Mathematical optimization

Mandatory assignment

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Deadline: 14:30 - Thursday October 20, 2016

To get this assignment approved you need to solve about 50% of the problems correctly, and have made a reasonable attempt in most of the others. You are encouraged to discuss these problems with other students in the class, but you must state who you have worked with and write down your own solutions independently.

1. Show that $\log(n!) = \Theta(n \log n)$. (*Hint: Use some calculus.*)

2. Suppose that G is a (non-oriented) graph with no cycles. Show that if G has n vertices, m edges, and c connected components then $n = m + c$. Use this to prove that a graph G is a tree if and only if G has $n - 1$ edges and it is acyclic or connected.

3. Show that T is a minimum spanning tree if and only if any edge e of T has minimum cost among all the edges in the graph that join the two connected components of $T \setminus \{e\}$. Use this to show that Prim's algorithm for finding minimum spanning trees works.

4. Let G be a directed graph, and let s, t be two distinct vertices of G. Show that if a subset S of the edges is a cut (i.e., $S = \delta(R)$ for some $R \subset V$ with $s \in R$ and $t \notin R$) then there is no directed path in $G \setminus S$ from s to t. Moreover, show that any subset S of edges which is minimal with respect to this property (that there is no path from s to t in $G \setminus S$) must be a cut.

5. Use König's theorem to prove the following result.

   Hall's theorem: If G is a bipartite graph with vertex bipartition $V(G) = P \sqcup Q$, then G contains a matching of size $|P|$ if and only if for any subset A of P, its set of neighbors $N(A) := \{v \in Q : \{a, v\} \in E(G) \text{ for some } a \in A\}$ has size at least $|A|$.

6. Let $P \subset \mathbb{R}^n$ be a polyhedron of the form $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_i \cdot \mathbf{x} \leq b_i \text{ for } i = 1, 2, \ldots, m\}$. A *Chebychev center* of P is defined as the center of an Euclidean ball of maximum radius completely contained in P (it might not be unique). Express the problem of finding a Chebychev center of P as a linear optimization problem.

7. Let G be a directed graph with non-negative edge costs $(c_e)_{e \in E}$, and let s, t be two vertices of G. A vector $f \in \mathbb{R}^V$ is called a *feasible potential* if for any directed edge (v, w) of G we have $f_w \leq f_v + c_{(v,w)}$. The *cost* of the feasible potential is $c(f) := f_t - f_s$. Express the problem of finding a feasible potential of maximum cost as a linear optimization problem. Compute the dual linear optimization problem. Finally, use complementary slackness to show that the maximum cost of a feasible potential is equal to the minimum cost of a path from s to t.

8. A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ is called affinely independent if there are no scalars $\lambda_1, \ldots, \lambda_m$, at least one different from 0, such that $\sum_{j=1}^m \lambda_j = 0$ and $\sum_{j=1}^m \lambda_j \mathbf{v}_j = \mathbf{0}$. Show that for any $X \subset \mathbb{R}^n$, the maximum size of an affinely independent subset of X is equal to the dimension of the affine hull of X plus 1.

9. Let $P \subset \mathbb{R}^n$ be the polytope obtained as the convex hull of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ in $\mathbb{R}^n$. If $\mathbf{c} \in \mathbb{R}^n$, show that the face $\text{face}_{\mathbf{c}}(P)$ of P consisting of the points in P minimizing the functional $\mathbf{c} \cdot \mathbf{x}$ is equal to

$$\text{face}_{\mathbf{c}}(P) = \text{convex}\{\mathbf{v}_i : \mathbf{c} \cdot \mathbf{v}_i = \min_j (\mathbf{c} \cdot \mathbf{v}_j)\}.$$

10. The n-dimensional *cross-polytope* P_n is the polytope in $\mathbb{R}^n$ defined as

$$P_n := \text{convex}\{\mathbf{e}_1, -\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_2, \ldots, \mathbf{e}_n, -\mathbf{e}_n\} \subset \mathbb{R}^n.$$

Draw the cross-polytopes of dimension less than or equal to 3. For any $\mathbf{c} \in \mathbb{R}^n$, compute the face $\text{face}_{\mathbf{c}}(P_n)$ of P_n that minimizes the functional $\mathbf{c} \cdot \mathbf{x}$. What are all the faces of P_n, and how many are there in total?