MAT4300: Solutions to the exam fall 2010

Problem 1: The function $|f|$ is integrable, and $|\mathbf{1}_{[-n,n]}f|$ is bounded by $|f|$ for all $n \in \mathbb{N}$. Hence by the Lebesgue's Dominated Convergence Theorem

$$\lim_{n\to\infty} \int_{[-n,n]} f\, d\lambda = \lim_{n\to\infty} \int \mathbf{1}_{[-n,n]} f\, d\lambda = \int \lim_{n\to\infty} \mathbf{1}_{[-n,n]} f\, d\lambda = \int f\, d\lambda$$

Problem 2: a) For $n = 1$ there is nothing to prove, and for $n = 2$ this is just property (ii) in the definition. Assume the property holds for $n = k$, we shall show that it holds for $n = k + 1$:

$$I(f_1 + f_2 + \cdots + f_{k+1}) = I((f_1 + f_2 + \cdots + f_k) + f_{k+1}) =$$

$$= I(f_1 + f_2 + \cdots + f_k) + I(f_{k+1}) = I(f_1) + I(f_2) + \cdots + I(f_k) + I(f_{k+1})$$

where we first used property (ii) and then the induction hypothesis.

b) Since $g - f \in \mathcal{M}^+$, we have

$$I(g) = I((g - f) + f) = I(g - f) + I(f) \geq I(f)$$

since $I(g - f) \geq 0$.

c) We have to check that $\mu(\emptyset) = 0$ and that $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n\in\mathbb{N}} \mu(E_n)$ for all disjoint sequences $\{E_n\}$ of sets from $\mathcal{A}$.

For the first part, observe that

$$\mu(\emptyset) = I(\mathbf{1}_\emptyset) = I(0 \cdot \mathbf{1}_\emptyset) = 0 \cdot I(\mathbf{1}_\emptyset) = 0$$

where we have used property (i). For the second part, we note that

$$\mu(\bigcup_{n\in\mathbb{N}} E_n) = I(\mathbf{1}_{\bigcup_{n\in\mathbb{N}} E_n}) = I(\lim_{N\to\infty} \mathbf{1}_{\bigcup_{n=1}^N E_n}) = \lim_{N\to\infty} I(\mathbf{1}_{\bigcup_{n=1}^N E_n}) =$$

$$= \lim_{N\to\infty} I(\sum_{n=1}^N \mathbf{1}_{E_n}) = \lim_{N\to\infty} \sum_{n=1}^N I(\mathbf{1}_{E_n}) = \sum_{n=1}^\infty I(\mathbf{1}_{E_n}) = \sum_{n=1}^\infty \mu(E_n)$$

where we have used property (iii) to pull the limit outside I and part a) to get I inside the finite sum.

d) If $f = \sum_{i=1}^n \alpha_i \mathbf{1}_{E_i}$ is a positive, simple function

$$I(f) = I(\sum_{i=1}^n \alpha_i \mathbf{1}_{E_i}) = \sum_{i=1}^n I(\alpha_i \mathbf{1}_{E_i}) = \sum_{i=1}^n \alpha_i I(\mathbf{1}_{E_i}) = \sum_{i=1}^n \alpha_i \mu(E_i) = \int f\, d\mu$$

where we have used a), (i), c) and the definition of the integral for simple functions.

e) Let $\{f_n\}$ be an increasing sequence of simple functions converging to f. By Beppo Levi's Theorem, $\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$ and by property (iii), $I(f) = \lim_{n\to\infty} I(f_n)$. By the previous point, $\int f_n\,d\mu = I(f_n)$ and hence $\int f\,d\mu = I(f)$

Problem 3: a) The sets are equal if $\mathbf{a} = \mathbf{b}$. If one of the sequences is an extension of the other, then the set belonging to the the longer sequence is contained in the other. The only remaining possibility is that $a_i \neq b_i$ for some i, and in this case $C_{\mathbf{a}}$ and $C_{\mathbf{b}}$ are disjoint.

b) We have to check the three points in the definition of a semi-ring:

(i) $\emptyset \in \mathcal{C}$ by definition.

(ii) Assume $S, T \in \mathcal{C}$. According to a), the intersection $S \cap T$ is either empty or equal to either S or T. In both cases $S \cap T \in \mathcal{C}$.

(iii) Assume $S, T \in \mathcal{C}$. The only situation in a) where $S \setminus T$ is nonempty, is when T is contained in S. In this case $S = C_{\mathbf{a}}$ and $T = C_{\mathbf{b}}$ where $\mathbf{b}$ is an extension of $\mathbf{a}$. But then

$$S \setminus T = \bigcup_{\mathbf{e}\in E} C_{\mathbf{e}}$$

where E is the set of all other extensions of $\mathbf{a}$ with the same length as $\mathbf{b}$.

c) Obviously, $C_{\mathbf{a}} = C_{\mathbf{a}0} \cup C_{\mathbf{a}1}$. Any other cylinder set contained in $C_{\mathbf{a}}$ must be properly contained in either $C_{\mathbf{a}0}$ or $C_{\mathbf{a}0}$, and cannot make up all of $C_{\mathbf{a}}$ with just one other cylinder set.

d) From c) we know that if $C = C_{\mathbf{a}}$, then D and E must be $C_{\mathbf{a}0}$ and $C_{\mathbf{a}1}$. If $\mathbf{a}$ has length n, then $\rho(C) = 2^{-n}$, $\rho(D) = 2^{-(n+1)}$, $\rho(E) = 2^{-(n+1)}$, and hence $\rho(C) = \rho(D) + \rho(E)$.

For the general case, we use the induction hypothesis:

$P(k)$: If a cylinder set C is the disjoint union of k or fewer cylinder sets $C_1, C_2, \ldots, C_i$, then $\rho(C) = \rho(C_1) + \rho(C_2) + \cdots + \rho(C_i)$

We have already seen that $P(2)$ holds. Assume that $P(k)$ holds, and that the cylinder set C is the union of $k+1$ cylinder set:

$$C = C_1 \cup C_2 \cup \ldots \cup C_{k+1}$$

If $C = C_{\mathbf{a}}$, the sets $C_1, C_2, \ldots, C_{k+1}$ fall into two groups; those that are subsets of $C_{\mathbf{a}0}$, and those that are subsets of $C_{\mathbf{a}1}$. In each category, there are k or less sets, and by the induction hypothesis, $\rho(C_{\mathbf{a}0})$ and $\rho(C_{\mathbf{a}1})$ are the sum of ρ applied to their respective subsets. But then

$$\rho(C_{\mathbf{a}}) = \rho(C_{\mathbf{a}0}) + \rho(C_{\mathbf{a}1}) = \rho(C_1) + \rho(C_2) + \cdots + \rho(C_{k+1})$$

e) By Caratheodory's Extension Theorem we only need to check that $\rho(\emptyset) = 0$ and that whenever a disjoint, countable union $\bigcup_{n\in\mathbb{N}} C_n$ of sets in $\mathcal{C}$ happens to be in $\mathcal{C}$, then

$$\rho(\bigcup_{n\in\mathbb{N}} C_n) = \sum_{n\in\mathbb{N}} \rho(C_n)$$

The first condition is part of the definition of ρ, and the second follows from the claim and the previous point since a disjoint union $\bigcup_{n\in\mathbb{N}} C_n$ can only belong to $\mathcal{C}$ when it is actually finite.

f) Assume we have a potential winner $\mathbf{a}$ with extensions in $C \setminus \bigcup_{n=1}^{N} C_n$ for all N. Then either $\mathbf{a}0$ of $\mathbf{a}1$ (or both) must be a potential winner — if not, there would be numbers $N_0, N_1 \in \mathbb{N}$ such that $\mathbf{a}0$ had no extensions in $C \setminus \bigcup_{n=1}^{N_0} C_n$ and $\mathbf{a}1$ had no extensions in $C \setminus \bigcup_{n=1}^{N_1} C_n$, and then $\mathbf{a}$ would have no extensions in $C \setminus \bigcup_{n=1}^{N} C_n$ where $N = \max\{N_0, N_1\}$. Using this argument inductively, we get a sequence of potential winners, $\{\mathbf{a}_k\}$, each extending the previous. This sequence defines an element $\mathbf{a} = \{a_1, a_2, a_3, \ldots\} \in X$. For all N, $\mathbf{a} \in C \setminus \bigcup_{n=1}^{N} C_n$ (this is because each $\mathbf{a}_k$ has extensions in $C \setminus \bigcup_{n=1}^{N} C_n$, and since $C, C_1, \ldots, C_n$ are cylinder sets, all sequences that agree on sufficiently large initial segments, are either both inside $C \setminus \bigcup_{n=1}^{N} C_n$ or both outside). Consequently, $\mathbf{a} \in C \setminus \bigcup_{n=1}^{\infty} C_n$ and we have our contradiction.