of the points $(1,1)$, $(-1,1)$, $(1,-1)$, $(-1,-1)$ in $\mathbb{R}^2$, (iv) $C_4$ is the convex hull of all vectors in $\mathbb{R}^n$ having components that are either 1 or -1.*

## SUMMARY OF NEW CONCEPTS AND RESULTS:

- nearest point
- projection operator
- hyperplane, normal vector
- supporting hyperplane
- separation, strong separation

# Chapter 4

# Representation of convex sets

We recall that in Corollary 3.2.4 we saw how any closed convex set may be represented as the intersection of all its supporting halfspaces. One of the topics of this chapter is to find other ("interior") representations of a convex set. In fact, having different such representations is useful for analysis or optimization in connection with convex sets.

## 4.1 Faces of convex sets

This section deals with faces of convex sets. As an illustration consider the unit cube in $\mathbb{R}^3$ (see Fig. 4.1): $C = \{x \in \mathbb{R}^3 : 0 \leq x_i \leq 1 \text{ for } i = 1, 2, 3\}$. Then $C$ has eight faces of dimension zero, these are the points $(x_1, x_2, x_3)$ where $x_i \in \{0, 1\}$ for $i = 1, 2, 3$. $C$ has twelve faces of dimension one, for instance $[(0, 0, 0), (1, 0, 0)]$ and $[(1, 1, 0), (1, 1, 1)]$. Finally, it has six faces of dimension two. These are the ones you look at when you throw dice, so for instance, a two-dimensional face is the convex hull of the points $(0, 0, 0), (1, 0, 0), (0, 1, 0)$ and $(1, 1, 0)$.

Let $C$ be a convex set in $\mathbb{R}^n$. We say that a convex subset $F$ of $C$ is a *face* of $C$ whenever the following condition holds: if $x_1, x_2 \in C$ is such that $(1-\lambda)x_1 + \lambda x_2 \in F$ for some $0 < \lambda < 1$, then $x_1, x_2 \in F$. In other words the condition says: if a relative interior point of the line segment between two points of $C$ lies in $F$, then the whole line segment between these two points lies in $F$. It is also convenient to let the empty set and $C$ itself be faces of $C$, these are called the *trivial faces* of $C$.

`face`

`trivial faces`

**Example 4.1.1.** *(Faces)* Consider the unit square in $\mathbb{R}^2$: $C = \{(x_1, x_2) : 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 1\}$. Let $F_1 = \text{conv}(\{(0, 0), (1, 1)\}$ which is a diagonal of the square. $F_1$ is a convex subset of $C$, but it is not a face of $C$. To see this, consider the point $z = (1/2, 1/2) \in F_1$ and note that $z = (1/2)(1, 0) + (1/2)(0, 1)$. If $F_1$ were

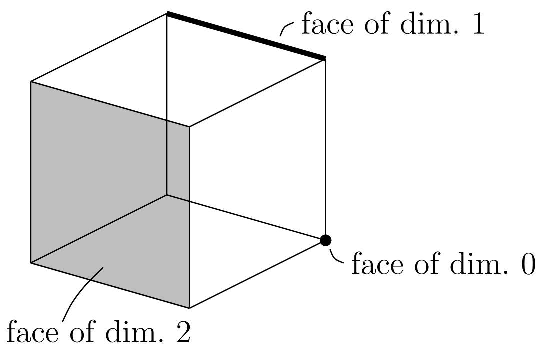


Figure 4.1: Some faces of the unit cube

a face, the face definition would give that $(1,0),(0,1) \in F_1$ which is not true. Therefore $F$ is not a face of $C$. On the other hand, the set $F_2 = \text{conv}(\{(0,0),(1,0)\}$ is a face of $C$. To verify this assume that $x_1, x_2 \in C$ and $z := (1-\lambda)x_1 + \lambda x_2 \in F_2$ for some $0 < \lambda < 1$. Since $z \in F_2$ we must have $z_2 = 0$, so $(1-\lambda)x_{1,2} + \lambda x_{2,2} = 0$ (where e.g., $x_{2,1}$ is the first component of the vector $x_2$). But here $x_{1,1}, x_{2,1} \geq 0$ (as $x_1, x_2 \in C$) and because $0 < \lambda < 1$ we must have that $x_{1,2} = x_{2,2} = 0$ and therefore $x_1, x_2 \in F_2$ as desired.

**Exercise 4.1.** *Consider the polytope $P \subset \mathbb{R}^2$ being the convex hull of the points $(0,0)$, $(1,0)$ and $(0,1)$ (so $P$ is a simplex in $\mathbb{R}^2$). (i) Find the unique face of $P$ that contains the point $(1/3,1/2)$. (ii) Find all the faces of $P$ that contain the point $(1/3,2/3)$. (iii) Determine all the faces of $P$.*

**Exercise 4.2.** *Explain why an equivalent definition of face is obtained using the condition: if whenever $x_1, x_2 \in C$ and $(1/2)(x_1+x_2) \in F$, then $x_1, x_2 \in F$.*

Every nontrivial face of a convex set is a (convex) subset of the boundary of $C$, see Exercise 4.19. Thus, in the example above the diagonal $F_1$ could not be a face of $C$ as it is not a subset of the boundary of $C$.

There is another face-concept which is closely related to the one above. Let $C \subseteq \mathbb{R}^n$ be a convex set and $H$ a supporting hyperplane of $C$. Then the intersection $C \cap H$ is called an *exposed face* of $C$, see Fig. 4.2. We also consider the empty set and $C$ itself as (trivial) exposed faces of $C$. Note that every exposed face of $C$ is convex, as the intersection of convex sets. Exposed faces are of great interest in optimization. Consider the optimization problem

`exposed face`

$$\max\{c^T x : x \in C\}$$

where $C \subseteq \mathbb{R}^n$ is a closed convex set and $c \in \mathbb{R}^n$. This includes linear programming. Assume that the optimal value $v$ is finite. Then the set of optimal solutions

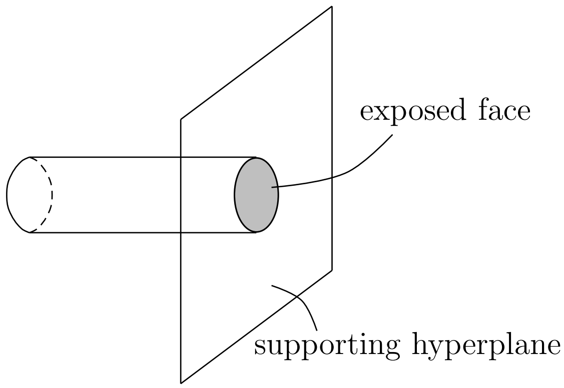


Figure 4.2: Exposed face

is

$$\{x \in C : c^T x = v\}$$

which is an exposed face of $C$. This is so because the hyperplane defined by $c^T x = v$ is supporting, and each point in $C$ also lies in the halfspace defined by $c^T x \leq v$. In fact, the *exposed faces are precisely the set of optimal solutions in problems of maximizing a (nontrivial) linear function over* $C$.

**Proposition 4.1.1** (Relation between faces and exposed faces)**.** *Let $C$ be a nonempty convex set in $\mathbb{R}^n$. Then each exposed face of $C$ is also a face of $C$.*

*Proof.* Let $F$ be an exposed face of $C$. Then, for some suitable vector $c \in \mathbb{R}^n$, $F$ is the set of optimal solutions in the problem of maximizing $c^T x$ over $C$. Define $v = \max\{c^T x : x \in C\}$. Thus, $F = \{x \in C : c^T x = v\}$. We noted above that $F$ is convex, and we now verify the remaining face property. Let $x_1, x_2 \in C$ and assume that $(1-\lambda)x_1 + \lambda x_2 \in F$ for some $0 < \lambda < 1$. Then $c^T((1-\lambda)x_1 + \lambda x_2) = v$ and, moreover,

$$\begin{array}{ll} \text{(i)} & c^T x_1 \leq v \\ \text{(ii)} & c^T x_2 \leq v \end{array}$$

by the definition of the optimal value $v$. Assume now that at least one of the inequalities (i) and (ii) above is strict. We multiply inequality (i) by $1-\lambda$, multiply inequality (ii) by $\lambda$ and add the resulting inequalities. Since $\lambda$ lies strictly between 0 and 1 we then obtain

$$v > (1-\lambda)c^T x_1 + \lambda c^T x_2 = c^T((1-\lambda)x_1 + \lambda x_2) = v.$$

From this contradiction, we conclude that $c^T x_1 = v$ and $c^T x_2 = v$, so both $x_1$ and $x_2$ lie in $F$ and we are done. $\square$

**Remark.** The technique we used in the previous proof is often used in convexity and optimization: if a convex combination (with strictly positive coefficients) of some inequalities holds with equality, then each of the individual inequalities must also hold with equality.

For polyhedra the two notions face and exposed face coincide, but this may not be so for more general convex sets, see Exercise 4.20.

Consider a face $F_1$ of a convex set $C$. Then $F_1$ is convex so it makes sense to look at some face $F_2$ of $F_1$. Then $F_2 \subseteq C$ and $F_2$ is convex. But is it a face of $C$?

**Proposition 4.1.2** (Face of face). *Let $C \subseteq \mathbb{R}^n$ be a nonempty convex set. Let $F_1$ be a face of $C$ and $F_2$ a face of $F_1$. Then $F_2$ is also a face of $C$.*

**Exercise 4.3.** *Prove this proposition!*

extreme point

edge

extreme halfline

extreme ray

Consider again a convex set $C$ in $\mathbb{R}^n$. There are certain faces of $C$ that have special interest. Recall that, by definition, every face is also a convex set, and the faces may have different dimensions. A face of $F$ of $C$ with $\dim(F) = 0$ consists of a single point, and it is called an *extreme point*. Thus, $x \in C$ is an extreme point of $C$ if and only if whenever $x_1, x_2 \in C$ satisfies $x = (1/2)(x_1 + x_2)$, then $x_1 = x_2 = x$. The set of all extreme points of $C$ is denoted by $\text{ext}(C)$. A bounded face of $C$ that has dimension 1 is called an *edge*. Consider next an unbounded face $F$ of $C$ that has dimension 1. Since $F$ is convex, $F$ must be either a line or a halfline (i.e., a set $\{x_0 + \lambda z : \lambda \geq 0\}$). If $F$ is a halfline, we call $F$ an *extreme halfline* of $C$. The union of all extreme halflines of $C$ is a set which we denote by $\text{exthl}(C)$. Note that if $C$ contains no line, then all unbounded faces of dimension one must be extreme halflines. Moreover, if $C$ is a convex cone, then every extreme halfline is also a ray and it is therefore often called an extreme ray. See Fig. 4.3 for an illustration of the concepts extreme point and extreme halfline.

**Exercise 4.4.** *Define $C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq 1\}$. Why does $C$ not have any extreme halfline? Find all the extreme points of $C$.*

**Exercise 4.5.** *Consider a polytope $P \subset \mathbb{R}^n$, say $P = \text{conv}(\{x_1, \ldots, x_t\})$. Show that if $x$ is an extreme point of $P$, then $x \in \{x_1, \ldots, x_t\}$. Is every $x_j$ necessarily an extreme point?*

## 4.2 The recession cone

In this section we study how unbounded convex sets "behave towards infinity". To introduce the ideas, consider the unbounded convex set $C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 1,\ x_2 \geq 2\}$. Consider the halfline $R$ starting in the point $(4, 3)$ and having

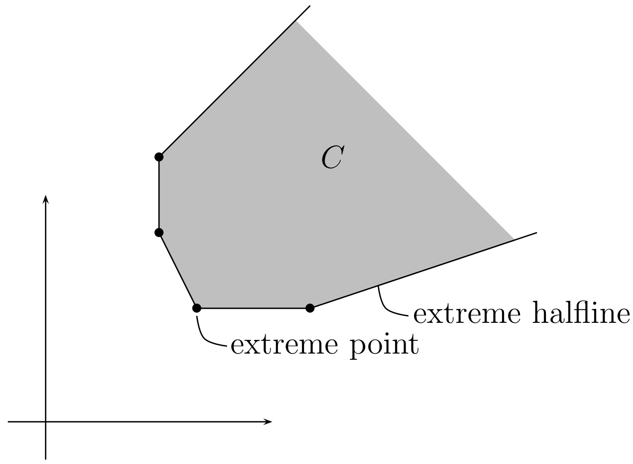


Figure 4.3: Extreme point and halfline

direction vector $z = (1, 1)$. We see that $R$ is contained in $C$. Consider another halfline now starting in $(2, 6)$ and again with direction vector $z$. This halfline also lies in $C$. In fact, testing out different start point but using the same direction vector $z$ you probably get convinced that all these halflines lie entirely in $C$. Thus, only the direction matters, not the starting point. This property, which certainly does not hold for all sets, turns out to hold for all convex sets as we shall see below. In our example, we see that all direction vectors that are nonnegative have this interesting property. We then say that the recession cone of $C$ is the convex cone $\mathbb{R}^2_+$.

Let $C$ hereafter be a nonempty closed convex set in $\mathbb{R}^n$. Define the set

$$\mathrm{rec}(C, x) = \{z \in \mathbb{R}^n : x + \lambda z \in C \ \text{ for all } \lambda \geq 0\}.$$

This is the set of directions of halflines from $x$ that lie in $C$.

**Exercise 4.6.** *Show that* $\mathrm{rec}(C, x)$ *is a closed convex cone. First, verify that* $z \in C$ *implies that* $\mu z \in C$ *for each* $\mu \geq 0$*. Next, in order to verify convexity you may show that*

$$\mathrm{rec}(C, x) = \bigcap_{\lambda > 0} \frac{1}{\lambda}(C - x)$$

*where* $\frac{1}{\lambda}(C - x)$ *is the set of all vectors of the form* $\frac{1}{\lambda}(y - x)$ *where* $y \in C$*.*

The following important result says that the closed convex cone $\mathrm{rec}(C, x)$ actually does not depend on $x$, it only depends on $C$!

**Proposition 4.2.1** (On recession cones)**.** *The closed convex cone* $\mathrm{rec}(C, x)$ *does not depend on* $x$*.*

*Proof.* Let $x_1$ and $x_2$ be distinct points in $C$ and let $z \in \mathrm{rec}(C, x_1)$. Consider the two halflines $R_i = \{x_i + \lambda z : \lambda \geq 0\}$ for $i = 1, 2$. Then $R_1 \subseteq C$ and we want to prove that also $R_2 \subseteq C$. To do this consider a point $y_2 \in R_2$, where $y_2 \neq x_2$, and let $w \in [x_1, y_2)$ (this set is $[x_1, y_2] \setminus \{y_2\}$). The halfline from $x_2$ that goes through $w$ must intersect the halfline $R_1$ (as $R_1$ and $R_2$ lie in the same two-dimensional plane). But then, as both this intersection point and $x_2$ lie in $C$, convexity implies that $w \in C$. It follows that $[x_1, y_2) \subseteq C$ and since $C$ is closed this implies that $y_2 \in C$. We have therefore proved that $R_2 \subseteq C$. Since this holds for every $z \in \mathrm{rec}(C, x_1)$ we conclude that $\mathrm{rec}(C, x_1) \subseteq \mathrm{rec}(C, x_2)$. The converse inclusion follows by similar arguments (i.e., by symmetry), so $\mathrm{rec}(C, x_1) = \mathrm{rec}(C, x_2)$. □

recession cone (asymptotic cone)

Due to this proposition we may define $\mathrm{rec}(C) := \mathrm{rec}(C, x)$ and this set is called the *recession cone* (or *asymptotic cone*) of $C$.

Based on the recession cone we may produce another interesting set associated with our closed convex set $C$. Define

$$\mathrm{lin}(C) = \mathrm{rec}(C) \cap (-\mathrm{rec}(C)).$$

lineality space

The set is called the *lineality space* of $C$. Thus, $z \in \mathrm{lin}(C)$ means that both $z$ and $-z$ lie in $\mathrm{rec}(C)$. So $\mathrm{lin}(C)$ consist of all direction vectors of lines that are contained in $C$ (as well as the zero vector). We remark that the notions recession cone and lineality space also make sense for nonclosed convex sets. But in that case the recession cone may not be closed.

**Exercise 4.7.** *Consider a hyperplane $H$. Determine its recession cone and lineality space.*

**Exercise 4.8.** *What is* $\mathrm{rec}(P)$ *and* $\mathrm{lin}(P)$ *when $P$ is a polytope?*

**Exercise 4.9.** *Let $C$ be a closed convex cone in $\mathbb{R}^n$. Show that* $\mathrm{rec}(C) = C$.

**Exercise 4.10.** *Prove that* $\mathrm{lin}(C)$ *is a linear subspace of* $\mathbb{R}^n$.

**Exercise 4.11.** *Show that* $\mathrm{rec}(\{x : Ax \leq b\}) = \{x : Ax \leq O\}$.

line-free, pointed

orthogonal complement

Often the treatment of convex sets becomes simpler if the lineality space is trivial, so it just contains the zero vector. We say that a convex set $C$ is *line-free* if it contains no line, or equivalently, that $\mathrm{lin}(C) = \{O\}$. Sometimes, especially for polyhedra, the term *pointed* is used instead of line-free. From linear algebra we recall that if $L$ is a linear subspace of $\mathbb{R}^n$, then its *orthogonal complement*, denoted by $L^\perp$, is defined by

$$L^\perp = \{y \in \mathbb{R}^n : y \perp x \text{ for all } x \in L\}.$$

Here $y \perp x$ means that these vectors are orthogonal, i.e., that $y^T x = 0$.

It is useful to keep the following result in mind. Any closed convex set $C \subseteq \mathbb{R}^n$ may be written as

$$C = \text{lin}(C) + C' \tag{4.1}$$

where $\text{lin}(C)$ is the lineality space of $C$ and $C'$ is a line-free closed convex set which is contained in $\text{lin}(C)^{\perp}$. [Proof: Define $L = \text{lin}(C)$ and consider the orthogonal decomposition (from linear algebra) $\mathbb{R}^n = L \oplus L^{\perp}$. Let $x \in C$, so $x = x_1 + x_2$ for some (unique) $x_1 \in L$ and $x_2 \in L^{\perp}$. Then $x_2 = x - x_1 \in C$ (as $x_1 \in L = \text{lin}(C)$) so $C \subseteq L + (C \cap L^{\perp})$. Conversely, assume $x = x_1 + x_2$ where $x_1 \in L$ and $x_2 \in C \cap L^{\perp}$. Then $x \in C$ as $x_2 \in C$ and $x_1 \in L$. Therefore $C = L + (C \cap L^{\perp})$ which shows (4.1) with $C' = C \cap L^{\perp}$. Clearly $C'$ is closed and convex (as intersection of such sets), and it is easy to see that $C'$ is line-free.]

Thus, every vector in $C$ may be written uniquely as a sum of a vector in $C'$ and one in $L$. Therefore, we will mainly be interested in understanding the structure of line-free convex sets in the following.

**Exercise 4.12.** *Let $C$ be a line-free closed convex set and let $F$ be an extreme halfline of $C$. Show that then there is an $x \in C$ and a $z \in \text{rec}(C)$ such that $F = x + \text{cone}(\{z\})$.*

**Exercise 4.13.** *Decide if the following statement is true: if $z \in \text{rec}(C)$ then $x + \text{cone}(\{z\})$ is an extreme halfline of $C$.*

## 4.3 Inner representation and Minkowski's theorem

The goal in this section is to present and prove results saying that any closed convex set $C$ may be written as the convex hull $\text{conv}(S)$ of a certain subset $S$ of $C$. Naturally, one would like to have $S$ smallest possible. It turns out that a possible choice for the subset $S$ is the boundary of $C$, in fact, with some exceptions, one may even let $S$ be the relative boundary of $C$.

**Exercise 4.14.** *Consider again the set $C = \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq 1\}$ from Exercise 2.16. Convince yourself that $C$ equals the convex hull of its relative boundary. Note that we here have $\text{bd}(C) = C$ so the fact that $C$ is the convex hull of its boundary is not very impressive!*

**Exercise 4.15.** *Let $H$ be a hyperplane in $\mathbb{R}^n$. Prove that $H \neq \text{conv}(\text{rbd}(H))$.*

In order to come to our main result we warm up with the following geometrical lemma.

**Lemma 4.3.1.** *Let $C$ be a line-free closed convex set and assume that $x \in \text{rint}(C)$. Then there are two distinct points $x_1, x_2 \in \text{rbd}(C)$ such that $x \in [x_1, x_2]$.*

*Proof.* The affine hull of $C$, $\text{aff}(C)$, is parallel to a unique linear subspace $L$, so $\text{aff}(C) = L + x_0$ for some $x_0 \in C$. We observe that $\text{rec}(C) \subseteq L$ (for if $z \in \text{rec}(C)$, then $x_0 + z \in C \subseteq x_0 + L$, so $z \in L$). Moreover, $-\text{rec}(C) \subseteq L$ as $L$ is a linear subspace. Thus, $\text{rec}(C) \cup (-\text{rec}(C)) \subseteq L$. We now prove that this inclusion is strict. For if $\text{rec}(C) \cup (-\text{rec}(C)) = L$, then it is easy to see that both $\text{rec}(C)$ and $-\text{rec}(C)$ must contain some line, and this contradicts that $C$ is line-free. Therefore, there is a vector $z \in L \setminus (\text{rec}(C) \cup (-\text{rec}(C)))$. Consider the point $x(t) = x + tz$. Since $x \in \text{rint}(C)$ there is some $\lambda_0 > 0$ such that $x(t) \in C$ when $|t| \leq \lambda_0$. On the other hand, if $t$ is large enough $x(t) \notin C$ as $z \notin \text{rec}(C)$. Since $C$ is closed there is a maximal $t$, say $t = t_1$ such that $x(t)$ lies in $C$. Thus, $x(t_1) \in \text{rbd}(C)$. Similarly (because $z \notin -\text{rec}(C)$) we can find $t_2$ such that $x(t_2) \in \text{rbd}(C)$. Moreover, we clearly have that $x \in [x_1, x_2]$ and the proof is complete. $\square$

The inner representation of convex sets becomes at its nicest when we consider convex sets not containing any line, i.e., line-free convex sets. But having in mind the decomposition in (4.1) we realize that the general case is easy to deduce from the line-free case. At this point recall that $\text{ext}(C)$ is the set of all extreme points of $C$ and $\text{exthl}(C)$ is the union of all extreme halflines of $C$. The following theorem is of great importance in convexity and its applications.

**Theorem 4.3.2** (Inner description of closed convex sets)**.** *Let $C \subseteq \mathbb{R}^n$ be a nonempty and line-free closed convex set. Then $C$ is the convex hull of its extreme points and extreme halflines, i.e.,*

$$C = \text{conv}(\text{ext}(C) \cup \text{exthl}(C)).$$

*Proof.* We prove this by induction on $d = \dim(C)$. If $\dim(C) = 0$, $C$ must be a one-point set, and then the result is trivial. Assume that the result holds for all line-free closed convex sets in $\mathbb{R}^n$ having dimension strictly smaller than $d$. Let $C$ be a line-free closed convex set with $d = \dim(C) > 0$. Let $x \in C$. We treat two possible cases separately.

First, assume that $x \in \text{rbd}(C)$. From Proposition 3.2.3 $C$ has a nontrivial supporting hyperplane $H$ at $x$. Then the exposed face $C' := C \cap H$ of $C$ is a strict subset of $C$ and $\dim(C') < d$ (as $C$ contains a point which does not lie in $\text{aff}(C') \subseteq H$). Since $C'$ is convex and has dimension less than $d$, and $x \in C'$, we conclude from our induction hypothesis that $x$ may be written as a convex combination of points in $\text{ext}(C') \cup \text{exthl}(C')$. But $\text{ext}(C') \subseteq \text{ext}(C)$ and $\text{exthl}(C') \subseteq \text{exthl}(C)$ so we are done.

The remaining case is when $x \in \text{rint}(C)$ (for $\text{rbd}(C) = \text{cl}(C) \setminus \text{rint}(C) = C \setminus \text{rint}(C)$ as $C$ is closed). We now use Lemma 4.3.1 and conclude that $x$ may be

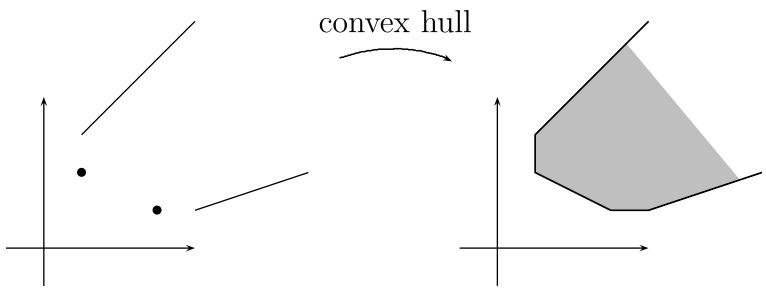


Figure 4.4: Minkowski's theorem, inner description

written as a convex combination of two points $x_1$ and $x_2$ lying on the relative boundary of $C$. But, by the first part of our proof, both $x_1$ and $x_2$ may be written as convex combinations of points in $\text{ext}(C') \cup \text{exthl}(C')$. A small calculation then proves that also $x$ is a convex combination of points in $\text{ext}(C') \cup \text{exthl}(C')$ and we are done. $\square$

The geometry of Theorem 4.3.2 is illustrated in Fig. 4.4.

The previous theorem may be reformulated in a more convenient form.

**Corollary 4.3.3** (Inner description – another version). *Let $C \subseteq \mathbb{R}^n$ be a nonempty and line-free closed convex set. Choose a direction vector $z$ for each extreme halfline of $C$ and let $Z$ be the set of these direction vectors. Then we have that*

$$C = \text{conv}(\text{ext}(C)) + \text{rec}(C) = \text{conv}(\text{ext}(C)) + \text{cone}(Z).$$

*Proof.* Note first that $Z \subseteq \text{rec}(C)$ (see Exercise 4.12) and by convexity $\text{cone}(Z) \subseteq \text{rec}(C)$ (why?). Let $x \in C$, so by Theorem 4.3.2 we may write $x$ as a convex combination of points $v_1, \ldots, v_t$ in $\text{ext}(C)$ and points $w_1, \ldots, w_r$ in $\text{exthl}(C)$. But every point $w_j$ in $\text{exthl}(C)$ may be written $w_j = x_j + z_j$ for some $x_j \in C$ and $z_j \in Z$. From this we obtain that $x$ is a convex combination of $v_1, \ldots, v_t$ plus a nonnegative combination of points $z_1, \ldots, z_r$ in $Z$. This proves that $C \subseteq \text{conv}(\text{ext}(C)) + \text{cone}(Z) \subseteq \text{conv}(\text{ext}(C)) + \text{rec}(C)$ (the last inclusion is due to what we noted initially). Moreover, the inclusion $\text{conv}(\text{ext}(C)) + \text{rec}(C) \subseteq C$, follows directly from the fact that $C$ is convex and the definition of the recession cone. This proves the desired equalities. $\square$

If a convex set is bounded, then clearly $\text{rec}(C) = \{0\}$. Therefore we get the following very important consequence of Corollary 4.3.3.

**Corollary 4.3.4** (Minkowski's theorem)**.** *If $C \subseteq \mathbb{R}^n$ is a compact convex set, then $C$ is the convex hull of its extreme points, i.e., $C = \text{conv}(\text{ext}(C))$.*

## 4.4 Polytopes and polyhedra

The goal of this section is to continue the investigations that led to Minkowski's theorem on inner representation of convex sets. But now we concentrate on polyhedra and for these special convex sets there are some very nice, and important, results. Our presentation is influenced by [10].

We consider a

$(\diamond)$ *nonempty line-free (pointed) polyhedron* $P = \{x \in \mathbb{R}^n : Ax \leq b\}$

where $A \in \mathbb{R}^{m,n}$ and $b \in \mathbb{R}^m$. We let $a_i$ denote the $i$th row in $A$ (treated as a column vector though). It follows from our assumption $(\diamond)$ that $\text{rank}(A) = n$ and $m \geq n$ where $\text{rank}(A)$ denotes the rank of $A$. In fact, if $\text{rank}(A) < n$ there would be a nonzero vector $z \in \mathbb{R}^n$ such that $Az = O$ (as the dimension of the nullspace (kernel) of $A$ is $n - \text{rank}(A)$). But then, for $x_0 \in P$, we get $A(x_0 + \lambda z) = Ax_0 + \lambda az \leq b$, so $P$ would contain the line through $x_0$ having direction vector $z$. This contradicts that $P$ is line-free, so we conclude that $\text{rank}(A) = n$ and therefore that $m \geq n$.

vertex

Our first job is to understand the concepts of extreme point and extreme halfline better. A point $x_0 \in P$ is called a *vertex* of a polyhedron $P$ if $x_0$ is the (unique) solution of $n$ linearly independent equations from the system $Ax = b$. This means that $x_0$ lies in $P$ and that we can find $n$ distinct row indices $i_1, \ldots, i_n$ (recall that $m \geq n$) such that $a_{i_1}, \ldots, a_{i_n}$ are linearly independent and $a_{i_1}^T x_0 = b_{i_1}, \ldots, a_{i_n}^T x_0 = b_{i_n}$. These selected rows form a nonsingular $n \times n$ submatrix $A_0$ of $A$ and $b_0$ is the corresponding subvector of $b$. Thus we have that $A_0 x_0 = b_0$. Note that the linear independence (i.e., $A_0$ nonsingular) assures that $x_0$ is the unique solution of this system of equations. It turns out that a vertex is nothing more than a good old extreme point!

**Lemma 4.4.1** (Extreme point = vertex)**.** *Let $x_0 \in P$. Then $x_0$ is a vertex of $P$ if and only if $x_0$ is an extreme point of $P$.*

*Proof.* Let $x_0$ be a vertex of $P$, so there is a $n \times n$ submatrix $A_0$ of $A$ and a corresponding subvector $b_0$ of $b$ such that $A_0 x_0 = b_0$. Assume that $x_0 = (1/2)x_1 + (1/2)x_2$ where $x_1, x_2 \in P$. Let $a_i$ be a row of $A_0$. We have that $a_i^T x_1 \leq b_i$ and $a_i^T x_2 \leq b_i$ (as both points lie in $P$). But if one of these inequalities were strict we would get $a_i^T x_0 = (1/2)a_i^T x_1 + (1/2)a_i^T x_2 < b_i$ which is impossible (because $A_0 x_0 = b_0$). This proves that $A_0 x_1 = b_0$ and $A_0 x_2 = b_0$. But $A_0$ is nonsingular so we get $x_1 = x_2 = x_0$. This shows that $x_0$ is an extreme point of $P$.

Conversely, assume that $x_0 \in P$ is not a vertex of $P$, and consider all the indices $i$ such that $a_i^T x_0 = b_i$. Let $A_0$ be the submatrix of $A$ containing the corresponding rows $a_i$, and let $b_0$ be the corresponding subvector of $b$. Thus, $A_0 x_0 = b_0$ and, since $x_0$ is not a vertex, the rank of $A_0$ is less than $n$, so the kernel (nullspace) of $A_0$ contains a nonzero vector $z$, i.e., $A_0 z = O$. We may now find a "small" $\epsilon > 0$ such that the two points $x_1 = x_0 + \epsilon \cdot z$ and $x_2 = x_0 - \epsilon \cdot z$ both lie in $P$. (For each row $a_i$ which is not in $A_0$ we have $a_i^T x_0 < b_i$ and a small $\epsilon$ assures that $a_i^T(x_0 \pm \epsilon z) < b_i$). But $x_0 = (1/2)x_1 + (1/2)x_2$ so $x_0$ is not an extreme point. □

**Exercise 4.16.** *Consider a polyhedral cone $C = \{x \in \mathbb{R}^n : Ax \leq O\}$ (where, as usual, $a$ is a real $m \times n$-matrix). Show that $O$ is the unique vertex of $C$.*

We now turn to extreme halflines and how they may be viewed. Consider a face $F$ of $P$ with dimension one. Due to ($\diamond$) this set $F$ is not a line so it is an extreme halfline, say

$$F = x_0 + \text{cone}(\{z\}) = \{x_0 + \lambda z : \lambda \geq 0\}$$

for some $x_0 \in P$ and $z \in \text{rec}(P)$. Moreover, the extremeness property means that we cannot find two nonparallel vectors $z_1, z_2 \in \text{rec}(P)$ such that $z = z_1 + z_2$. The following result is analogous to Lemma 4.4.1 and tells us how extreme halflines may be found directly from the matrix $a$. We leave the proof as an exercise (it is very similar to the proof of Lemma 4.4.1).

**Lemma 4.4.2** (Extreme halfline)**.** *Let $R = x_0 + \text{cone}(\{z\})$ be a halfline in $P$. Then $R$ is an extreme halfline if and only if $A_0 z = O$ for some $(n-1) \times n$ submatrix $A_0$ of $a$ with $\text{rank}(A_0) = n - 1$.*

The previous two lemmas are useful when one wants to determine all (or some) of the vertices (or extreme halflines) of a polyhedron. Another consequence is that the number of vertices and extreme halflines is finite.

**Corollary 4.4.3** (Finiteness)**.** *Each pointed polyhedron has a finite number of extreme points and extreme halflines.*

*Proof.* According to Lemma 4.4.1 each vertex of a pointed polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is obtained by setting $n$ linearly independent inequalities among the $m$ inequalities in the defining system $Ax \leq b$ to equality. But there are only a finite number of such choices of subsystems (in fact, at most $\begin{pmatrix} m \\ n \end{pmatrix} = m!/(n!(m-n)!)$), so the number of vertices is finite. For similar reasons the number of extreme halflines is finite (at most $m!/((n-1)!(m-n+1)!)$). □

**Remark.** As we saw in the proof, an upper bound on the number of vertices of a polyhedron $Ax \leq b$ (where $A \in \mathbb{R}^{m,n}$) is $m!/(n!(m-n)!)$. This is the number of different ways of choosing $n$ different objects (active inequalities) from $m$. The following much better (smaller) upper bound was shown by P. McMullen:

$$\binom{m-\lfloor (n+1)/2 \rfloor}{m-n} + \binom{m-\lfloor (n+2)/2 \rfloor}{m-n}.$$

For instance, when $m = 16$ and $n = 8$ the upper bound of our proof is $16!/(8!{\cdot}8!) = 12870$ while McMullens bound is 660. For a further discussion on the number of vertices, see e.g. [2].

We are now ready to prove a very important result for polyhedra. It is known under several names: *the main theorem for polyhedra* or *the representation theorem for polyhedra* or *Motzkin's representation theorem*. T.S. Motzkin proved the result in its general form in 1936, and earlier more specialized versions are due to G. Farkas, H. Minkowski and H. Weyl. The theorem holds for all polyhedra, also non-pointed polyhedra.

**Theorem 4.4.4** (Main theorem for polyhedra)**.** *Each polyhedron $P \subseteq \mathbb{R}^n$ may be written as*

$$P = \mathrm{conv}(V) + \mathrm{cone}(Z)$$

*for finite sets $V, Z \subset \mathbb{R}^n$. In particular, if $P$ is pointed, we may here let $V$ be the set of vertices and let $Z$ consist of a direction vector of each extreme halfline of $P$.*

*Conversely, if $V$ and $Z$ are finite sets in $\mathbb{R}^n$, then the set $P = \mathrm{conv}(V) + \mathrm{cone}(Z)$ is a polyhedron. i.e., there is a matrix $A \in \mathbb{R}^{m,n}$ and a vector $b \in \mathbb{R}^m$ for some $m$ such that*

$$\mathrm{conv}(V) + \mathrm{cone}(Z) = \{x \in \mathbb{R}^n : Ax \leq b\}.$$

*Proof.* Consider first a pointed polyhedron $P \subseteq \mathbb{R}^n$. Due to Corollary 4.4.3 $P$ has a finite number of extreme halflines. Moreover the set $V$ of vertices is finite (Lemma 4.4.1). Let $Z$ be the finite set consisting of a direction vector of each of these extreme halflines. It follows from Corollary 4.3.3 that

$$P = \mathrm{conv}(V) + \mathrm{cone}(Z).$$

This proves the first part of the theorem when $P$ is pointed. If $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is not pointed, we recall from (4.1) that $P = P' \oplus \mathrm{lin}(P)$ where $\mathrm{lin}(P)$ is the lineality space of $P$ and $P'$ is pointed. In fact, we may let $P'$ be the pointed polyhedron

$$P' = \{x \in \mathbb{R}^n : Ax \leq b,\ Bx = O\}$$

where the rows $b_1, \ldots, b_k \in \mathbb{R}^n$ of the $k \times n$-matrix $B$ is a basis of the linear subspace $\mathrm{lin}(P)$. By the first part of the theorem (as $P'$ is pointed) there are finite sets $V$ and $Z'$ such that $P = \text{conv}(V) + \text{cone}(Z')$. We now note that $\text{lin}(P) = \text{cone}(b_0, b_1, \ldots, b_k)$ where $b_0 = -\sum_{j=1}^k b_j$ (see Exercise 4.31). But then it is easy to check that $P = \text{conv}(V) + \text{cone}(Z)$ where $Z = Z' \cup \{b_0, b_1, \ldots, b_k\}$ so the first part of the theorem is shown.

We shall prove the second part by using what we just showed in a certain (clever!) way.

First, we prove the result for convex cones, so assume that $C$ is a finitely generated cone, say $C = \text{cone}(\{z_1, \ldots, z_t\})$. We introduce the set

$$C^\circ = \{a \in \mathbb{R}^n : z_j^T a \le 0 \ \text{ for } j = 1, \ldots, t\}.$$

The main observation is that $C^\circ$ is a polyhedral cone (and therefore a polyhedron) in $\mathbb{R}^n$: it is defined by the linear and homogeneous inequalities $z_j^T a \le 0$ for $j = 1, \ldots, t$ (here $a$ is the variable vector!). Thus, by the first part of the theorem, there is a finite set of vectors $a_1, \ldots, a_s \in C^\circ$ such that $C^\circ = \text{cone}(\{a_1, \ldots, a_s\})$ (because any polyhedral cone has only one vertex, namely $O$, see Exercise 4.16). We shall prove that

$$(*) \quad C = \{x \in \mathbb{R}^n : a_i^T x \le 0 \ \text{ for } i = 1, \ldots, s\}.$$

If $x_0 \in C$, then $x_0 = \sum_{j=1}^t \mu_j z_j$ for some $\mu_j \ge 0$, $j \le t$. For each $i \le s$ and $j \le t$ we have from the definition of $C^\circ$ that $a^T z_j \le 0$ and therefore $a^T x_0 = \sum_{j=1}^t \mu_j a^T z_j \le 0$. This shows the inclusion "$\subseteq$" in $(*)$. Assume next that $x_0 \notin C$. As $C = \text{cone}(\{z_1, \ldots, z_t\})$ it follows from Farkas' lemma (Theorem 3.2.5) that there is a vector $y \in \mathbb{R}^n$ such that $y^T x_0 > 0$ and $y^T z_j < 0$ for each $j$. Therefore $y \in C^\circ$ so $y = \sum_i \lambda_i a_i$ for nonnegative numbers $\lambda_i$ $(i \le s)$. But $x_0$ violates the inequality $y^T x \le 0$ (as $y^T x_0 > 0$). This implies that $x_0 \notin \{x \in \mathbb{R}^n : a_i^T x \le 0 \ \text{ for } i = 1, \ldots, s\}$. This proves $(*)$ and we have shown that every finitely generated convex cone is polyhedral.

More generally, let $P = \text{conv}(V) + \text{cone}(Z)$ where $V$ and $Z$ are finite sets in $\mathbb{R}^n$. Let $V = \{v_1, \ldots, v_k\}$ and $Z = \{z_1, \ldots, z_m\}$. Let $C = \text{cone}(\{(v_1, 1), \ldots, (v_k, 1), (z_1, 0), \ldots, (z_m, 0)\})$, so this is a finitely generated convex cone in $\mathbb{R}^{n+1}$. By what we just showed this cone is polyhedral, so there is a matrix $A \in \mathbb{R}^{m,n+1}$ such that $C = \{(x, x_{n+1}) : \sum_{j=1}^n a^j x_j + a^{n+1} x_{n+1} \le O\}$ (here $a^j$ is the $j$th column of $A$). Note that $x \in P$ if and only if $(x, 1) \in C$ (see Exercise 4.32). Therefore, $x \in P$ if and only if $Ax + a^{n+1} \cdot 1 \le O$, i.e., $Ax \le b$ where $b = -a^{n+1}$. This proves that $P$ is a polyhedron and our proof is complete. $\square$

A very important consequence of the Main theorem for polyhedra is the following. The proof is an exercise.

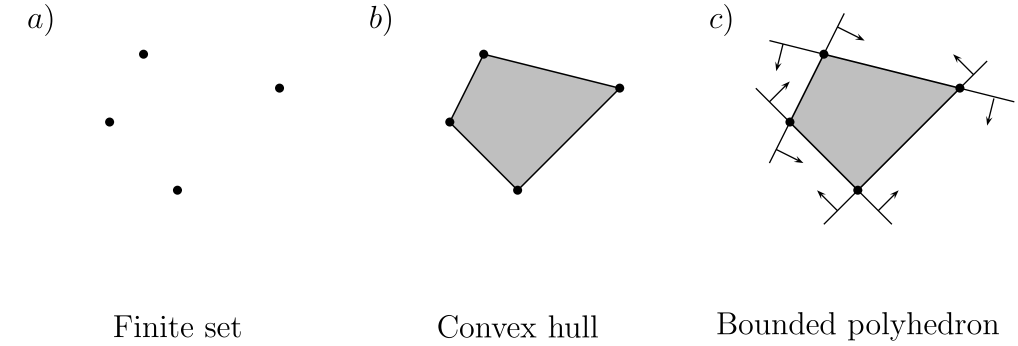


Figure 4.5: Polytope = bounded polyhedron

**Corollary 4.4.5** (Polytopes)**.** *A set is a polytope if and only if it is a bounded polyhedron.*

In the plane, the geometrical contents of Theorem 4.3.2 is quite simple, see Fig. 4.5. But it is certainly a nontrivial, and important, fact that the result holds in any dimensions.

Consider an LP problem $\max\{c^T x : x \in P\}$ where $P$ is a bounded polyhedron, which, therefore, contains no line or halfline. Then the previous proposition shows that we can solve the LP problem by comparing the objective function for all the vertices, and there is a finite number of these. This is not a good algorithm in the general case, because the number of vertices may be huge, even with a rather low number of inequalities in the defining system $Ax \leq b$. However, the fact that LP's may be solved by searching through vertices only has been the starting point for another algorithm, the *simplex method*, which, in practice, is a very efficient method for solving linear programming problems.

**Example 4.4.1.** *(Combinatorial optimization and* $(0,1)$*-polytopes)* The area of combinatorial optimization deals with optimization problems associated with typically highly structured finite (or discrete) sets. Most of these problems involve a feasible set which may be viewed as a certain subset $S$ of $\{0,1\}^m$. Thus, $S$ consists of certain $(0,1)$-vectors and each vector $x \in S$ specifies a subset of $\{1,\ldots,m\}$, namely $F = \{j : x_j = 1\}$. We say that $x$ is the incidence vector of the set $F$ and it is also denoted by $\chi^F$. As an example, consider the famous Traveling Salesman Problem (TSP): given a set of cities and intercity distances, find a tour visiting each city exactly once such that the total distance traveled is as small as possible. If we have $n$ cities there are $m = n(n-1)/2$ possible city pairs and a tour is to choose $n$ such consecutive pairs, say $\{c_1,c_2\},\{c_2,c_3\},\ldots,\{c_{n-1},c_n\},\{c_n,c_1\}$

(where no city is visited more than once). Thus, every tour may be viewed as a certain subset of "the set of city pairs" $\{1, \ldots, m\}$. Equivalently, the tours correspond to a certain subset $S_{TSP}$ of $\{0,1\}^m$. The problem of finding a shortest tour may now be seen as the problem of minimizing a linear function over the set

$$P_{TSP} = \text{conv}(S_{TSP}).$$

The linear function of interest here is $c^T x = \sum_{j=1}^m c_j x_j$ where $c_j$ is the distance of the $j$th city pair. The set $P_{TSP}$ is, by definition, a polytope, and it is usually called the *Traveling salesman polytope*. The extreme points of $P_{TSP}$ are the incidence vectors of tours (see Exercise 4.27). Since the minimum of a linear function is attained in a vertex of the polytope, say $x = \chi^F$, we obtain $c^T x = c^T \chi^F = \sum_{j \in F} c_j$ which is the total length of the tour. This explains why $P_{TSP}$ is of interest in connection with the TSP. Now, due to Corollary 4.4.5, $P_{TSP}$ is a bounded polyhedron. Therefore there *exists* a linear system $Ax \le b$ such that

$$P_{TSP} = \{x \in \mathbb{R}^m : Ax \le b\}.$$

As a consequence, in theory, the TSP may be viewed as a linear programming problem $\max\{c^T x : Ax \le b\}$. A difficulty is that the structure of $P_{TSP}$ is very complex, and lots of research papers have been written on the facial structure of TSP polytopes (see e.g., [5] in a wonderful book on the TSP). This approach has been very fruitful in gaining a mathematical understanding of the TSP and also solving TSP problems computationally. We should say that the TSP is a so-called *NP*-hard optimization problem, which loosely speaking means that, most likely, for this problem there is no efficient algorithm that is guaranteed to find an optimal solution. Note the very important fact that the approach we have sketched here for the TSP may also be applied to other combinatorial optimization problems. This is the topic of the area of *polyhedral combinatorics* in which convexity, and polyhedral theory in particular, play important roles. Thus, for polyhedral combinatorics, Corollary 4.4.5 may be seen as a main driving force!

We now turn to a study of faces of polyhedra. There will be two main goals: one is to show that notions of exposed face and face coincide for polyhedra, and the second goal is to give a useful description of faces in terms of linear systems. First, we shall go for the second goal! A *subsystem* $A'x \le b'$ of a linear system $Ax \le b$ is obtained by deleting some (possibly none) of the constraints in $Ax \le b$. The $i$th row of the matrix $A$ will be denoted by $a_i^T$ below. We say that an inequality $c^T x \le \alpha$ is *valid* for a set $P \subseteq \mathbb{R}^n$ if $P \subseteq \{x \in \mathbb{R}^n : c^T x \le \alpha\}$, i.e., each point $x$ in $P$ satisfies $c^T x \le \alpha$. `valid inequality`

**Theorem 4.4.6.** *Consider a polyhedron $P = \{x \in \mathbb{R}^n : Ax \le b\}$. A nonempty set $F$ is an exposed face of $P$ if and only if*

$$F = \{x \in P : A'x = b'\} \tag{4.2}$$

*for some subsystem $A'x \le b'$ of $Ax \le b$.*

*Proof.* Let $a_i$ denote the $i$th row of $A$ (viewed as a column vector). Let $F$ be a nonempty exposed face of $P$, so $F = \{x \in P : c^T x = \alpha\}$ where $c^T x = \alpha$ defines a supporting halfspace for $P$. Then the optimal value $v^*$ of the linear programming problem (P) max $\{c^T x : Ax \le b\}$ satisfies $v^* = \alpha < \infty$ and we have $F = \{x \in P : c^T x = v^*\}$. From the LP duality theorem we have that

$$v^* = \max\{c^T x : Ax \le b\} = \min\{y^T b : y^T A = c^T, \;\; y \ge O\}$$

and the dual LP must be feasible (as the primal optimal value is finite). Let $y^*$ be an optimal dual solution, so $(y^*)^T A = c^T$, $y^* \ge O$ and $(y^*)^T b = v^*$, and define $I' = \{i \le m : y_i > 0\}$. We claim that (4.2) holds with the subsystem $a'x \le b'$ consisting of the inequalities $a_i^T x \le b_i$ for $i \in I'$. To see this, note that for each $x \in P$ we have

$$c^T x = (y^*)^T Ax = \sum_{i \in I} y_i^* (Ax)_i = \sum_{i \in I'} y_i^* (Ax)_i \le \sum_{i \in I'} y_i^* b_i = v^*.$$

Thus we have $c^T x = v^*$ if and only if $a_i^T x = b_i$ for each $i \in I'$, and (4.2) holds.

Conversely, assume that the set $F$ satisfies (4.2) for some subsystem $A'x \le b'$ of $Ax \le b$, say that $A'x \le b'$ consists of the inequalities $a_i^T x \le b_i$ for $i \in I'$ (where $I' \subseteq \{1, \ldots, m\}$). Let $c = \sum_{i \in I'} a_i$ and $\alpha = \sum_{i \in I'} b_i$. Then $c^T x \le \alpha$ is a valid inequality for $P$ (it is a sum of other valid inequalities, see Exercise 4.33). Furthermore $F$ is the face induced by $c^T x \le \alpha$, i.e., $F = \{x \in P : c^T x = \alpha\}$. In fact, a point $x \in P$ satisfies $c^T x = \alpha$ if and only if $a_i^T x = b_i$ for each $i \in I'$. $\square$

Thus, Theorem 4.4.6 says that any (exposed) face of a polyhedron is obtained by setting some of the valid inequalities to equality. This result is often used to determine all (or some of) the faces of different polyhedra or polytopes. We now apply the theorem to show the announced result that exposed faces and faces coincide for polyhedra.

**Theorem 4.4.7.** *Let $P$ be a polyhedron in $\mathbb{R}^n$. Then every face of $P$ is also an exposed face of $P$. Thus, for polyhedra, these two notions coincide.*

*Proof.* In Proposition 4.1.1 we showed that every exposed face is also a face (for any convex set), so we only need to prove the converse when $P$ is a polyhedron. So, let $F$ be a face of a polyhedron $P = \{x \in \mathbb{R}^n : Ax \le b\}$. Let $I'$ be the set of indices $i$ such that $a_i^T x = b_i$ holds for *every* $x \in F$. Therefore

$$(*) \quad F \subseteq G := \{x \in P : a_i^T x = b_i \;\text{ for all } i \in I'\}.$$

Now, due to Theorem 4.4.6, $G$ is an exposed face of $P$, so we are done if we can prove that there is equality in $(*)$. Assume, to the contrary, that the inequality is strict. $G$ is a polyhedron which is the convex hull of a set $Z$ (consisting of its extreme points and extreme halflines). Then, by assumption, we can find at least one point $z \in Z$ which is outside $F$ and one may also find a convex combination $w$ of points in $Z$ with positive weight for $z$ such that $w \in F$ (explain why!). But this contradicts that $F$ is a face of $P$ which proves that $F = G$ and we are done. $\square$

A *facet* of a polyhedron $P \subseteq \mathbb{R}^n$ may be defined as a face $F$ of $P$ with $\dim(F) = \dim(P) - 1$. Consider a full-dimensional polytope $P$ in $\mathbb{R}^3$. It has nontrivial faces of dimension zero (vertices), of dimension one (edges) and, finally, of dimension two (facets). In 1752 the great Swiss mathematician L. Euler found a beautiful, and simple, relation between the number of these faces. Let $v$, $e$ and $f$ denote the number of vertices, edges and facets, respectively. *Euler's relation* says that

facet

Euler's relation

$$v - e + f = 2.$$

Later, in 1899, Poincaré found a generalization of this relation to arbitrary dimensions! To present this result, consider a polytope $P$ of dimension $r$ and let $f_k(P)$ be the number of faces of $P$ of dimension $k$ for $k = -1, 0, 1, \ldots, r$. Here we define $f_{-1}(P) = f_r(P) = 1$. With this notation we have the generalized *Euler's relation* (or *Euler-Poincaré relation*) saying that

$$\sum_{k=-1}^{r} (-1)^{k+1} f_k(P) = 0$$

so the alternating sum of the numbers $f_k(P)$ is zero!! For a proof of this relation we refer to [16].

Facets of polyhedra are important for finding minimal linear systems that define a polyhedron $P$, i.e., representations of $P$ without any redundant inequalities. Roughly speaking, each facet $F$ of $P$ requires an inequality in such a minimal system, and, moreover $F$ consists precisely of those points in $P$ satisfying that particular inequality with equality. We omit the details, but simply point out that a study of facets of different types of combinatorial polyhedra is a theme in the area of polyhedral combinatorics. For more about this and a full treatment of the theory of polyhedra, we strongly recommend the book [13].

## 4.5 Exercises

**Exercise 4.17.** *Let $F$ be a face of a convex set $C$ in $\mathbb{R}^n$. Show that every extreme point of $F$ is also an extreme point of $C$.*

**Exercise 4.18.** *Find all the faces of the unit ball in* $\mathbb{R}^2$*. What about the unit ball in* $\mathbb{R}^n$*?*

**Exercise 4.19.** *Let* $F$ *be a nontrivial face of a convex set* $C$ *in* $\mathbb{R}^n$*. Show that* $F \subseteq \mathrm{bd}(C)$ *(recall that* $\mathrm{bd}(C)$ *is the boundary of* $C$*). Is the stronger statement* $F \subseteq \mathrm{rbd}(C)$ *also true? Find an example where* $F = \mathrm{bd}(C)$*.*

**Exercise 4.20.** *Consider the convex set* $C = B + ([0,1] \times \{0\})$ *where* $B$ *is the unit ball (of the Euclidean norm) in* $\mathbb{R}^2$*. Find a point on the boundary of* $C$ *which is a face of* $C$*, but not an exposed face.*

**Exercise 4.21.** *Let* $P \subset \mathbb{R}^2$ *be the polyhedron being the solution set of the linear system*

$$\begin{array}{rcrl} x & - & y & \leq 0; \\ -x & + & y & \leq 1; \\ & & 2y & \geq 5; \\ 8x & - & y & \leq 16; \\ x & + & y & \geq 4. \end{array}$$

*Find all the extreme points of* $P$*.*

**Exercise 4.22.** *Find all the extreme halflines of the cone* $\mathbb{R}^n_+$*.*

**Exercise 4.23.** *Determine the recession cone of the set* $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 \geq 1/x_1\}$*. What are the extreme points?*

**Exercise 4.24.** *Let* $B$ *be the unit ball in* $\mathbb{R}^n$ *(in the Euclidean norm). Show that every point in* $B$ *can be written as a convex combination of two of the extreme points of* $C$*.*

**Exercise 4.25.** *Let* $C$ *be a compact convex set in* $\mathbb{R}^n$ *and let* $f : C \to \mathbb{R}$ *be a function satisfying*

$$f(\sum_{j=1}^{t} \lambda_j x_j) \leq \sum_{j=1}^{t} \lambda_j f(x_j)$$

*whenever* $x_1, \ldots, x_t \in C$*,* $\lambda_1, \ldots, \lambda_t \geq 0$ *and* $\sum_{j=1}^{t} \lambda_j = 1$*. Such a function is called convex, see chapter* 5*. Show that* $f$ *attains its maximum over* $C$ *in an extreme point. Hint: Minkowski's theorem.*

**Exercise 4.26.** *Prove Corollary* 4.4.5 *using the Main theorem for polyhedra.*

**Exercise 4.27.** *Let* $S \subseteq \{0,1\}^n$*, i.e.,* $S$ *is a set of* $(0,1)$*-vectors. Define the polytope* $P = \mathrm{conv}(S)$*. Show that* $x$ *is a vertex of* $P$ *if and only if* $x \in S$*.*

**Exercise 4.28.** *Let* $S \subseteq \{0,1\}^3$ *consist of the points* $(0,0,0)$*,* $(1,1,1)$*,* $(0,1,0)$ *and* $(1,0,1)$*. Consider the polytope* $P = \mathrm{conv}(S)$ *and find a linear system defining it.*

**Exercise 4.29.** *Let* $P_1$ *and* $P_2$ *be two polytopes in* $\mathbb{R}^n$*. Prove that* $P_1 \cap P_2$ *is a polytope.*

**Exercise 4.30.** *Is the sum of polytopes again a polytope? The sum of two polytopes* $P_1$ *and* $P_2$ *in* $\mathbb{R}^n$ *is the set* $P_1 + P_2 = \{p_1 + p_2 : p_1 \in P_1, p_2 \in P_2\}$*.*

**Exercise 4.31.** *Let* $L = \mathrm{span}(\{b_1, \ldots, b_k\})$ *be a linear subspace of* $\mathbb{R}^n$*. Define* $b_0 = -\sum_{j=1}^{k} b_j$*. Show that* $L = \mathrm{cone}(\{b_0, b_1, \ldots, b_k\})$*. Thus, every linear subspace is a finitely generated cone, and we know how to find a set of generators for L (i.e., a finite set with conical hull being L).*

**Exercise 4.32.** *Let* $P = \mathrm{conv}(\{v_1, \ldots, v_k\}) + \mathrm{cone}(\{z_1, \ldots, z_m\}) \subseteq \mathbb{R}^n$*. Define new vectors in* $\mathbb{R}^{n+1}$ *by adding a component which is 1 for all the* $v_{\cdot}$*-vectors and a component which is 0 for all the* $z_{\cdot}$*-vectors, and let* $C$ *be the cone spanned by these new vectors. Thus,*

$$C = \mathrm{cone}(\{(v_1, 1), \ldots, (v_k, 1), (z_1, 0), \ldots, (z_m, 0)\})$$

*Prove that* $x \in P$ *if and only if* $(x, 1) \in C$*. The cone* $C$ *is said to be obtained by homogenization of the polyhedron* $P$*. This is sometimes a useful technique for translating results that are known for cones into similar results for polyhedra, as in the proof of Theorem* 4.4.4*.* `homogenization`

**Exercise 4.33.** *Show that the sum of valid inequalities for a set* $P$ *is another valid inequality for* $P$*. What about weighted sums? What can you say about the properties of the set*

$$\{(a, \alpha) \in \mathbb{R}^{n+1} : a^T x \leq \alpha \text{ is a valid inequality for } P\}.$$

**SUMMARY OF NEW CONCEPTS AND RESULTS:**

- face, exposed face
- extreme point, extreme halfline, ray
- recession cone
- inner representation, Minkowski's theorem
- main theorem for polyhedra

# Chapter 5

# Convex functions

This chapter treats convex functions, both in the univariate and the multivariate case. In order to give an idea of what a convex function is, we give a small example. Consider two or more linear functions, say of a single variable, and let $f$ be the pointwise maximum of these functions. This function $f$ is convex (loosely speaking) as "its graph bends upwards". We also observe that all the points in the plane lying on or above the graph of $f$ is a convex set (an unbounded polyhedron).

We start our treatment with functions of a single variable before we pass on to the multivariate case.

## 5.1 Convex functions of one variable

Consider a function $f : \mathbb{R} \to \mathbb{R}$. We say that $f$ is *convex* if the inequality `convex function`

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y) \tag{5.1}$$

holds for every $x, y \in \mathbb{R}$ and every $0 \leq \lambda \leq 1$.

**Example 5.1.1.** *(A convex function)* Let $f$ be given by $f(x) = x^2$. Let us verify that $f$ is convex according to our definition. Let $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$. After some calculation we get

$$f((1-\lambda)x + \lambda y) - ((1-\lambda)f(x) + \lambda f(y)) = -(1-\lambda)\lambda(x-y)^2$$

Since $0 \leq \lambda \leq 1$ the last expression is nonpositive, and and the desired inequality (5.1) holds. So $f$ is convex (as you probably knew already).

Our analytical definition has a nice geometrical interpretation. It says that the *line segment between every pair of points* $(x, f(x))$ *and* $(y, f(y))$ *on the graph of* $f$ *lies above the graph of* $f$ *in the interval between* $x$ *and* $y$, see Fig. 5.1. This

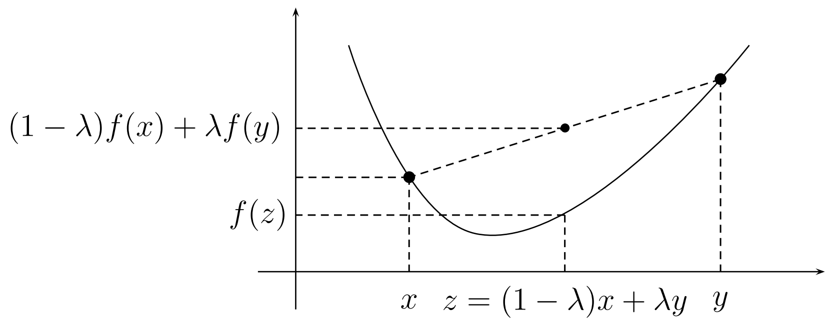


Figure 5.1: Convex function

geometrical viewpoint also leads to an understanding of the growth of convex functions. To describe this we introduce some simple notation. Let $f : \mathbb{R} \to \mathbb{R}$ be any function. For every $x \in \mathbb{R}$ we define the point $P_x = (x, f(x))$ which then lies on the graph of $f$. When $x < y$ we let $\mathrm{slope}(P_x, P_y)$ denote the slope of the line segment $P_xP_y$ between $P_x$ and $P_y$, so

$$\mathrm{slope}(P_x, P_y) = (f(y) - f(x))/(y - x).$$

The following lemma is illustrated in Fig. 5.2.

**Lemma 5.1.1** (Slopes)**.** *Let $x_1 < x_2 < x_3$. Then the following statements are equivalent:*

*(i) $P_{x_2}$ is below the line segment $P_{x_1}P_{x_3}$;*

*(ii)* $\mathrm{slope}(P_{x_1}, P_{x_2}) \leq \mathrm{slope}(P_{x_1}, P_{x_3})$*;*

*(iii)* $\mathrm{slope}(P_{x_1}, P_{x_3}) \leq \mathrm{slope}(P_{x_2}, P_{x_3})$*.*

**Exercise 5.1.** *Prove this lemma.*

A direct consequence of the lemma is the following characterization of a convex function defined on the real line.

**Proposition 5.1.2** (Increasing slopes)**.** *A function $f : \mathbb{R} \to \mathbb{R}$ is convex if and only if for each $x_0 \in \mathbb{R}$ the slope function*

$$x \to (f(x) - f(x_0))/(x - x_0).$$

*is increasing on $\mathbb{R} \setminus \{x_0\}$.*

**Exercise 5.2.** *Show that the sum of convex functions is a convex function, and that $\lambda f$ is convex if $f$ is convex and $\lambda \geq 0$ (here $\lambda f$ is the function given by $(\lambda f)(x) = \lambda f(x)$).*

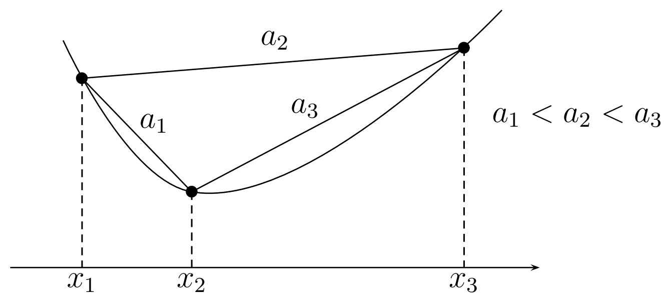


Figure 5.2: Increasing slopes

Sometimes one meets convex functions that are not defined on the whole real line. Let $I \subseteq \mathbb{R}$ be an interval (whether the endpoints of $I$ are contained in $I$ or not does not matter). Let $f : I \to \mathbb{R}$ be a function defined on $I$. We extend our definition of convexity by saying that $f$ is convex if the inequality (5.1) holds for every $x, y \in I$ and $0 \leq \lambda \leq 1$.

A function $f : I \to \mathbb{R}$ (where $I$ is an interval) is called *concave* if the function $-f$ is convex. This means that $f((1-\lambda)x+\lambda y) \geq (1-\lambda)f(x)+\lambda f(y)$ holds whenever $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$. Any result for convex functions may be reformulated in terms of concave functions by proper adjustments. Thus, we shall restrict the attention to convex functions here. `concave`

A convex function need not be differentiable, consider for instance the function $f(x) = |x|$ which is convex but not differentiable in 0. However, convex functions do have one-sided derivatives and are differentiable "almost everywhere" as discussed next. Recall that the *left-sided derivative* of $f$ at $x_0$ is defined by `one-sided derivative`

$$f'_-(x_0) := \lim_{x \to x_0^-} (f(x) - f(x_0))/(x - x_0).$$

provided this limit exists (here $x$ tends towards $x_0$ "from the left", i.e., through values $x$ that are strictly smaller than $x_0$). The *right-sided derivative* of $f$ at $x_0$ is defined similarly and it is denoted by $f'_+(x_0)$.

**Theorem 5.1.3** (One-sided derivatives)**.** *Let $f : I \to \mathbb{R}$ be a convex function defined on an interval $I$. Then $f$ has both left-and right-sided derivatives at every interior point of $I$. Moreover, if $x, y \in I$ and $x < y$ it holds that*

$$f'_-(x) \leq f'_+(x) \leq (f(y) - f(x))/(y - x) \leq f'_-(y) \leq f'_+(y).$$

*In particular, both $f'_-$ and $f'_+$ are increasing functions.*

*Proof.* Let $z$ be an interior point of $I$ and let $x, y \in I$ satisfy $x < z < y$. It follows from Proposition 5.1.2 that if $x$ increases to $z$ from the left, then $(f(x) - f(z))/(x - z)$ increases and is bounded above by $(f(y) - f(z))/(y - z)$. This implies that $f'_-(z)$ exists and that

$$f'_-(z) \le (f(y) - f(z))/(y - z).$$

In this inequality, we now decrease $y$ and conclude that $f'_+(z)$ exists and that $f'_-(z) \le f'_+(z)$. This proves the first and the last inequality of the theorem (letting $z = x$ and $z = y$, respectively) and the two remaining inequalities are obtained from Proposition 5.1.2. □

How can we decide if a function is convex? One way is to check the definition, but this is often a lot of work. So it is useful to have convexity criteria that are more convenient to check. If $f$ is differentiable, a criterion for convexity is that $f'$ is increasing. We now prove a more general version of this which, in fact, is a converse of Theorem 5.1.3.

**Theorem 5.1.4** (Increasing derivative)**.** *Let $f : I \to \mathbb{R}$ be a continuous function defined on an open interval $I$. Assume that $f$ has an increasing left-derivative, or an increasing right-derivative, on $I$. Then $f$ is convex.*

*If $f$ is differentiable, then $f$ is convex if and only if $f'$ is increasing. If $f$ is two times differentiable, then $f$ is convex if and only if $f'' \ge 0$ (i.e., $f''(x) \ge 0$ for all $x \in I$).*

*Proof.* Assume that $f$ has an increasing right-derivative $f'_+$ on $I$. Let $x, y \in I$ satisfy $x < y$ and let $0 < \lambda < 1$. We shall show that (5.1) holds (when $\lambda$ is 0 or 1, this inequality trivially holds). Define $z = (1 - \lambda)x + \lambda y$. Note that $x < z < y$ and that $z - x = \lambda(y - x)$ and $y - z = (1 - \lambda)(y - x)$.

By the the mean-value theorem (for functions having a right-derivative) there is a point $a$ with $x < a < z$ such that $f'_+(a) = (f(z) - f(x))/(z - x)$. This implies that $(f(z) - f(x))/(z - x) \le \sup_{x<t<z} f'_+(t)$. Similarly we obtain the inequality $(f(y) - f(z))/(y - z) \ge \inf_{z<t<y} f'_+(t)$ and therefore

$$(*) \quad (f(z) - f(x))/(z - x) \le \sup_{x<t<z} f'_+(t) \le \inf_{z<t<y} f'_+(t) \le (f(y) - f(z))/(y - z).$$

Here the second inequality follows from our assumption that $f'_+$ is increasing. We now multiply the inequality $(f(z) - f(x))/(z - x) \le (f(y) - f(z))/(y - z)$ by $y - x$ and (as $z - x = \lambda(y - x)$ and $y - z = (1 - \lambda)(y - x)$) obtain $(f(z) - f(x))/\lambda \le (f(y) - f(z))/(1 - \lambda)$. By rewriting this inequality we get $f((1 - \lambda)x + \lambda y) \le (1 - \lambda)f(x) + \lambda f(y)$ which proves that $f$ is convex. The proof is similar when $f$ has a left-sided derivative. The last two statements of the theorem are now easy consequences. □

**Exercise 5.3.** *Prove that the following functions are convex: (i) $f(x) = x^2$, (ii) $f(x) = |x|$, (iii) $f(x) = x^p$ where $p \geq 1$, (iv) $f(x) = e^x$, (v) $f(x) = -\ln(x)$ defined on $\mathbb{R}_+$.*

We may now prove that every convex function is continuous on the interior of its domain. In fact, a stronger Lipschitz continuity holds.

**Corollary 5.1.5** (Continuity)**.** *Let $f : I \to \mathbb{R}$ be convex and let $a, b \in \mathrm{int}(I)$ where $a < b$. Define $M = \max\{-f'_+(a), f'_-(b)\}$ (which is finite by Theorem 5.1.1). Then*

$$|f(y) - f(x)| \leq M \cdot |y - x| \quad \text{for all } x, y \in [a, b].$$

*In particular, $f$ is continuous at every interior point of $I$.*

*Proof.* If $x = y$ the desired inequality is trivial so assume that $x \neq y$. If $a < x < y < b$ it follows from Theorem 5.1.3 that $f'_+(a) \leq (f(y) - f(x))/(y - x) \leq f'_-(b)$ which implies that $|(f(y) - f(x))/(y - x)| \leq \max\{-f'_+(a), f'_-(b)\} = M$. It is easy to see that this inequality holds for every $x, y$ lying strictly between $a$ and $b$ which proves the desired Lipschitz inequality. This directly implies that $f$ is continuous at every interior point of $I$ (why?). □

Thus, a convex function defined on an interval $[a, b]$ (where $a < b$) is continuous at every point x with $a < x < b$. Concerning the endpoints $a$ and $b$, we can only say that $f(a) \geq \lim_{x \to a^+} f(x)$ and $f(b) \geq \lim_{x \to b^-} f(x)$ and any of these inequalities may be strict.

Another consequence concerns the set of points for which a convex function is not differentiable.

**Corollary 5.1.6** (Sets of Differentiability)**.** *Let $f : I \to \mathbb{R}$ be convex and let $Z$ be the set of points for which $f$ is not differentiable. Then $Z$ is countable.*

*Proof.* The idea is to find a one-to-one mapping $r$ from $Z$ to the set of rational numbers. Since the latter set is countable, this will imply that $Z$ is countable. (Recall that a function $r$ is one-to-one if distinct elements are mapped to distinct elements).

For each $z \in Z$ we have that $f'_-(z) < f'_+(z)$ (confer Theorem 5.1.3) so we may select a *rational* number $r(z)$ satisfying $f'_-(z) < r(z) < f'_+(z)$. This defines our function $r$. If $z < z'$ we have that

$$f'_-(z) < r(z) < f'_+(z) \leq f'_-(z') < r(z') < f'_+(z')$$

so $r(z) < r(z')$. This proves that the function $r$ is one-to-one as desired. □

We now look further at derivatives of convex functions. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. For each $x \in \mathbb{R}$ we associate the closed interval

$$\partial f(x) := [f'_-(x), f'_+(x)].$$

sub-differential sub-derivative

which is called the *subdifferential* of $f$ at $x$. Each point $s \in \partial f(x)$ is called a *subderivative* of $f$ at $x$. Note that, due to Theorem 5.1.3, $\partial f(x)$ is a nonempty and finite (closed) interval for each $x \in \mathbb{R}$. Moreover, $f$ is differentiable at $x$ if and only if $\partial f(x)$ contains a single point, namely the derivative $f'(x)$.

**Corollary 5.1.7** (Subdifferential)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function and let $x_0 \in \mathbb{R}$. Then, for every $s \in \partial f(x_0)$, the inequality*

$$f(x) \geq f(x_0) + s \cdot (x - x_0)$$

*holds for every $x \in \mathbb{R}$.*

*Proof.* Let $s \in \partial f(x_0)$. Note first that if $x = x_0$ the inequality is trivial. Due to Theorem 5.1.3 the following inequality holds for every $x < x_0$:

$$(f(x) - f(x_0))/(x - x_0) \leq f'_-(x_0) \leq s.$$

Thus, $f(x) - f(x_0) \geq s \cdot (x - x_0)$. Similarly, if $x > x_0$ then

$$s \leq f'_+(x_0) \leq (f(x) - f(x_0))/(x - x_0)$$

so again $f(x) - f(x_0) \geq s \cdot (x - x_0)$ and we are done. □

Corollary 5.1.7 says that the (affine) function $h : x \to f(x_0) + s \cdot (x - x_0)$ *supports* $f$ at $x_0$, i.e., $h(x_0) = f(x_0)$ and $h(x) \leq f(x)$ for every $x$. Note that $h$ can be seen as a linear approximation to $f$ at $x_0$, see Fig. 5.3. Thus, a convex function has a support at every point. There is a converse of this result that also holds: if a function $f : \mathbb{R} \to \mathbb{R}$ has a support at every point $x$, then $f$ is convex.

global minimum

Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. We are interesting in minimizing $f$ over $\mathbb{R}$. Recall that $x_0 \in \mathbb{R}$ is called a *global minimum* of this optimization problem if

$$f(x_0) \leq f(x) \quad \text{for all } x \in \mathbb{R}.$$

local minimum

A weaker requirement is that the inequality holds for all $x$ in some (sufficiently small) neighborhood of $x_0$. In that case we have a *local minimum*. In general, when $g : \mathbb{R} \to \mathbb{R}$ is any function (no convexity assumed), a global minimum is clearly also a local minimum, but the converse may not hold. However, for convex function this converse does hold! For an interpretation of the following corollary, see Fig. 5.4.

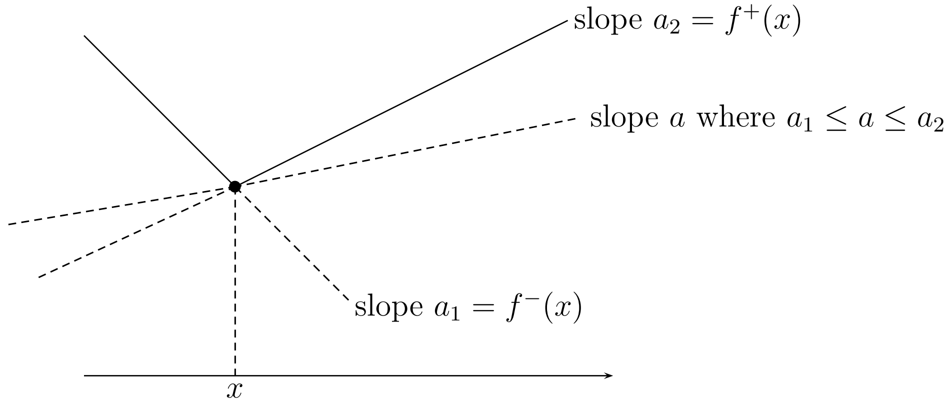


Figure 5.3: Subdifferential and linear approximation

**Corollary 5.1.8** (Global minimum)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function and let $x_0 \in \mathbb{R}$. Then the following three statements are equivalent.*

*(i)* $x_0$ *is a local minimum for* $f$.

*(ii)* $x_0$ *is a global minimum for* $f$.

*(iii)* $0 \in \partial f(x_0)$.

*Proof.* Assume that (iii) holds. Then, by Corollary 5.1.7 (with $s = 0$) we have that

$$f(x) \geq f(x_0) + s \cdot (x - x_0) = f(x_0)$$

for all $x \in \mathbb{R}$ which means that $x_0$ is a global minimum and (ii) holds. As mentioned above, the implication from (ii) to (i) is trivial. Finally, assume that (i) holds. Then there is a positive number $\epsilon$ such that

$$(*) \;\; f(x) \geq f(x_0) \quad \text{for all } x \in [x_0 - \epsilon, x_0 + \epsilon].$$

If $f'_+(x_0) < 0$, we could find a $z$ with $x_0 < z < x_0 + \epsilon$ such that $(f(z) - f(x_0))/(z - x_0) < 0$. But then $f(z) < f(x_0)$ contradicting $(*)$. Therefore $f'_+(x_0) \geq 0$. Similarly we prove that $f'_-(x_0) \leq 0$. Consequently $0 \in \partial f(x_0)$ and (iii) holds. ☐

Note that, when $f$ is differentiable, the optimality condition in Corollary 5.1.8 (statement (iii)) specializes into the statement that $f'(x_0) = 0$, i.e., that $x_0$ is a critical point.

There is a rich theory of subdifferentiability where many results from calculus are generalized very nicely. Here is an example (the proof is indicated in Exercise 5.23).

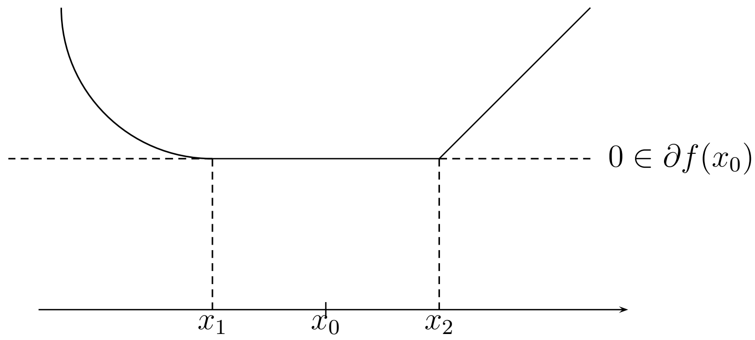


Figure 5.4: Minimum of a convex function

**Theorem 5.1.9** (Mean value theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then there exists a $c \in \langle a, b\rangle$ such that*

$$\frac{f(b) - f(a)}{b - a} \in \partial f(c).$$

We cannot go further into this area here, but refer the interested reader to a comprehensive treatment in [6].

Convex functions are central in several classical inequalities, like Jensen's inequality, Hölder's inequality, Minkowski's inequality and the arithmetic-geometric inequality etc. For a lot of material in this area we refer to a basic book on inequalities [9], see also [16]. As an illustration we prove Jensen's inequality and study an application.

**Theorem 5.1.10** (Jensen's inequality)**.** *Let $f : I \to \mathbb{R}$ be a convex function defined on an interval $I$. If $x_1, \ldots, x_r \in I$ and $\lambda_1, \ldots, \lambda_r \geq 0$ satisfy $\sum_{j=1}^{r} \lambda_j = 1$, then*

$$f(\sum_{j=1}^{r} \lambda_j x_j) \leq \sum_{j=1}^{r} \lambda_j f(x_j). \tag{5.2}$$

*Proof.* This proof is very similar to the proof of Proposition 2.1.1. We use induction on $r$ and note first that for $r = 2$ the result is simply the definition of convexity. Assume that (5.2) holds for any set of $r$ points and scalars, where $r$ is some fixed number satisfying $r \geq 2$. Let $x_1, \ldots, x_{r+1} \in I$ and $\lambda_1, \ldots, \lambda_{r+1} \geq 0$ satisfy $\sum_{j=1}^{r+1} \lambda_j = 1$. At least one of the $\lambda_j$'s must be smaller than 1 (as $r \geq 2$), say that $\lambda_{r+1} < 1$. We define $\lambda = 1 - \lambda_{r+1}$ so $\lambda > 0$. Consider the point $y = (1/\lambda) \sum_{j=1}^{r} \lambda_j x_j$. By our induction hypothesis, we have that $f(y) \leq \sum_{j=1}^{r} (\lambda_j/\lambda) f(x_j)$. Combining this inequality with the convexity of $f$ we obtain

$f(\sum_{j=1}^{r+1} \lambda_j x_j) = f(\lambda y + \lambda_{r+1} x_{r+1}) \leq \lambda f(y) + \lambda_{r+1} f(x_{r+1}) \leq \sum_{j=1}^{r+1} (\lambda_j/\lambda) f(x_j)$. and the induction proof is complete. □

**Example 5.1.2.** *(Arithmetic and geometric means)* In statistics one has different notions of the average or mean of a set of numbers. Let $x_1, \ldots, x_r$ be given numbers (data). The *arithmetic mean* is defined to be

arithmetic mean

$$(1/r)\sum_{j=1}^{r} x_j$$

and the *geometric mean* is defined as

geometric mean

$$(\prod_{j=1}^{r} x_j)^{1/r}.$$

The *arithmetic-geometric inequality* relates these two concepts and says that the geometric mean is never greater than the arithmetic mean, i.e.,

$$(\prod_{j=1}^{r} x_j)^{1/r} \leq (1/r)\sum_{j=1}^{r} x_j$$

This inequality may be proved quite easily using convexity. Recall that the function $f(x) = -\ln(x)$ is convex on $\mathbb{R}_+$ (see Exercise 5.3). Using Jensen's inequality (5.2) on $f$ with $\lambda_j = 1/r$ for $r = 1, \ldots, r$ we get $-\ln(\sum_{j=1}^{r}(1/r)x_j) \leq -\sum_{j=1}^{r}(1/r)\ln(x_j) = -\ln((\prod_{j=1}^{r} x_j)^{1/r})$. Now, $f$ is a strictly decreasing function so we get the desired inequality $(\prod_{j=1}^{r} x_j)^{1/r} \leq (1/r)\sum_{j=1}^{r} x_j$.

**Exercise 5.4.** *Consider Example* 5.1.2 *again. Use the same technique as in the proof of arithmetic-geometric inequality except that you consider general weights* $\lambda_1, \ldots, \lambda_r$ *(nonnegative with sum one). Which inequality do you obtain? It involves the so-called weighted arithmetic mean and the weighted geometric mean.*

## 5.2 Convex functions of several variables

The notion of convex function also makes sense for real-valued functions of several variables. This section treats such convex functions and we shall see that many results from the univariate case extends to the general case of $n$ variables.

Consider a real-valued function $f : C \to \mathbb{R}$ where $C \subseteq \mathbb{R}^n$ is a convex set. We say that $f$ is *convex* provided that the inequality

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y) \tag{5.3}$$

holds for every $x, y \in \mathbb{R}^n$ and every $0 \leq \lambda \leq 1$. Note that, due to the convexity of $C$, the point $(1-\lambda)x + \lambda y$ lies in $C$ so the inequality makes sense. This definition has a simple geometrical interpretation in terms of the graph of $f$. The *graph* of the function $f$ is the set $\{(x, f(x)) : x \in \mathbb{R}^n\}$ which is a subset of $\mathbb{R}^{n+1}$. Now, the geometrical interpretation of convexity of $f$ is: whenever you take two points on the graph of $f$, say $(x, f(x))$ and $(y, f(y))$, the graph of $f$ lies below the line segment between the two chosen points.

`convex function`

**Exercise 5.5.** *Repeat Exercise* 5.2*, but now for convex functions defined on some convex set in* $\mathbb{R}^n$*.*

**Exercise 5.6.** *Verify that every linear function from* $\mathbb{R}^n$ *to* $\mathbb{R}$ *is convex.*

We now give a result which may be useful for proving that a given function is convex. A function $h : \mathbb{R}^n \to \mathbb{R}$ is called *affine* if it holds for every $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ that $h((1-\lambda)x + \lambda y) = (1-\lambda)h(x) + \lambda h(y)$. Thus, $h$ preserves affine combinations. One can show that every affine function is the sum of a linear function and a constant, i.e., if $h$ is affine, then $h(x) = c^T x + \alpha$ for some $c \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

`affine function`

**Proposition 5.2.1** (Composition)**.** *Assume that* $f : \mathbb{R}^n \to \mathbb{R}$ *is convex and* $h : \mathbb{R}^m \to \mathbb{R}^n$ *is affine. Then the composition* $f \circ h$ *is convex (where* $(f \circ h)(x) := f(h(x))$*).*

**Exercise 5.7.** *Prove Proposition* 5.2.1*.*

**Exercise 5.8.** *Let* $f : C \to \mathbb{R}$ *and let* $w \in \mathbb{R}^n$*. Show that the function* $x \to f(x + w)$ *is convex.*

Jensen's inequality extends directly to the multivariate case. The proof is quite similar, so it is omitted here.

**Theorem 5.2.2** (Jensen's inequality, more generally)**.** *Let* $f : C \to \mathbb{R}$ *be a convex function defined on a convex set* $C \subseteq \mathbb{R}^n$*. If* $x_1, \ldots, x_r \in C$ *and* $\lambda_1, \ldots, \lambda_r \geq 0$ *satisfy* $\sum_{j=1}^r \lambda_j = 1$*, then*

$$f(\sum_{j=1}^{r} \lambda_j x_j) \leq \sum_{j=1}^{r} \lambda_j f(x_j). \tag{5.4}$$

There are some interesting, and useful, connections between convex functions and convex sets, and we consider one such basic relation. Let $f$ be a real-valued function defined on a convex set $C \subseteq \mathbb{R}^n$. We define the following set in $\mathbb{R}^{n+1}$ associated with $f$:

$$\text{epi}(f) = \{(x, y) \in \mathbb{R}^{n+1} : y \geq f(x)\}.$$

`epigraph`

The set epi$(f)$ is called the *epigraph* of $f$. Thus, the epigraph is the set of points in $\mathbb{R}^{n+1}$ lying on or above the graph of $f$, see Fig. 5.5. For instance, if $n = 1$,

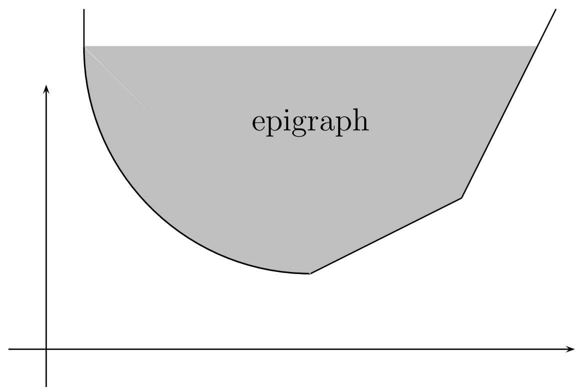


Figure 5.5: Convexity and epigraph

$f(x) = x^2$ and $C = \mathbb{R}$, then $\text{epi}(f) = \{(x, y) \in \mathbb{R}^2 : y \geq x^2\}$. Convexity of $f$ and convexity of $\text{epi}(f)$ turn out to be really closely connected.

**Theorem 5.2.3** (Epigraph)**.** *Let $f : C \to \mathbb{R}$ where $C \subseteq \mathbb{R}^n$ is a convex set. Then $f$ is a convex function if and only if* $\text{epi}(f)$ *is a convex set.*

**Exercise 5.9.** *Prove Theorem* 5.2.3 *(just apply the definitions).*

This theorem is useful because it means, for instance, that results for convex sets may be applied to $\text{epi}(f)$ in order to get results for a convex function $f$. Here is one such application which is very useful.

**Corollary 5.2.4** (Supremum of convex functions)**.** *Let $f_i$, for $i \in I$, be a nonempty family of convex functions defined on a convex set $C \subseteq \mathbb{R}^n$. Then the function $f$ given by*

$$f(x) = \sup_{i \in I} f_i(x) \quad \text{for } x \in C$$

*is convex.*

*Proof.* We see that $y \geq f(x)$ if and only if $y \geq f_i(x)$ for all $i \in I$. This implies that

$$\text{epi}(f) = \bigcap_{i \in I} \text{epi}(f_i).$$

Now $f_i$ is convex so each set $\text{epi}(f_i)$ is convex (Theorem 5.2.3). The intersection of (any family of) convex sets is a convex set (see Project 1.4), so $\text{epi}(f)$ is convex. This, again by Theorem 5.2.3, means that $f$ is convex. $\square$

So, for instance, the maximum of a finite number of convex functions is a convex function, see Fig. 5.6.

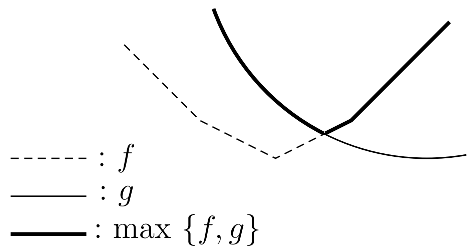


Figure 5.6: Maximum of convex functions

**Exercise 5.10.** *By the result above we have that if $f$ and $g$ are convex functions, then the function* $\max\{f, g\}$ *is also convex. Prove this result directly from the definition of a convex function.*

sum norm

**Example 5.2.1.** *(The sum norm)* The $l_1$-norm, or *sum norm*, of a vector $x \in \mathbb{R}^n$ is defined by

$$\|x\|_1 = \sum_{j=1}^{n} |x_j|.$$

This defines a function $f(x) = \|x\|_1$ from $\mathbb{R}^n$ to $\mathbb{R}$. This function is convex which can be seen as follows. First, we note that each of the functions $x \to x_j$ and $x \to -x_j$ are linear and therefore convex. Thus, the maximum of these functions, i.e., the function $x \to |x_j|$, is also convex, by Corollary 5.2.4. This holds for every $j$ and so the sum of these functions, namely $f$, is also convex.

support function

**Example 5.2.2.** *(The support function)* Let $P$ be a polytope in $\mathbb{R}^n$, say $P = \text{conv}(\{v_1, \ldots, v_t\})$. We are interested in LP problems given by

$$\psi_P(c) := \max\{c^T x : x \in P\}.$$

Thus, $\psi_P(c)$ denotes the optimal value of this LP problem, the function $\psi_P$ is called the *support function* (or *value function*) of $P$. We consider $P$ fixed, and want to examine the behavior of $\psi_P$. We claim that $\psi_P$ is a convex function. To see this, note first that each of the functions $f_j : c \to c^T v_j$ is linear and therefore convex. Thus, by Corollary 5.2.4, the maximum of these functions is also convex. So, what is this maximum? We have for each $c \in \mathbb{R}^n$ that

$$\max_j c^T v_j = \max\{c^T x : x \in P\} = \psi_P(c)$$

since the optimal value of every LP problem over $P$ is attained in one of the extreme points (which is a subset of $\{v_1, \ldots, v_t\}$). It follows that the support function $\psi_P$ is convex.

Similarly, for the problem of maximizing a linear function over a compact convex set $C$, the support function is convex. This is shown similarly, but we take the supremum of an infinite family of linear functions, namely one for each extreme point of $C$. Note here that we use Minkowski's theorem (Corollary 4.3.4) saying that a compact convex set is the convex hull of its extreme points.

We remark that the support function may also be defined for the problem of optimizing over any set (even unbounded), but then we must replace "max" by "sup" in the definition and allow infinite function values. This can be done, but one needs to take proper care of the arithmetic involving infinite values. We do not go into these matters here, but refer e.g. to [6].

**Exercise 5.11.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $\alpha \in \mathbb{R}$. Show that the set $\{x \in \mathbb{R}^n : f(x) \leq \alpha\}$ is a convex set. Each such set is called a sublevel set.*

**Example 5.2.3.** *(Vector norms)* Vector norms are used to measure the "length" or "size" of a vector. Thus, for instance, one can use norms to measure how close two vectors are, and this is important in many connections. We shall relate norms and convexity, and start by defining a norm in general terms. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called a *norm* on $\mathbb{R}^n$ if it satisfies the following properties for every $x, y \in \mathbb{R}^n$ and $\lambda \geq 0$: `vector norm`

$$\begin{array}{lll} \text{(i)} & f(x) \geq 0 & \text{(nonnegative);} \\ \text{(ii)} & f(x) = 0 \text{ if and only if } x = O & \text{(positive);} \\ \text{(iii)} & f(\lambda x) = |\lambda| f(x) & \text{(homogeneous);} \\ \text{(iv)} & f(x+y) \leq f(x) + f(y) & \text{(triangle inequality).} \end{array} \tag{5.5}$$

`Euclidean norm`
`max norm`

Often, the norm function $f$ is written $\|\cdot\|$, i.e., $f(x) = \|x\|$ is the norm of the vector $x$. The $l_1$-norm as defined in Example 5.2.1 satisfies (5.5) so it is a norm on $\mathbb{R}^n$. Another example is, of course, the *Euclidean norm* (or $l_2$-norm) given by $\|x\|_2 = (\sum_{j=1}^n |x_j|^2)^{1/2}$. A third example is the $l_\infty$-norm, or *max norm*, defined by $\|x\|_\infty = \max_{j \leq n} |x_j|$. A large class of norms may be defined as follows. For any real number $p \geq 1$ the $l_p$-norm is given by $\|x\|_p = (\sum_{j=1}^n |x_j|^p)^{1/p}$. We see that for $p = 1$ and $p = 2$ we obtain the sum norm and the Euclidean norm, respectively. Moreover, it can be shown that

$$\lim_{p \to \infty} \|x\|_p = \|x\|_\infty$$

for every $x \in \mathbb{R}^n$. Thus, the max norm is the (pointwise) limit of the $l_p$-norms.

How can we *prove* that the $l_p$-norms are really norms as defined in (5.5). It is quite easy to see that the properties (i)–(iii) all hold, but the triangle inequality is not straightforward. In fact, the triangle inequality for $l_p$-norms is known as *Minkowski's inequality* and it says that

Minkowski's inequality

$$\|x + y\|_p \le \|x\|_p + \|y\|_p \quad \text{for all } x, y \in \mathbb{R}^n. \tag{5.6}$$

We shall prove Minkowski's inequality below in connection with some other convexity results. First, however, we observe that every norm is a convex function. To check this, let $x, y \in \mathbb{R}^n$ and $0 \le \lambda \le 1$. From the norm properties (5.5) we obtain

$$\|(1-\lambda)x + \lambda y\| \le \|(1-\lambda)x\| + \|\lambda y\| = (1-\lambda)\|x\| + \lambda\|y\|$$

which shows the convexity of $\|\cdot\|$. The reader who wants to read more about vector (and matrix) norms, may consult [7].

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *positively homogeneous* if $f(\lambda x) = \lambda f(x)$ holds for every $x \in \mathbb{R}^n$ and $\lambda \ge 0$.

**Exercise 5.12.** *Verify that the function $x \to \|x\|_p$ is positively homogeneous.*

**Exercise 5.13.** *Consider the support function of an optimization problem with a linear objective function, i.e., let $f(c) := \max\{c^T x : x \in S\}$ where $S \subseteq \mathbb{R}^n$ is a given nonempty set. Show that $f$ is positively homogeneous. Therefore (due to Example 5.2.2), the support function is convex and positively homogeneous when $S$ is a compact convex set.*

positively homogeneous

A positively homogeneous function need not be convex, but the next theorem gives the additional property needed to ensure convexity.

**Theorem 5.2.5** (Positively homogeneous convex functions)**.** *Let $f$ be a positive homogeneous function defined on a convex cone $C \subseteq \mathbb{R}^n$. Then $f$ is convex if and only if $f(x+y) \le f(x) + f(y)$ for all $x, y \in \mathbb{R}^n$ (this property is called subadditivity).*

*Proof.* First, assume that $f$ is convex. Then $(1/2)f(x+y) = f((1/2)x+(1/2)y) \le (1/2)f(x) + (1/2)f(y)$, so $f(x+y) \le f(x) + f(y)$. Next, assume that $f(x+y) \le f(x) + f(y)$ holds for all $x, y \in \mathbb{R}^n$. Let $x, y \in C$ and $\lambda \in [0,1]$. Then $f((1-\lambda)x+\lambda y) \le f((1-\lambda)x) + f(\lambda y) = (1-\lambda)f(x) + \lambda f(y)$ so $f$ is convex. $\square$

The next theorem gives a convexity criterion which is of interest in connection with norms.

**Corollary 5.2.6** (A convexity criterion)**.** *Let $f$ be a nonnegative and positive homogeneous function defined on a convex cone $C \subseteq \mathbb{R}^n$. Assume that the set $U := \{x \in C : f(x) \le 1\}$ is convex. Then $f$ is a convex function on $C$.*

*Proof.* Due to Theorem 5.2.5 it suffices to show subadditivity ($f(x+y) \leq f(x)+f(y)$). Let $x, y \in C$ and let $a$ and $b$ be positive numbers satisfying $a > f(x)$ and $b > f(y)$. Then, as $f$ is positive homogeneous $f((1/a)x) = f(x)/a \leq 1$ and $f(y)/b \leq 1$. Thus, $(1/a)x, (1/b)y \in U$ and, as $U$ is assumed convex, any convex combination of these two points also lies in $U$. Therefore

$$\frac{1}{a+b}f(x+y) = f(\frac{1}{a+b}(x+y)) = f(\frac{a}{a+b} \cdot \frac{1}{a}x + \frac{b}{a+b} \cdot \frac{1}{b}y) \leq 1$$

so $f(x+y) \leq a+b$. This holds for every $a > f(x)$ and $b > f(y)$. We may here choose $a$ arbitrarily close to $f(x)$ (and $b$ arbitrarily close to $f(y)$) as $f(x), f(y) \geq 0$, so it follows that $f(x+y) \leq f(x)+f(y)$. Therefore $f$ is convex on $C$. □

*Proof of Minkowski's inequality.* Let $p \geq 1$ and let $C = \mathbb{R}^n_+$ (which is a convex cone). Consider the function $f(x) = (\sum_{j=1}^n x_j^p)^{1/p}$ for $x \geq O$. First, we see that the function $f^p$ ($f$ to the power of $p$) is convex: it is the sum of the convex functions $x \to x_j^p$ defined on $\mathbb{R}_+$. Then, by Exercise 5.11, the set

$$\{x \in C : f^p(x) \leq 1\} = \{x \in C : f(x) \leq 1\}$$

is convex. It now follows from Corollary 5.2.6 that $f$ is convex (on $C$). Finally, since $f$ is convex, we may conclude from Theorem 5.2.5, that $f(x+y) \leq f(x)+f(y)$ for $x, y \geq O$. This implies Minkowski's inequality (by replacing $x_j$ by $|x_j|$ which is nonnegative). □

## 5.3 Continuity and differentiability

In this section we study continuity and differentiability properties of convex functions.

We first need to recall the concept of directional derivative. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and let $x_0 \in \mathbb{R}^n$ and $z \in \mathbb{R}^n \setminus \{O\}$. If the limit

$$\lim_{t \to 0}(f(x_0+tz) - f(x_0))/t$$

exists, it is called the *directional derivative* of $f$ at $x_0$ in the direction $z$, and this limit is denoted by $f'(x_0; z)$. This number tells us the local growth rate of the function when we move from the point $x_0$ in the direction $z$. When we let $z = e_j$ (the $j$'th unit vector) we get the $j$'th partial derivative: $f'(x_0; e_j) = \lim_{t \to 0}(f(x_0+te_j) - f(x_0))/t = \partial f(x)/\partial x_j$. Below we will also consider one-sided directional derivatives.

`directional derivative`

**Exercise 5.14.** *Let $f(x) = x^T x = \|x\|^2$ for $x \in \mathbb{R}^n$. Show that the directional derivative $f'(x_0; z)$ exists for all $x$ and nonzero $z$ and that $f'(x_0; z) = 2z^T x$.*

If we restrict a given (convex) function $f : \mathbb{R}^n \to \mathbb{R}$ to some line $L$ in $\mathbb{R}^n$ we get a new function of a single variable (this variable describes the position on the line). This construction will be useful for understanding the behavior of convex functions of several variables. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and consider a line $L = \{x_0 + \lambda z : \lambda \in \mathbb{R}\}$ where $x_0$ is a point on the line and $z$ is the direction vector of $L$. Define the function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) = f(x_0 + tz) \quad \text{for } t \in \mathbb{R}.$$

We now prove that $g$ is a convex function (of a single variable). Let $t_1, t_2 \in \mathbb{R}$ and let $0 \le \lambda \le 1$. We have $f((1-\lambda)t_1 + \lambda t_2) = f(x_0 + ((1-\lambda)t_1 + \lambda t_2)z) = f((1-\lambda)(x_0+t_1 z)+\lambda(x_0+t_2 z)) \le (1-\lambda)f(x_0+t_1 z)+\lambda f(x_0+t_2 z) = (1-\lambda)g(t_1)+\lambda g(t_2)$ and therefore $g$ is convex.

Thus, the restriction $g$ of a convex function $f$ to any line is another convex function. A first consequence of this result is that a convex function $f : \mathbb{R}^n \to \mathbb{R}$ has one-sided directional derivatives. In fact, since $g$ is convex it has a right-sided derivative at 0 (due to Theorem 5.1.3) so

$$g'_+(0) = \lim_{t \to 0^+} (g(t) - g(0))/t = \lim_{t \to 0^+} (f(x_0 + tz) - f(x_0))/t = f'_+(x_0; z)$$

`one-sided directional derivative`

This shows that $f'_+(x_0; z)$, the *right-sided directional derivative* of $f$ at $x_0$ in the direction $z$, exists and that it equals $g'_+(0)$. Clearly, a similar statement holds for the left-sided directional derivative of $f$. This is true for any point $x_0$ and nonzero direction vector $z$. Note that a similar differentiability result holds for a convex function defined on an open convex set $C \subseteq \mathbb{R}^n$.

Recall that a convex function of a single variable is continuous on the interior of its domain. A similar result holds for convex functions of several variables.

**Theorem 5.3.1** (Continuity)**.** *Let $f : C \to \mathbb{R}$ be a convex function defined on an open convex set $C \subseteq \mathbb{R}^n$. Then $f$ is continuous on $C$.*

*Proof.* Let $x_0 \in C$. Then we can find a full-dimensional polytope $P$ having vertices $y_1, \ldots, y_t$ such that $x_0 \in \text{int}(P) \subset P \subset C$ (why?). Moreover, for a suitably small $r > 0$ we have that $B(x_0, r) \subseteq P$. (Recall that $B(x_0, r)$ is the closed ball consisting of all points $x$ satisfying $\|x - x_0\| \le r$.)

We first prove that $f$ is bounded above on $B(x_0, r)$. Let $x \in B(x_0, r)$ and define $M = \max_{j \le t} f(y_j)$. Since $x \in B(x_0, r) \subseteq P$ there are nonnegative numbers $\lambda_1, \ldots, \lambda_t$ with $\sum_j \lambda_j = 1$ such that $x = \sum_j \lambda_j y_j$ and therefore, due to the convexity of $f$, $f(x) \le \sum_j \lambda_j f(y_j) \le \sum_j \lambda_j M = M$.

Consider a point $x \in B(x_0, r)$ with $x \neq x_0$. Define the function $g : [-r, r] \to \mathbb{R}$ by $g(t) = f(x_0 + tz)$ where $z = (x - x_0)/\|x - x_0\|$. Then (see above) $g$ is convex and it is bounded above by $g(t) \leq M$ for $-r \leq t \leq r$. According to Proposition 5.1.2 $g$ has an increasing slope function so we obtain

$$\begin{aligned} -(M - g(0))/r \leq \quad & (g(-r) - g(0))/(-r) \leq (g(\|x - x_0\|) - g(0))/\|x - x_0\| \leq \\ & (g(r) - g(0))/r \leq (M - g(0))/r. \end{aligned}$$

Thus, $|g(\|x - x_0\|) - g(0)| \leq ((M - g(0))/r) \cdot \|x - x_0\|$ which becomes, using the definition of $g$,

$$|f(x) - f(x_0)| \leq ((M - f(x_0))/r) \cdot \|x - x_0\|.$$

This proves that $f$ is continuous at $x_0$ and the proof is complete. $\square$

How can we decide if a given function of several variables is convex? We recall that in the one-variable case, when $f$ is differentiable, the answer is to check the sign of the second derivative. The function is convex if and only if $f''(x) \geq 0$ for all $x$. We shall now prove an extension of this result and obtain a criterion in terms of the second-order partial derivatives. First, we establish a useful lemma.

**Lemma 5.3.2** (One-variable characterization)**.** *Let $f : C \to \mathbb{R}$ be a real-valued function defined on an open convex set $C \subseteq \mathbb{R}^n$. For each $x \in C$ and $z \in \mathbb{R}^n$ we define the interval $I = \{t \in \mathbb{R} : x + tz \in C\}$ and the function $g : I \to \mathbb{R}$ given by $g(t) = f(x + tz)$.*

*Then $f$ is convex if and only if each such function $g$ (for all $x \in C$ and $z \in \mathbb{R}^n$) is convex.*

*Proof.* In the beginning of this section we proved that the convexity of $f$ implies that each $g$ is convex. To prove the converse let $x, y \in C$ and $0 \leq \lambda \leq 1$. Define $z = x - y$. From the convexity of $g$ we obtain $f((1 - \lambda)x + \lambda y) = f(x + \lambda z) = g((1-\lambda) \cdot 0 + \lambda \cdot 1) \leq (1-\lambda)g(0) + \lambda g(1) = (1-\lambda)f(x) + \lambda f(y)$ so $f$ is convex. $\square$

The previous lemma says that a function of several variables is convex if and only if it is "convex in every direction". This means that certain results known for convex functions of one variable have extensions to the multivariate case. We now consider the mentioned convexity characterization in terms of second-order partial derivatives.

Let $f$ be a real-valued function defined on an open convex set $C \subseteq \mathbb{R}^n$. We assume that $f$ has continuous second-order partial derivatives at every point in $C$. Note that this implies that $\partial^2 f/\partial x_i \partial x_j = \partial^2 f/\partial x_j \partial x_i$ holds everywhere. Let $H_f(x) \in \mathbb{R}^{n \times n}$ be the matrix whose $(i, j)$th entry is $\partial^2 f(x)/\partial x_i \partial x_j$. The square matrix $H_f(x)$ is called the *Hessian matrix* of $f$ at $x$. This matrix is symmetric as just noted. We now recall a concept from linear algebra: a symmetric matrix $A \in \mathbb{R}^{n\times n}$ is *positive semidefinite* if $x^T Ax = \sum_{i,j} a_{i,j}x_i x_j \geq 0$ for each $x \in \mathbb{R}^n$. A useful fact is that $a$ is positive semidefinite if and only if all the eigenvalues of $a$ are (real and) nonnegative. Hessian matrix

**Theorem 5.3.3** (Characterization via the Hessian)**.** *Let $f$ be a real-valued function defined on an open convex set $C \subseteq \mathbb{R}^n$ and assume that $f$ has continuous second-order partial derivatives on $C$.*

*Then $f$ is convex if and only if the Hessian matrix $H_f(x)$ is positive semidefinite for each $x \in C$.*

*Proof.* Due to Lemma 5.3.2 $f$ is convex if and only if each function $g(t) = f(x+tz)$ is convex. Since $f$ has second-order partial derivatives on $C$, the function $g$ is twice differentiable on the interior of $I = \{t : x + tz \in C\}$. We use the chain rule for calculating the first- and second-order derivatives of $g$, and they are

$$\begin{aligned} g'(t) &= \textstyle\sum_{j=1}^n (\partial f/\partial x_j) z_j \\ g''(t) &= \textstyle\sum_{i=1}^n \sum_{j=1}^n (\partial^2 f/\partial x_i \partial x_j) z_i z_j \end{aligned}$$

for each $t \in I$ and where the partial derivatives are evaluated at $x + tz$. From Theorem 5.1.4 we have that $g$ is convex if and only if $g'' \geq 0$, and the desired result follows. $\square$

**Example 5.3.1.** *(A quadratic function)* Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix which is positive semidefinite and consider the function $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x) = x^T Ax = \sum_{i,j} a_{i,j} x_i x_j.$$

Then it is easy to check that $H_f(x) = A$ for each $x \in \mathbb{R}^n$. Therefore, $f$ is a convex function.

**Exercise 5.15.** *A quadratic function is a function of the form*

$$f(x) = x^T Ax + c^T x + \alpha$$

*for some (symmetric) matrix $A \in \mathbb{R}^{n\times n}$, a vector $c \in \mathbb{R}^n$ and a scalar $\alpha \in \mathbb{R}$. Discuss whether $f$ is convex.*

Sometimes we can determine that a symmetric matrix $a$ is positive semidefinite by a very simple test. A (real) symmetric $n\times n$ matrix $a$ is called *diagonally dominant* if $|a_{i,i}| \geq \sum_{j\neq i} |a_{i,j}|$ for $i = 1,\ldots,n$. If all these inequalities are strict, $a$ is *strictly diagonally dominant*. These matrices arise in many applications, e.g. splines and differential equations. It can be shown that every symmetric diagonally dominant

diagonally dominant

matrix is positive semidefinite. For a simple proof of this fact using convexity, see [3]. Thus, we get a useful test for convexity of a function: check if the Hessian matrix $H_f(x)$ is diagonally dominant for each $x$. Note, however, that this criterion is only sufficient for $f$ to be convex (so even if the Hessian is not diagonally dominant the function $f$ may still be convex).

We now turn to the notion of differentiability of a function. We recall that a function $f$ defined on an open set in $\mathbb{R}^n$ is said to be *differentiable* at a point $x_0$ in its domain if there is a vector $d$ (which may depend on $x_0$) such that differ-entiable

$$\lim_{h \to O} (f(x_0 + h) - f(x_0) - d^T h)/\|h\| = 0.$$

More precisely this means the following: for every $\epsilon > 0$ there is a $\delta > 0$ such that if $h$ satisfies $0 < \|h\| < \delta$ then it also holds that $|f(x_0+h) - f(x_0) - d^T h|/\|h\| < \epsilon$. If $f$ is differentiable at $x_0$, the vector $d$ is unique and it is called the *gradient* of $f$ at $x_0$. gradient

Assume that $f$ is differentiable at $x_0$ and the gradient at $x_0$ is $d$. Then we have for every nonzero vector $z$ that

$$\lim_{t \to 0} |f(x_0+tz) - f(x_0) - d^T tz|/\|tz\| = (1/\|z\|) \lim_{t \to 0} |(f(x_0+tz) - f(x_0))/t - d^T z| = 0.$$

Here the existence of the first limit is due to the differentiability assumption. This proves that $f$ has a directional derivative at $x_0$ in direction $z$ and that

$$f'(x_0; z) = d^T z.$$

In particular, we see that the function $z \to f'(x_0; z)$ is linear. Note that usually the gradient $d$ of $f$ at $x_0$ is denoted by $\nabla f(x_0)$ and one can verify that

$$\nabla f(x_0)^T = [\partial f(x_0)/\partial x_1, \ldots, \partial f(x_0)/\partial x_n] .$$

We have just shown that differentiability at $x_0$ is a stronger notion than the existence of directional derivatives at that point. In fact, there exist nondifferentiable functions that have directional derivatives in every direction at some point. We now show that this does not happen for convex functions: even the existence of all partial derivatives at a point turns out to imply that the function is differentiable at that point.

**Theorem 5.3.4** (Partial derivatives and differentiability)**.** *Let $f$ be a real-valued convex function defined on an open convex set $C \subseteq \mathbb{R}^n$. Assume that all the partial derivatives $\partial f(x)/\partial x_1, \ldots, \partial f(x)/\partial x_n$ exist at a point $x \in C$. Then $f$ is differentiable at $x$.*

*Proof.* Since the partial derivatives exist at the point $x \in C$ we may define the function $g : \mathbb{R}^n \to \mathbb{R}$ by

$$g(h) = f(x+h) - f(x) - \nabla f(x)^T h.$$

where $\nabla f(x_0)^T = [\partial f(x_0)/\partial x_1, \ldots, \partial f(x_0)/\partial x_n]$. We shall show that $f$ is differentiable at $x$, i.e., that $\lim_{h\to O} g(h)/\|h\| = 0$. Note that $g(O) = 0$. Moreover, $g$ has partial derivates and they are given by

$$\partial g(h)/\partial h_j = \partial f(x+h)/\partial x_j - \partial f(x)/\partial x_j$$

so, in particular, $\partial g(O)/\partial h_j = 0$.

We observe that $g$ is convex (see Proposition 5.2.1). Let $x \in \mathbb{R}^n$. As $h = \sum_{j=1}^n h_i e_i$ we obtain

$$\begin{array}{ll}(*)\ \ g(h) = & g(\sum_{i=1}^n h_i e_i) = g((1/n)\sum_{i=1}^n n h_i e_i) \leq \\ & \sum_{i=1}^n (1/n) g(n h_i e_i) = \sum_{i=1}^n h_i \cdot (g(n h_i e_i)/(n h_i)) \leq \\ & \|h\| \sum_{i=1}^n |(g(n h_i e_i)/(n h_i))|.\end{array}$$

Here, in the second last equality each summand with $h_i = 0$ must be replaced by zero, and the equality holds as $g(O) = 0$. To obtain the last equality we use the Cauchy-Schwarz inequality and the triangle inequality as follows (for any vector $w$): $\sum_{i=1}^n h_i w_i = h^T w \leq \|h\|\|w\| \leq \|h\|\|\sum_j w_i e_i\| \leq \|h\| \sum_j |w_i|\|e_i\| = \|h\| \sum_j |w_i|$. An inequality similar to $(*)$ holds when we replace $h$ by $-h$.

Furthermore, it follows from the convexity of $g$ that $g(h) + g(-h) \geq g(O) = 0$ so $-g(-h) \leq g(h)$. Therefore we have shown the following inequalities

$$-\sum_{i=1}^n |(g(-n h_i e_i)/(n h_i))| \leq -g(-h)/\|h\| \leq g(h)/\|h\| \leq \sum_{i=1}^n |(g(n h_i e_i)/(n h_i))|.$$

Here, as $h \to O$, both the left-most and the right-most term tend to $\partial g(O)/\partial h_j$ which is zero (see above). Thus, $\lim_{h\to O} g(h)/\|h\| = 0$, as desired. □

A convex function may not be differentiable everywhere, but it is differentiable "almost everywhere". More precisely, for a convex function defined on an open convex set in $\mathbb{R}^n$ the set of points for which $f$ is not differentiable has Lebesgue measure zero. We do not go into further details on this here, but refer to e.g. [6] for a proof and a discussion.

Another characterization of convex functions that involves the gradient may now be presented.

**Theorem 5.3.5** (Characterization via gradients)**.** *Let $f : C \to \mathbb{R}$ be a differentiable function defined on an open convex set $C \subseteq \mathbb{R}^n$. Then the following conditions are equivalent:*

*(i)* $f$ *is convex.*

*(ii)* $f(x) \geq f(x_0) + \nabla f(x_0)^T(x - x_0)$ for all $x, x_0 \in C$.

*(iii)* $(\nabla f(x) - \nabla f(x_0))^T(x - x_0) \geq 0$ for all $x, x_0 \in C$.

*Proof.* Assume that (i) holds. Let $x, x_0 \in C$ and $0 \leq \lambda \leq 1$. We have that

$$f(x_0 + \lambda(x - x_0)) = f((1-\lambda)x_0 + \lambda x) \leq (1-\lambda)f(x_0) + \lambda f(x)$$

This implies that

$$(f(x_0+\lambda(x-x_0)) - f(x_0))/\lambda - \nabla f(x_0)^T(x-x_0) \leq f(x) - f(x_0) - \nabla f(x_0)^T(x-x_0).$$

If we let $\lambda \to 0^+$, then the left side of this inequality tends to 0, and therefore $f(x) - f(x_0) - \nabla f(x_0)^T(x - x_0) \geq 0$, i.e., (ii) holds.

Assume next that (ii) holds. Then, for $x, x_0 \in C$, we have that $f(x) \geq f(x_0) + \nabla f(x_0)^T(x - x_0)$ and (by symmetry) $f(x_0) \geq f(x) + \nabla f(x)^T(x_0 - x)$. Adding these two inequalities gives (iii).

Finally, assume that (iii) holds. Let $x, x_0 \in C$ and consider the function $g(t) = f(x_0 + t(x - x_0))$ for $0 \leq t \leq 1$. Let $0 \leq t_1 \leq t_2 \leq 1$. Then, using the chain rule we get

$$g'(t_2) - g'(t_1) = \nabla f(x_0 + t_1(x - x_0))^T(x - x_0) - \nabla f(x_0 + t_2(x - x_0))^T(x - x_0) \geq 0$$

due to (iii). This shows that $g'$ is increasing on $[0, 1]$ and it follows from Lemma 5.3.2 that $f$ is convex, so (i) holds. □

This theorem is very important. Property (ii) says that the first-order Taylor approximation of $f$ at $x_0$ (which is the right-hand side of the inequality in (ii)) always underestimates $f$. This result has interesting consequences for optimization, see section 6.

Finally, we mention a result related to Corollary 5.1.7. Consider a convex function $f$ and an affine function $h$, both defined on a convex set $C \subseteq \mathbb{R}^n$. We say that $h : \mathbb{R}^n \to \mathbb{R}$ *supports* $f$ at $x_0$ if $h(x) \leq f(x)$ for every $x$ and $h(x_0) = f(x_0)$. Thus, the mentioned Taylor approximation $h$ above supports $f$ at $x_0$, see Fig. 5.7. Note, however, that the concept of supporting function also makes sense for nondifferentiable convex functions. In fact, the following general result holds, and it can be proved using Corollary 3.2.4.

`supporting function`

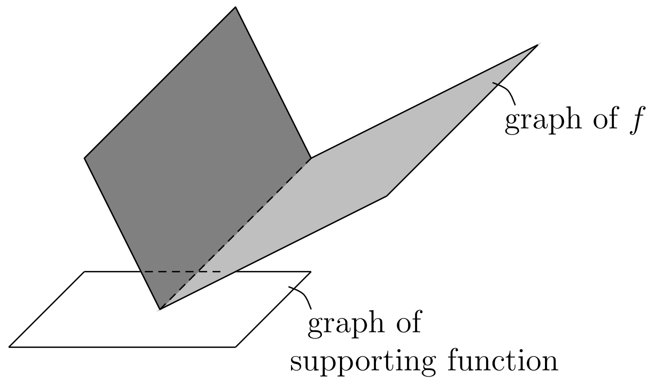


Figure 5.7: Supporting function

**Theorem 5.3.6.** *Let $f : C \to \mathbb{R}$ be a convex function defined on a convex set $C \subseteq \mathbb{R}^n$. Then $f$ has a supporting (affine) function at every point. Moreover, $f$ is the pointwise supremum of all its (affine) supporting functions.*

We remark that the last statement of this theorem represents a converse to the fact known from Corollary 5.2.4, i.e., that every supremum of convex (and therefore affine) functions is convex.

## 5.4 Exercises

**Exercise 5.16.** *Assume that $f$ and $g$ are convex functions defined on an interval $I$. Determine which of the functions following functions that are convex or concave: (i) $\lambda f$ where $\lambda \in \mathbb{R}$, (ii) $\min\{f, g\}$, (iii) $|f|$.*

**Exercise 5.17.** *Let $f, g : I \to \mathbb{R}$ where $I$ is an interval. Assume that $f$ and $f + g$ both are convex. Does this imply that $g$ is convex? Or concave? What if $f + g$ is convex and $f$ concave?*

**Exercise 5.18.** *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Show that*

$$\max\{f(x) : x \in [a, b]\} = \max\{f(a), f(b)\}$$

*i.e., a convex function defined on closed real interval attains its maximum in one of the endpoints.*

**Exercise 5.19.** *Let $f : I \to \mathbb{R}$ be a convex function defined on a bounded interval $I$. Prove that $f$ must be bounded below (i.e., there is a number $L$ such that $f(x) \geq L$ for all $x \in I$). Is $f$ also bounded above?*

**Exercise 5.20.** *Let* $f, g : \mathbb{R} \to \mathbb{R}$ *be convex functions and assume that* $f$ *is increasing. Prove that the composition* $f \circ h$ *is convex.*

**Exercise 5.21.** *Find the optimal solutions of the problem* $\min\{f(x) : a \leq x \leq b\}$ *where* $a < b$ *and* $f : \mathbb{R} \to \mathbb{R}$ *is a differentiable convex function.*

**Exercise 5.22.** *Let* $f : \langle 0, \infty\rangle \to \mathbb{R}$ *and define the function* $g : \langle 0, \infty\rangle \to \mathbb{R}$ *by* $g(x) = xf(1/x)$. *Prove that* $f$ *is convex if and only if* $g$ *is convex. Hint: Prove that*

$$\frac{g(x) - g(x_0)}{x - x_0} = f(1/x_0) - \frac{1}{x_0} \cdot \frac{f(1/x) - f(1/x_0)}{1/x - 1/x_0}$$

*and use Proposition* 5.1.2. *Why is the function* $x \to xe^{1/x}$ *convex?*

**Exercise 5.23.** *Prove Theorem* 5.1.9 *as follows. Consider the function*

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

*Explain why* $g$ *is convex and that it has a minimum point at some* $c \in \langle a, b\rangle$ *(note that* $g(a) = g(b) = 0$ *and* $g$ *is not constant). Then verify that*

$$\partial g(c) = \partial f(c) - \frac{f(b) - f(a)}{b - a}$$

*and use Corollary* 5.1.8.

**Exercise 5.24.** *Let* $f : \mathbb{R} \to \mathbb{R}$ *be an increasing convex function and let* $g : C \to \mathbb{R}$ *be a convex function defined on a convex set* $C$ *in* $\mathbb{R}^n$. *Prove that the composition* $f \circ g$ *(defined on* $C$*) is convex.*

**Exercise 5.25.** *Prove that the function given by* $h(x) = e^{x^T ax}$ *is convex when* $a$ *is positive definite.*

**Exercise 5.26.** *Let* $f : C \to \mathbb{R}$ *be a convex function defined on a compact convex set* $C \subseteq \mathbb{R}^n$. *Show that* $f$ *attains its maximum in an extreme point. Hint: use Minkowski's theorem (Corollary* 4.3.4*).*

**Exercise 5.27.** *Let* $C \subseteq \mathbb{R}^n$ *be a convex set and consider the distance function* $d_C$ *defined in (*3.1*), i.e.,* $d_C(x) = \inf\{\|x - c\| : c \in C\}$. *Show that* $d_C$ *is a convex function.*

**Exercise 5.28.** *Prove Corollary* 6.1.1 *using Theorem* 5.3.5.

**Exercise 5.29.** *Compare the notion of support for a convex function to the notion of supporting hyperplane of a convex set (see section* 3.2*). Have in mind that* $f$ *is convex if and only if* $\mathrm{epi}(f)$ *is a convex set. Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be convex and consider a supporting hyperplane of* $\mathrm{epi}(f)$. *Interprete the hyperplane in terms of functions, and derive a result saying that every convex function has a support at every point.*

**SUMMARY OF NEW CONCEPTS AND RESULTS:**

- convex function
- subdifferential, subderivative
- continuity and subdifferentiability of convex functions
- criteria for convexity
- Jensen's inequality and the arithmetic-geometric inequality
- epigraph
- support function
- vector norms
- Minkowski's inequality
- directional derivative, differentiability
- gradient, Hessian matrix
- convexity criteria

# Chapter 6

# Nonlinear and convex optimization

Problems of minimizing a function in several variables arise in many areas of mathematics and its applications. Several books and papers discuss this important class of problems. We shall here give a very brief presentation of this area and the focus is on convex optimization, i.e. to minimize a convex function over a convex set. Our presentation is based on the book [1] which is an excellent reference for both theory and algorithms. This book also dicusses nonconvex optimization (where many of the same ideas are used, but where it may be hard to find a globally optimal solution). Another book that can be recommended is [6] which treats this subject, both from a theoretical and a computational perspective. Moreover, the recent book [4] treats convexity and optimization in detail with a focus on theory.

Secions 6.1 and 6.2 concentrate on convex optimization while the remaining sections also discuss nonlinear optimization more generally.

## 6.1 Local and global minimum in convex optimization

Recall the notions of local and global minimum defined in Section 5.1. Exactly the same definitions are valid for the problem of minimizing a real-valued function of several variables (defined on $\mathbb{R}^n$ or some proper subset). The following result is a direct consequence of Theorem 5.3.5.

**Corollary 6.1.1** (Global minimum). *Let $f : C \to \mathbb{R}$ be a differentiable convex function defined on an open convex set $C \subseteq \mathbb{R}^n$. Let $x^* \in C$. Then the following three statements are equivalent.*

*(i)* $x^*$ *is a local minimum for* $f$*.*

*(ii)* $x^*$ *is a global minimum for* $f$*.*

*(iii)* $\nabla f(x^*) = O$ *(i.e., all partial derivatives at* $x^*$ *are zero).*

We leave the proof as an exercise. Note that in this corollary the domain of $f$ is an *open* convex set. This means that optimal solutions, if they exist, cannot lie on the boundary of $C$, simply because each point on the boundary lies outside $C$. Note also that when $C$ is open the problem $\min\{f(x) : x \in C\}$ may not have any optimal solution at all, even though $f$ is bounded on $C$. A simple example is when $f(x) = x$ and $C = \{x \in \mathbb{R} : 1 < x < 3\}$. If, however, $C = \mathbb{R}^n$, we have an unconstrained problem and such complications do not occur. Below we treat convex optimization problems where $C$ is a closed convex set.

Due to Corollary 6.1.1, say when $C = \mathbb{R}^n$, numerical algorithms for minimizing a differentiable convex function $f$ search for a stationary point $x^*$, i.e., a point where $\nabla f(x^*) = O$. For instance, consider the quadratic function

$$f(x) = (1/2)\, x^T Ax - b^T x + \alpha$$

defined for $x \in \mathbb{R}^n$. When $A$ is symmetric and positive semidefinite this function is convex (see Example 5.3.1 and Exercise 5.15). We calculate that

$$\nabla f(x) = Ax - b$$

so $x$ is a stationary point if and only if $Ax = b$. Thus, the problem of finding a local (and therefore global) minimum of a convex quadratic function boils down to solving a linear system of equations. For other convex functions, the system $\nabla f(x) = O$ may be a nonlinear system of equations. Thus, many numerical algorithms for solving this problem are based on Newton's method for nonlinear systems of equations.

Our next goal is to extend the result of Corollary 6.1.1 to the more typical situation where $C$ is a closed set. We consider the problem of minimizing a convex function $f$ over a (nonempty) closed convex set $C$ in $\mathbb{R}^n$, i.e.,

$$\text{minimize } f(x) \text{ subject to } x \in C. \tag{6.1}$$

In this case optimal solutions may lie on the (relative) boundary of $C$. Note that the next result does not require $f$ to be differentiable.

**Lemma 6.1.2** (Global minimum in constrained problems)**.** *Let $f : C \to \mathbb{R}$ be a convex function defined on a closed convex set $C \subseteq \mathbb{R}^n$. Then, in problem (6.1), each local minimum is also a global minimum. Moreover, the set of minima (optimal solutions) in (6.1) is a closed convex subset of $C$.*

*Proof.* Assume that $x^*$ is a local minimum of problem (6.1). Then there is an $r > 0$ such that $f(x^*) \le f(x)$ for all $x \in B(x^*, r) \cap C$. Let $y \in C \setminus B(x^*, r)$ and assume that $f(y) < f(x^*)$. Consider a point $z$ lying in the relative interior of the line segment between $x^*$ and $y$, so $z = (1-\lambda)x^* + \lambda y$ where $0 < \lambda < 1$. Due to the convexity of $f$ we have

$$\begin{aligned} f(z) = \; & f((1-\lambda)x^* + \lambda y) \leq (1-\lambda)f(x^*) + \lambda f(y) < \\ & (1-\lambda)f(x^*) + \lambda f(x^*) = f(x^*). \end{aligned}$$

By choosing $\lambda$ sufficently small we assure that $z$ lies in $B(x^*, r)$ (and $z \in C$ as $C$ is convex) and then we have the contradiction $f(z) < f(x^*)$. This proves that $f(x^*) \leq f(y)$ for each $y \in C$, so $x^*$ is a globally optimal solution. To prove the last statement, let $v^* = f(x^*)$ denote the optimal value in problem (6.1). Then the set of optimal solutions is the set $C^* = \{x \in C : f(x) \leq v^*\}$ which is closed and convex as both $C$ and the sublevel set $\{x : f(x) \leq v^*\}$ are closed and convex (see Exercise 5.11). □

In Section 3.1 we discussed the problem of finding a nearest point in a closed convex set $C$ to a given point, say $z$. This is an example of a convex optimization problem. Actually, it corresponds to minimizing a convex quadratic function:

$$\text{minimize } f(x) := \|x - z\|^2 \text{ subject to } x \in C.$$

Moreover, Theorem 3.1.2 showed that there is a unique nearest point and that this point may be characterized in terms of certain inequalities. This chacterization was in fact an optimality condition. The next section studies optimality conditions for our general problem of minimizing a convex function $f$ over a convex set $C$. Such conditions lie behind numerical algorithms for solving these optimization problems.

## 6.2 Optimality conditions for convex optimization

In order to minimize a function $f : \mathbb{R}^n \to \mathbb{R}$ one usually needs optimality conditions that somehow describe optimal solutions. An algorithm can then be set up to search for a point satisfying those conditions. This is familiar for a differentiable function of a single variable: we look for a point where the derivative is zero. Our goal here is to find general optimality conditions for convex functions defined on a convex set.

Throughout the section we consider the following situation

- $f : C \to \mathbb{R}$ is a convex function defined on a convex set $C \subseteq \mathbb{R}^n$.
- $f$ is continuously differentiable

and the problem of interest is

$$\text{minimize } f(x) \text{ subject to } x \in C.$$

(In Section 6.5 we consider the more specific situation where $C$ is the solution set of a system of equations and inequalities.)

At a point $x$ we know that the function $f$ increases (locally) fastest in the direction of the gradient $\nabla f(x)$. Similarly $f$ decreases fastest in the direction $-\nabla f(x)$. Thus, in order to decrease the function value (if possible) we want to go from $x$ to another feasible point $x'$ (so $x' \in C$) for which the direction $d := x' - x$ satisfies $\nabla f(x)^T d < 0$. Due to Taylor's formula (first order approximation) this condition assures that $f(x') < f(x)$ provided that we make a suitably small step in this direction.

convex optimality condition

The following theorem gives optimality conditions for our problem.

**Theorem 6.2.1** (Optimality condition)**.** *Let $x^* \in C$. Then $x^*$ is a (local and therefore global) minimum of $f$ over $C$ if and only if*

$$\nabla f(x^*)^T(x - x^*) \geq 0 \quad \text{for all } x \in C. \tag{6.2}$$

*Proof.* Assume first that $\nabla f(x^*)^T(x - x^*) < 0$ for some $x \in C$. Consider the function $g(\epsilon) = f(x^* + \epsilon(x - x^*))$ and apply the mean value theorem to this function. Thus, for every $\epsilon > 0$ there exists an $s \in [0, 1]$ with

$$f(x^* + \epsilon(x - x^*)) = f(x^*) + \epsilon \nabla f(x^* + s\epsilon(x - x^*))^T(x - x^*).$$

Since $\nabla f(x^*)^T(x - x^*) < 0$ and the gradient function is continuous (our standard assumption!) we have for sufficiently small $\epsilon > 0$ that $\nabla f(x^* + s\epsilon(x - x^*))^T(x - x^*) < 0$. This implies that $f(x^* + \epsilon(x - x^*)) < f(x^*)$. But, as $C$ is convex, the point $x^* + \epsilon(x - x^*)$ also lies in $C$ and so we conclude that $x^*$ is not a local minimum. This proves that (6.2) is necessary for $x^*$ to be a local minimum of $f$ over $C$.

Next, assume that (6.2) holds. Using Theorem 5.3.5 we then get

$$f(x) \geq f(x^*) + \nabla f(x^*)^T(x - x^*) \geq f(x) \quad \text{for every } x \in C$$

so $x^*$ is a (global) minimum. $\square$

It is quite instructive to see how the convexity of both the feasible set $C$ and the function $f$ enters this proof. We understand that the convexity of $C$ simplifies the optimality condition.

stationary point

A point $x$ which satisfies (6.2) is called a *stationary point*. Thus, for these kind of optimization problems, the goal is to find a stationary point since it is a global optimal solution.

**Remark.** The condition (6.2) is also a *necessary optimality condition* for a (local) minimum of a general differentiable, but nonconvex function $f$. This is useful since we may use the condition to find a set of candidates for being a local minimum point. But, unfortunately, the condition may not be sufficient for $x^*$ to be a minimum when $f$ is nonconvex. (For instance, maximum points also satisfy this condition.)

**Example 6.2.1.** Consider the problem to minimize $f(x)$ subject to $x \geq O$. So here $C = \{x \in \mathbb{R}^n : x \geq O\}$ is the nonnegative orthant. Then the optimality condition (6.2) becomes

$$\begin{aligned} \partial f(x^*)/\partial x_i = 0 \quad & \text{for all } i \leq n \text{ with } x_i^* > 0, \text{ and} \\ \partial f(x^*)/\partial x_i \geq 0 \quad & \text{for all } i \leq n \text{ with } x_i^* = 0. \end{aligned}$$

## 6.3 Feasible direction methods

We now consider the convex optimization problem

$$\text{minimize } f(x) \text{ subject to } x \in C$$

where we assume that $f$ is continuously differentiable and $C$ is a nonempty closed convex set on $\mathbb{R}^n$. The method we discuss is a *primal method* meaning that it produces a sequence of primal feasible points, i.e., points in $C$. The algorithm is iterative and generates a sequence $\{x^k\}$ in $C$ according to

`primal method`

$$x^{k+1} = x^k + \alpha^k d^k$$

where $x^0$ is some starting point in $C$. Here $x^k$ is the current solution, $d^k$ is the *search direction* and $\alpha^k$ is the *step length.* The problem in iteration $k$ is to verify optimality of the currect solution or to find a suitable (nonzero) search direction which leads to a new feasible point with smaller $f$-value. Thus the algorithm has to perform two tasks in every iteration:

`search direction`
`step length`

- find a search direction $d^k$
- line search: find a suitable step length $\alpha^k$ in direction $d^k$.

The vector $d^k$ is a *feasible direction* in $x^k$ if $x^k + \alpha d^k$ lies in $C$ for all positive and suitably small $\alpha$. Moreover, if $d^k$ is also a *descent direction* meaning that

`descent direction`

$$\nabla f(x^k)^T d^k < 0$$

then for suitable $\alpha^k > 0$ we have $f(x^k + \alpha^k d^k) < f(x^k)$ so a better solution has been found. This general procedure is called a *feasible direction method.*

`feasible direction method`

Since $C$ is convex one can see that any feasible direction $d^k$ at $x^k$ has the form $d^k = \bar{x}^k - x^k$ for some $\bar{x}^k \in C$ and that $x^k + \alpha(\bar{x}^k - x^k) \in C$ for all suitably small $\alpha$, say $\alpha \leq \hat{\alpha}$. Note here that $\hat{\alpha} \geq 1$ due to the convexity of $C$. Thus, if $x^k$ is nonstationary, i.e. (6.2) does not hold, then for some $\bar{x}^k \in C$ we have

$$\nabla f(x^k)^T(\bar{x}^k - x^k) < 0$$

so we have a decent direction. The next problem is to choose the step length $\alpha$. Any choice with $0 < \alpha < \hat{\alpha}$ will give a new feasible solution, so we want to find one with small, perhaps smallest possible, $f$-value in this range. There are several *stepsize selection rules* for this purpose. Some of them are "constant stepsize" (meaning that $\alpha^k = 1$), "Armijo rule"[1], and "limited minimization" (minimize exactly for $\alpha \in [0, 1)$; for further rules, see [1]. Note that this problem is a univariate convex minimization problem.

conditional gradient method

The *conditional gradient method* , also called the *Frank-Wolfe method*, is a feasible direction method where we choose the feasible direction at $x^k$ by solving the problem

$$\begin{array}{ll} \text{minimize} & \nabla f(x^k)^T(x - x^k) \\ \text{subject to} & x \in C \end{array}$$

This corresponds to linearizing the objective function (first order Taylor approximation at $x^k$) and minimizing it over the whole feasible region $C$. If $\bar{x}^k$ is the optimal soluton found to this problem, the new search direction is $d^k = \bar{x}^k - x^k$. This method makes sense if finding the new direction is a rather easy problem. For instance, this is the case if $C$ is a polyhedron for then we get a linear programming problem which can be solved efficiently.

The following proposition says that, under some technical assumptions, feasible direction methods work, i.e. they converge to a minimum point.

**Proposition 6.3.1.** *Let $\{x^k\}$, where $x^{k+1} = x^k + \alpha^k d^k$, be a sequence determined by a feasible direction method. If $d^k$ is gradient related and $\alpha^k$ is chosen using the limited minimization rule (or the Armijo rule), then every limit point is a stationary point.*

The assumption that $d^k$ is gradient related is a technical assumption which prevents the search direction $d^k$ to be nearly orthogonal to the gradiant $\nabla f(x^k)$. For if $\nabla f(x^k)^T d^k/(\|\nabla f(x^k)\|\|d^k\|) \to 0$, then the algorithm might get stuck at a nonstationary point. For an algorithm where $x^k$ determines $d^k$ one says that $d^k$ is gradient related if the following holds: for any subsequence $\{x^k\}_{k\in K}$ that converges to a nonstationary point, the corresponding subsequence $\{d^k\}_{k\in K}$ is

1. Armijo rule: one chooses numbers $0 < \beta, \sigma < 1$ and $s$ and let $\alpha^k = \beta^m s$ where $m_k$ is the smallest nonnegative integer such that $f(x^k) - f(x^k + \beta^m d^k) \geq -\sigma\beta^m \nabla f(x^k)^T d^k$.

bounded and satisfies $\lim_{k\to\infty} \sup \nabla f(x^k)^T d^k < 0$. One can prove that $d^k$ is gradient related in the conditional gradient method (see [1]) so this method converges to a stationary point as desired.

Unfortunately, the speed of convergence may not be good for the conditional gradient method. For instance, the convergence may be slow when $C$ is a polyhedron. For certain nonpolyhedral sets it is known that the method has linear convergence rate, i.e. for suitable numbers $q$, $K$ and $0 < \beta < 1$ the distance to an optimal point $x^*$ satisfies

$$\|x^k - x^*\| \leq q\beta^k \quad \text{for all } k \geq K.$$

Although the convergence speed may not be very good, these methods are simple and quite popular. Moreover, the computational task in every iteration may be small.

Another method, which we do not discuss in detail, is called the *constrained Newton's method*. It works for problems where $f$ is twice continuously differentiable. The method is based on minimizing a second order Taylor expansion of $f$ near $x^k$ in order to find the search direction $d^k$. Thus, this minimization problem may take longer than the corresponing linear problem (in the conditional gradient method). However, one improves on the convergence speed and it can be shown that the constrained Newton's method converges superlinearly (meaning that for every $0 < \beta < 1$ there is a $q$ such that the error is no more than $q\beta^k$ for all suitably large $k$).

There are many other methods around and their suitability relates to the structure of the constraint set and also the complexity of the objective function $f$. Again we refer to [1]) for further reading.

## 6.4 Nonlinear optimization and Lagrange multipliers

We now change focus and discuss more general nonlinear optimization problems.

Consider a nonlinear optimization problem

$$\text{minimize } \; f(x) \quad \text{subject to} \quad x \in S$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $S \subseteq \mathbb{R}^n$. In this section we leave the "convex setting" so we have no convexity assumption on $f$ or $S$. The feasible set $S$ is usually described in terms of certain equations and inequalities. This opens up for the introduction of auxiliary variables called *Lagrangian multipliers* that provide a

Lagrangian multipliers

powerful tool for developing theory as well as methods for solving the original problem.

To be specific we first consider the nonlinear optimization problem with equality constraints

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & \\ & h_i(x) = 0 \quad \text{for } i = 1, \ldots, m \end{array} \tag{6.3}$$

where $f$ and $h_1, h_2, \ldots, h_m$ are continuously differentiable functions from $\mathbb{R}^n$ into $\mathbb{R}$.

We want to establish necessary optimality conditions for this problem. They are useful for numerical algorithms. Such necessary conditions are contained in the following theorem. Recall that $H_f(x)$ denotes the Hessian matrix of $f$ at $x$, i.e. it contains the second order partial derivatives.

**Theorem 6.4.1** (Lagrange mulipliers - necessary condition)**.** *Let $x^*$ be a local minimum in problem (6.3) and assume that the corresponding gradients $\nabla h_i(x^*)$ ($i \leq m$) are linearly independent. Then there is a unique vector $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*)$ such that*

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla h_i(x^*) = O. \tag{6.4}$$

*If $f$ and each $h_i$ are twice continuously diffrentiable, then the following also holds*

$$y^T[H_f(x^*) + \sum_{i=1}^{m} \lambda_i^* H_{h_i}(x^*)]y \geq 0 \quad \text{for all } y \text{ with } \nabla h_i(x^*)^T y = 0 \;\; (i \leq m). \tag{6.5}$$

The numbers $\lambda_i^*$ in this theorem are called the *Lagrangian multipliers*. Note that the Lagrangian multiplier vector $\lambda$ is unique; this follows directly from the linear independence assumption (although more arguments are needed to prove the existence of $\lambda^*$).

We may interpret the theorem in the following way. At the point $x^*$ the linear subspace $L(x^*)$ of first order feasible variations are vectors $y$ satisfying $\nabla h_i(x^*)^T y = 0$ $i \leq m$. Note that, if each $h_i$ is linear, then $L(x^*)$ consists of those $y$ such that $x^* + y$ is feasible, i.e., $h_i(x^* + y) = 0$ for each $i \leq m$. Thus, (6.4) says that in the local minimum $x^*$ the gradient $\nabla f(x^*)$ of the objective function is orthogonal to the subspace $L(x^*)$ of first order feasible variations. This is reasonable since otherwise there would be a feasible direction in which $f$ would decrease. Note that this necessary optimality condition corresponds to the condition $\nabla f(x^*) = O$ in the unconstrained case.

One may prove the theorem by eliminating variables based on the equations and thereby reducing the problem to an unconstrained one. Another proof, which we shall briefly look at, is the *penalty approach*. This approach is interesting as is related to algorithms for actually solving the problem.

*Proof.* Let $h = (h_1, h_2, \ldots, h_m)$, so $h(x) = (h_1(x), h_2(x), \ldots, h_m(x))$. For $k = 1, 2, \ldots$ consider the modified objective function

$$F^k(x) = f(x) + (k/2)\|h(x)\|^2 + (\alpha/2)\|x - x^*\|^2$$

where $x^*$ is the local minimum under consideration. The second term is a penalty term for violating the constraints and the last term is there for proof technical reasons. As $x^*$ is a local minimum there is an $\epsilon > 0$ such that $f(x^*) \le f(x)$ for all $x \in S$ where $S = \{x : \|x - x^*\| \le \epsilon\}$. Choose now an optimal solution $x^k$ of the problem $\min\{F^k(x) : \|x - x^*\| \le \epsilon\}$; the existence here follows from Weierstrass' theorem. For every $k$ we have

$$F^k(x^k) = f(x^k) + (k/2)\|h(x^k)\|^2 + (\alpha/2)\|x^k - x^*\|^2 \le F^k(x^*) = f(x^*).$$

By letting $k \to \infty$ in this inequality we conclude that $\lim_{k\to\infty} \|h(x^k)\| = 0$ (here we use that $f(x^k)$ is bounded in the set $\{x : \|x - x^*\| \le \epsilon\}$). So every limit point $\bar{x}$ of the sequence $\{x^k\}$ satisfies $h(\bar{x}) = O$. The inequality above also implies (by dropping a term on the left-hand side) that $f(x^k) + (\alpha/2)\|x^k - x^*\|^2 \le f(x^*)$ for all $k$, so by passing to the limit we get

$$f(\bar{x}) + (\alpha/2)\|\bar{x} - x^*\|^2 \le f(x^*) \le f(\bar{x})$$

where the last inequality follows from the facts that $\|x - x^*\| \le \epsilon$ and $h(\bar{x}) = O$. Clearly, this gives $\bar{x} = x^*$. We have therefore shown that the sequence $\{x^k\}$ converges to the local minimum $x^*$. Since $x^*$ is the center of the ball $S$ the points $x^k$ lie in the interior of $S$ for suitably large $k$. The conclusion is then that $x^k$ is the *unconstrained* minimum of $F^k$ when $k$ is sufficiently large. We may therefore apply Corollary 6.1.1 so $\nabla F^k(x^k) = O$. Thus, by calculation of gradient we obtain

$$O = \nabla F^k(x^k) = \nabla f(x^k) + k\nabla h(x^k)h(x^k) + \alpha(x^k - x^*). \tag{6.6}$$

For suitably large $k$ the matrix $\nabla h(x^k)^T \nabla h(x^k)$ is nonsingular (as the columns of $\nabla h(x^k)^T$ are linearly independent due to $\mathrm{rank}(\nabla h(x^*)) = m$ and a continuity argument). Multiply equation (6.6) by $(\nabla h(x^k)^T \nabla h(x^k))^{-1} \nabla h(x^k)^T$ to obtain

$$kh(x^k) = -(\nabla h(x^k)^T \nabla h(x^k))^{-1} \nabla h(x^k)^T (\nabla f(x^k) + \alpha(x^k - x^*)).$$

Letting $k \to \infty$ we see that the sequence $\{kh(x^k)\}$ is convergent and its limit point $\lambda^*$ is given by

$$\lambda^* = -(\nabla h(x^*)^T \nabla h(x^*))^{-1} \nabla h(x^*)^T \nabla f(x^*).$$

Finally, by passing to the limit in (6.6) we get

$$O = \nabla f(x^*) + \nabla h(x^*)\lambda^*$$

This proves the first part of the theorem; we omit proving the second part (it can be found in [1]). □

The first order necessary condition (6.4) along with the constraints $h(x) = O$ is a system of $n+m$ equations in the $n+m$ variables $x_1, x_2, \ldots, x_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_m$. One may use e.g. Newton's method for solving these equations and find a candidate for an optimal solution.

Lagrangian function

The *Lagrangian function* associated with the problem (6.3) is the function $L : \mathbb{R}^{n+m} \to \mathbb{R}$ defined by

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x)$$

Note that the equation (6.4) is equivalent to $\nabla_x L(x^*, \lambda^*) = O$ while $h(x^*) = O$ corresponds to $\nabla_\lambda L(x^*, \lambda^*) = O$. Thus, in order to solve (6.3) we may look for a suitable Lagrangian multiplier vector $\lambda^*$ and a corresponding $x^*$ satisfying $h(x^*) = O$ and such that $x^*$ is a stationary point of the Lagrangian function, i.e., $\nabla_x L(x^*, \lambda^*) = O$.

Necessary optimality conditions are used for finding a candidate solutions for being optimal. In order to verify optimality we need sufficient optimality conditions.

**Theorem 6.4.2** (Lagrange muliplier - sufficient condition)**.** *Assume that $f$ and $h$ are twice continuously differentiable functions. Moreover, let $x^*$ be a point satisfying the first order necesary optimality condition (6.4) and the following condition*

$$y^T H_L(x^*, \lambda^*) y > 0 \quad \text{for all } y \neq O \text{ with } \nabla h(x^*)^T y = 0 \tag{6.7}$$

*where $H_L(x^*, \lambda^*)$ is the Hessian of the Lagrangian functions with second order partial derivatives with respect to $x$. Then $x^*$ is a (strict) local minimum of $f$ subject to $h(x) = O$.*

This theorem may be proved (see [1] for details) by considering the *augmented Lagrangian function*

$$L_c(x, \lambda) = f(x) + \lambda^T h(x) + (c/2)\|h(x)\|^2 \tag{6.8}$$

where $c$ is a positive scalar. This is in fact the Lagrangian function in the modified problem

$$\text{minimize} \;\; f(x) + (c/2)\|h(x)\|^2 \quad \text{subject to} \quad h(x) = O \tag{6.9}$$

and this problem must have the same local minima as the problem of minimizing $f(x)$ subject to $h(x) = O$. The objective function in (6.9) contains the *penalty term* $(c/2)\|h(x)\|^2$ which may be interpreted as a penalty (increased function value) for violating the constraint $h(x) = O$. In connection with the proof of Theorem 6.4.2 based on the augmented Lagrangian one also obtains the following interesting and useful fact: *if $x^*$ and $\lambda^*$ satisfy the sufficient conditions in Theorem* 6.4.2 *then there exists a positive $\bar{c}$ such that for all $c \geq \bar{c}$ the point $x^*$ is also a local minimum of the augmented Lagrangian $L_c(\cdot, \lambda^*)$.* Thus, the original constrained problem has been converted to an unconstrained one involving the augmented Lagrangian. And, as we know, unconstrained problems are easier to solve (solve the equations saying that the gradient is equal to zero).

penalty term

## 6.5 Nonlinear optimization: inequality constraints

We now discuss the general nonlinear programming problem where there are both equality and inequality constraints. The problem is then

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & \\ & h_i(x) = 0 \quad \text{for } i = 1, \ldots, m \\ & g_j(x) \leq 0 \quad \text{for } j = 1, \ldots, r. \end{array} \tag{6.10}$$

We assume, as usual, that all these functions are are continuously differentiable real-valued functions defined on $\mathbb{R}^n$. In short form we write the constraints as $h(x) = O$ and $g(x) \leq O$ where we let $h = (h_1, h_2, \ldots, h_m)$ and $g = (g_1, g_2, \ldots, g_r)$.

A main difficulty in problems with inequality constraints is to determine which of the inequalities that are active in an optimal solution. For, if we knew the active inequalities, we would essentially have a problem with only equality constraints, $h(x) = O$ plus the active equalities, i.e. a problem as discussed in the previous section. For small problems (solvable by hand-calculation) a direct method is to consider all possible choices for active inequalities and solve the corresponding equality-constrained problem by looking at the Lagrangian function.

Interestingly, one may also transform the problem (6.10) into the following equality-constrained problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & \\ & h_i(x) = 0 \qquad \text{for } i = 1, \ldots, m \\ & g_j(x) + z_j^2 = 0 \quad \text{for } j = 1, \ldots, r. \end{array} \tag{6.11}$$

We have introduced artificial variables $z_j$, one for each inequality. These square of these variables represent slack in each of the original inequalities. Note that there is no sign constraint on $z_j$. Clearly, the problems (6.10) and (6.11) are equivalent. This transformation is useful computationally; see below. Moreover, it is useful theoretically as one may apply the optimality conditions from the previous section to problem (6.11). We omit the details in this derivation (see [1]), but the result is a set of optimality conditions for problem (6.10) called the *Karush-Kuhn-Tucker conditions*, or simply *KKT conditions.* In order to present the KKT conditions we introduce the Lagrangian function

KKT conditions

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{r} \mu_j g_j(x).$$

The Hessian matrix of $L$ at $(x, \lambda, \mu)$ containing second order partial derivatives of $L$ with respect to $x$ will be denoted by $\nabla_{xx} L(x, \lambda, \mu)$. Finally, the indices of the active inequalities at $x$ is denoted by $J(x)$, so $J(x) = \{j \leq r : g_j(x) = 0\}$.

**Theorem 6.5.1** (Karush-Kuhn-Tucker conditions)**.** *Consider problem (6.10) with the usual differentiability assumptions.*

*(i) Let $x^*$ be a feasible point which is regular, meaning that the gradients of $h_i$ and each active $g_j$ at $x^*$ are linearly independent. Then there are unique Lagrange multiplier vectors $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*)$ and $\mu^* = (\mu_1^*, \mu_2^*, \ldots, \mu_r^*)$ such that*

$$\begin{aligned} \nabla_x L(x^*, \lambda^*, \mu^*) &= O, \\ \mu^* &\geq 0 \qquad (j \leq r), \\ \mu^* &= 0 \qquad (j \notin J(x^*)). \end{aligned} \tag{6.12}$$

*If $f$, $g$ and $h$ are twice continuously diffrentiable, then the following also holds*

$$y^T \nabla_{xx} L(x^*, \lambda^*, \mu^*) y \geq 0 \tag{6.13}$$

*for all $y$ with $\nabla h_i(x^*)^T y = 0$ $(i \leq m)$ and $\nabla g_j(x^*)^T y = 0$ $(j \in J(x^*))$.*

*(ii) Assume that $x^*$, $\lambda^*$ and $\mu$ are such that $x^*$ is a feasible point and (6.12) holds. Assume, moreover, that (6.13) holds with* strict *inequality for each $y$. Then $x^*$ is a (strict) local minimum in problem (6.10)*

We remark that the assumption that $x^*$ is a regular point may be too restrictive in some situations, for instance there may be more than $n$ active inequalities in $x^*$. There exist several other weaker assumptions that assure the existence of Lagrangian multipliers (and similar necessary conditions); see the discussion of so-called constraint qualifications in [1].

ne convex rogramming roblem

In the remaining aprt of this section we consider the convex optimization problem and the corresponding KKT conditions. The *convex programming problem* given by

$$
\begin{array}{lll}
\text{minimize} & f(x) & \\
\text{subject to} & & \\
\text{(i)} & a_i^T x = b_i & \text{for } i = 1, \ldots, m; \\
\text{(ii)} & g_k(x) \leq 0 & \text{for } k = 1, \ldots, p.
\end{array}
\tag{6.14}
$$

Here we assume that all the functions $f$ and $g_k$ are differentiable convex functions, and that $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for each $i$. Let $C$ denote the feasible set of problem (6.14), so

$$C = \{x \in \mathbb{R}^n : a_i^T x = b_i \ \text{ for } i = 1, \ldots, m, \ \ g_k(x) \leq 0 \ \text{ for } k = 1, \ldots, p\}.$$

The feasible set $C$ is convex as the intersection of other convex sets. In fact, the solution set of the linear equations in (i) is affine (and therefore convex) and each sublevel set $\{x : g_k(x) \leq 0\}$ is also convex as $g_k$ is convex (see Exercise 5.11). Thus, $C$ is a convex set described by linear and convex functions. Moreover, $C$ is a closed set since it is the intersection between a finite number of sets, each being the inverse image of a continuous function. Note that, due to Corollary 6.1.1, we have that a local minimum in (6.14) is also a global minimum. Therefore, we may simply speak of a *minimum* below. Linear programming is a special case of the convex programming problem, see also Example 6.5.4.

The KKT conditions may be simplified in the case of the convex programming problem (6.14). We omit the proof, it can be found e.g. in [6] (see also [1], [16]).

**Theorem 6.5.2.** *(i) Assume that $x^*$ is optimal in (6.14). Then there are Lagrangian multiplier vectors $\lambda^*$ and $\mu^*$ such that (6.12) holds.*

*(ii) Assume that there is a point $x' \in C$ such that $g_j(x') < 0$ holds for each non-affine function $g_j$ $(j \leq r)$. Then the converse of (i) also holds, i.e., if there are vectors $\lambda^*$ and $\mu^*$ such that (6.12) holds, then $x^*$ is a minimum of the convex programming problem (6.14).*

The assumption stated in (ii) of the theorem (the existence of the vector $x'$) is called the *weak Slater assumption.*

The KKT conditions have a geometrical interpretation, see Fig. 6.1. They say that $-\nabla f(x^*)$ may be written as a nonnegative combination of the $a_i$'s plus a linear combination of the gradients of the active constraints at $x^*$. (This means that $-\nabla f(x^*)$ lies in the so-called normal cone of $C$ at $x^*$).

We shall now consider some examples of convex programming problems where we apply Theorem 6.5.2.

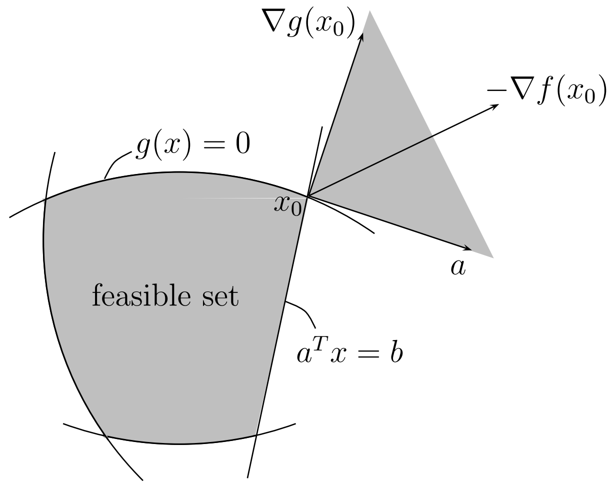


Figure 6.1: KKT conditions

**Example 6.5.1.** *(A one-variable problem)* We start with a very simple example, namely the one-variable problem: minimize $f(x)$ subject to $x \geq 0$. Here $f : \mathbb{R} \to \mathbb{R}$ is a differentiable convex function. We here let $g_1(x) = -x$ and $p = 1$, $m = 0$. In this case the KKT conditions become: there is a number $z$ such that $f'(x) - z = 0$, $z \geq 0$ and $z(-x) = 0$. This is one of the (rare) occasions where we can eliminate the Lagrangian variable $z$ via the equations $z = f'(x)$. So the optimality conditions are: $x \geq 0$ (feasibility), $f'(x) \geq 0$ and $x \cdot f'(x) = 0$. Thus, if $x > 0$ we must have $f'(x) = 0$ ($x$ is an interior point of the domain so the derivative must be zero), and if $x = 0$ we must have $f'(0) \geq 0$. From this, and the convexity of $f$, we see that there are two possibilities. First, if $f'(0) > 0$, then the unique optimal solution is $x = 0$. Second, if $f'(0) \leq 0$, then each point $x$ with $f'(x) = 0$ is optimal.

**Example 6.5.2.** *(Quadratic optimization with equality constraints)* Consider the following quadratic optimization problem with linear equality constraints

$$\begin{array}{ll} \text{minimize} & (1/2)\, x^T D x - q^T x \\ \text{subject to} & \\ & \qquad Ax = b \end{array}$$

where $D$ is positive semidefinite and $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$. This is a special case of (6.14) where $f(x) = (1/2)\, x^T D x - q^T x$ (so $p = 0$). We have that $\nabla f(x) = Dx - q$. Thus, the KKT conditions for this problem say that for some $y \in \mathbb{R}^m$ we have that $Dx - q + A^T y = O$. In addition, the vector $x$ is feasible so we have $Ax = b$. Thus, solving the quadratic optimization problem amounts to solving the linear system of equations

$$Dx + a^T y = q, \; Ax = b$$

which may be written as

$$(*) \quad \begin{bmatrix} D & A^T \\ A & O \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} q \\ b \end{bmatrix}.$$

Under the additional assumption that $D$ is positive definite and $A$ has full row rank, one can show that the coefficient matrix in $(*)$ is nonsingular so this system has a unique solution $x, y$. Thus, for this problem, we may write down an explicit solution (in terms of the inverse of the block matrix). Numerically, one finds $x$ (and the Lagrangian multiplier $y$) by solving the linear system $(*)$ by e.g. Gaussian elimination or some faster (direct or iterative) method.

**Example 6.5.3.** *(Quadratic programming)* We extend the problem in the previous example by allowing linear inequality constraints as well:

$$\begin{array}{ll} \text{minimize} & (1/2)\, x^T Dx - q^T x \\ \text{subject to} & \\ & \qquad Ax = b \\ & \qquad Cx \leq r. \end{array}$$

Here $D$, $A$ and $b$ are as above and $C \in \mathbb{R}^{p,n}$, $r \in \mathbb{R}^p$. We see that $\nabla f(x) = Dx - q$ (as above) and that $\nabla g_k(x) = -e_k$. Thus, the KKT conditions for this problem say that for some $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^n$ we have that $Dx - q + A^T y - z = O$, $z \geq O$ and $z_k \cdot (-x_k) = 0$ for each $k$. We may here eliminate $z$ from the first equation and obtain the equivalent condition: there is a $y \in \mathbb{R}^m$ such that $Dx + A^T y \geq q$ and $(Dx + A^T y - q)_k \cdot x_k = 0$ for each $k$. In addition, we have $Ax = b$, $Cx \leq r$. This problem may be solved numerically, for instance, by a so-called active set method, see [8].

**Example 6.5.4.** *(Linear programming)* As mentioned, linear programming is a special case of the convex programming problem (6.14). We discuss this in some detail. Consider the special case of (6.14) where $f(x) = c^T x$ for some vector $c \in \mathbb{R}^n$ and $g_k(x) = -x_k$ for $k = 1, \ldots, n$ (so $p = n$). Moreover we have the constraints $a_i^T x = b_i$ for $i = 1, \ldots, m$. Let $A$ be the $m \times n$ matrix whose $i$'th row is $a_i$, and let $b$ be the vector with $i$'th component being $b_i$. Then our optimization problem is

$$\text{minimize} \;\; c^T x \quad \text{subject to} \;\; Ax = b, \quad x \geq O$$

which is a general LP problem. Let us see what the KKT conditions become in this case. We calculate $\nabla f(x) = c$ and $\nabla g_k(x) = -e_k$. Let $x$ be a feasible solution of our LP. The KKT conditions state that there are vectors $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^n$ such that

$$\begin{array}{ll} \text{(i)} & c + A^T y - z = O; \\ \text{(ii)} & z \geq O; \\ \text{(iii)} & z_k(-x_k) = 0 \quad \text{for all } k = 1, \ldots, n. \end{array}$$

We may here eliminate the vector $z$ from equation (i) and obtain the equivalent set of KKT conditions: there is a vector $y \in \mathbb{R}^m$ such that

$$\begin{array}{ll} \text{(i)} & c + A^T y \geq O; \\ \text{(ii)} & (c + A^T y)_k \cdot x_k = 0 \quad \text{for all } k = 1, \ldots, n. \end{array} \tag{6.15}$$

These conditions relate to LP duality theory. The LP dual of our LP problem (minimize $c^T x$ subject to $Ax = b$, $x \geq O$) is the LP problem maximize $b^T w$ subject to $A^T w \leq c$. From LP duality theory we know that a feasible solution $x$ to the primal problem is optimal if and only if there is a feasible solution in the dual such that these two solutions satisfy the complementary slackness conditions. In the present situation this optimality condition says: there is a vector $w \in \mathbb{R}^m$ such that $A^T w \leq c$ and $(A^T w - c)_k \cdot x_k = 0$ for each $k = 1, \ldots, n$. But this is precisely the content of the KKT conditions (6.15); just let $y = -w$. Therefore, the KKT conditions, in the special case of linear programming, amount to the optimality conditions known from LP duality theory. Moreover, we see that the Lagrangian multiplier $y$ corresponds to the dual variable $w$ (and $w = -y$).

## 6.6 An augmented Lagrangian method

Consider again the nonlinear optimization problem with equality constraints (6.3), i.e.,

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & \\ & h_i(x) = 0 \quad \text{for } i = 1, \ldots, m \end{array} \tag{6.16}$$

where $f$ and each $h_i$ are continuously differentiable functions. We here briefly discuss a numerical method for solving this problem. Note that this method also applies to problems with inequality constraints; then we first apply the transformation to the equality-constrained problem (6.11).

The method is based on the augmented Lagrangian function

$$L_c(x, \lambda) = f(x) + \lambda^T h(x) + (c/2)\|h(x)\|^2$$

where $c$ is a positive scalar. Recall that if $x^*$ and $\lambda^*$ satisfy the sufficient conditions in Theorem 6.4.2 then there exists a positive $\bar{c}$ such that for all $c \geq \bar{c}$ the point $x^*$ is also a local minimum of the augmented Lagrangian $L_c(\cdot, \lambda^*)$. This fact is the basis of the algorithm which may be called the *quadratic penalty function method.*

quadratic penalty function method

**Proposition 6.6.1.** *Assume that $f$ and each $h_i$ are continuous and that there are feasible points in (6.3). Let $\{x^k\}$, $\{\lambda^k\}$ and $\{c^k\}$ be sequences satisfying*

- $x^k$ *is a global minimum of* $L_{c^k}(x, \lambda^k)$ *for* $x \in \mathbb{R}^n$,
- $\{\lambda^k\}$ *is bounded,*
- $0 < c^k < c^{k+1}$ *for* $k \geq 1$, *and* $c^k \to \infty$.

*Then every limit point of $\{x^k\}$ is a global minimum of the problem (6.16).*

We remark that this proposition holds more generally for a constrained problem where $x \in S$ provided that $\{x \in S : h(x) = O\}$ is nonempty; then we minimize the augmented Lagrangian over $S$.

A common approach is to use Newton's method for minimizing $L_{c^k}(x, \lambda^k)$. Moreover, one uses the previous solution $x^{k-1}$ as the starting point of the minimization in iteration $k$. A practical issue is how fast one should increase $c^k$. One needs a balance as too fast increase gives ill-conditioned problems and too slow increase of $c^k$ gives slow convergence of $x^k$ towards the minimum. Although the quadratic penalty function method works under extremely mild conditions on the multiplier vectors $\lambda^k$, simply boundedness, the method is faster if the multipliers are updated suitably. One approach, called the *method of multipliers*, is to update $\lambda^k$ according to the formula

method of multipliers

$$\lambda^{k+1} = \lambda^k + c^k h(x^k)$$

This is motivated by a fact (which we do not prove) saying that, if $x^*$ is regular, then the sequence $\{\lambda^k + c^k h(x^k)\}$ converges towards the corresponding Lagrangian multiplier vector $\lambda^*$.

We conclude at this point. But the story of convexity and optimization is much longer. So, for *fascinating mathematical theory, algorithms and a rich set of applications* we recommend further reading in these areas. And, by now, you know which books to consult! Good luck!

## 6.7 Exercises

**Exercise 6.1.** *Consider the least squares problem minimize $\|Ax - b\|$ over all $x \in \mathbb{R}^n$. From linear algebra we know that the optimal solutions to this problem* *are precisely the solutions to the linear system (called the normal equations)*

$$A^T Ax = A^T b.$$

*Show this using optimization theory by considering the function* $f(x) = \|Ax-b\|^2$.

**Exercise 6.2.** *Prove that the optimality condition is correct in Example* 6.2.1.

**Exercise 6.3.** *Consider the problem to minimize a (continuously differentiable) convex function* $f$ *subject to* $x \in C = \{x \in \mathbb{R}^n : O \leq x \leq p\}$ *where* $p$ *is some nonnegative vector. Find the optimality conditions for this problem. Suggest a numerical algorithm for solving this problem.*

**Exercise 6.4.** *Consider the optimization problem minimize* $f(x)$ *subject to* $x \geq O$, *where* $f : \mathbb{R}^n \to \mathbb{R}$ *is a differentiable convex function. Show that the KKT conditions for this problem are*

$$x \geq O,\ \nabla f(x) \geq O, \text{ and } x_k \cdot \partial f(x)/\partial x_k = 0 \text{ for } k = 1, \ldots, n.$$

*Discuss the consequences of these conditions for optimal solutions.*

**Exercise 6.5.** *Solve the problem: minimize* $(x + 2y - 3)^2$ *for* $(x, y) \in \mathbb{R}^2$ *and the problem minimize* $(x + 2y - 3)^2$ *subject to* $(x - 2)^2 + (y - 1)^2 \leq 1$.

**Exercise 6.6.** *Solve the problem: minimize* $x^2+y^2-14x-6y$ *subject to* $x+y \leq 2$, $x + 2y \leq 3$.

**Exercise 6.7.** *Solve the problem: minimize* $x^2 - y$ *subject to* $y - x \geq -2$, $y^2 \leq x$, $y \geq 0$.

## SUMMARY OF NEW CONCEPTS AND RESULTS:

- convex optimality condition
- stationary point
- descent direction
- feasible direction method
- conditional gradient method
- the convex programming problem
- Karush-Kuhn-Tucker conditions (KKT conditions)
- Lagrangian multipliers

# Bibliography


[1] D.P. Bertsekas. *Nonlinear Programming*. Athena Scientific, 1999.

[2] V. Chvatal. *Linear programming*. W.H. Freeman and Company, 1983.

[3] G. Dahl. A note on diagonally dominant matrices. *Linear Algebra and its Applications*, 317:217–224, 2000.

[4] D.P. Bertsekas et al. *Convex Analysis and Optimization*. Athena Scientific, 2003.

[5] M. Grötschel and M.W. Padberg. Polyhedral theory. In E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan, and D.B. Shmoys, editors, *The traveling salesman problem*, chapter 8, pages 251–361. Wiley, 1985.

[6] J.-B. Hiriart-Urruty and C. Lemaréchal. *Convex analysis and minimization algorithms I*. Springer, 1993.

[7] R.A. Horn and C.R. Johnson. *Matrix analysis*. Cambridge University Press, 1991.

[8] D.G. Luenberger. *Linear and nonlinear programming*. Addison-Wesley, 1984.

[9] A.W. Marshall and I. Olkin. *Inequalities: Theory of Majorization and Its Applications*. Academic Press, New York, 1979.

[10] W.R. Pulleyblank. Polyhedral combinatorics. In Nemhauser et al., editor, *Optimization*, volume 1 of *Handbooks in Operations Research and Management Science*, chapter 5, pages 371–446. North-Holland, 1989.

[11] R.T. Rockafellar. *Convex analysis*. Princeton, 1970.

[12] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 44 of *Encyclopedia of Mathematics and its applications*. Cambridge University Press, Cambridge, 1993.

[13] A. Schrijver. *Theory of linear and integer programming*. Wiley, Chichester, 1986.

[14] E. Torgersen. *Comparison of Statistical Experiments*, volume 36 of *Encyclopedia of Mathematics and its applications*. Cambridge University Press, Cambridge, 1992.

[15] R.J. Vanderbei. *Linear programming: foundations and extensions*. Kluwer, 1996.

[16] R. Webster. *Convexity*. Oxford University Press, Oxford, 1994.

[17] G. Ziegler. *Lectures on polytopes*. Springer, 1995.

# Index