Solution to exam in MAT3400/4400, Linear analysis with applications.
Exam date Monday, December 5, 2011.

Problem 1. The n-th Fourier coefficient of f for $n \neq 0$ is

$$\begin{aligned}
c_n(f) &= \int_{-\pi}^{\pi} f(x)e_{-n}(x)dx = \int_{-\pi}^{0} \frac{1}{\sqrt{2\pi}}(-e^{-inx})dx \\
&= -\frac{1}{\sqrt{2\pi}}\int_0^{\pi} e^{inx}dx = -\frac{1}{\sqrt{2\pi}}\left[\frac{e^{inx}}{in}\right]_0^{\pi} \\
&= \frac{1}{\sqrt{2\pi}in}(1-\cos(n\pi)) \\
&= \frac{1}{\sqrt{2\pi}in}(1-(-1)^n) \\
&= \begin{cases} 0 & \text{if } n = 2k, k \in \mathbb{Z} \\ \frac{2}{\sqrt{2\pi}in} & \text{if } n = 2k+1, k \in \mathbb{Z}. \end{cases}
\end{aligned}$$

We have $c_0(f) = \int_{-\pi}^{0} \frac{1}{\sqrt{2\pi}}(-1)dx = -\frac{\sqrt{2\pi}}{2}$. The Fourier series is

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} c_n(f)e_n(x) &= c_0(f) + \sum_{n=1}^{\infty} c_n(f)e_n(x) \sum_{n=1}^{\infty} c_{-n}(f)e_{-n}(x) \\
&= -\frac{1}{2} + \sum_{n=1, n \text{ odd}}^{\infty} \frac{1}{\pi in}(e^{inx} - e^{-inx}) \\
&= -\frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1}.
\end{aligned}$$

Problem 2. The operator K is an integral operator where the kernel $\kappa(x,y)$ is a continuous function on $[0,1]\times[0,1]$ such that $\overline{\kappa(y,x)} = \kappa(x,y)$. By the spectral theorem for compact self-adjoint operators on Hilbert space we know that the eigenvalues of K are real. We need to find the real numbers α for which $Kf = \alpha f$ has non-trivial solutions. Now $Kf = \alpha f \iff Kf(x) = \alpha f(x)$ for all $x \in [0,1]$ so, equivalently, we solve $\frac{1}{3}\int_0^1 f(y)dy - x\int_0^1 yf(y)dy = \alpha f(x)$ for all $x \in [0,1]$.

Denote $c_1 = \int_0^1 f(y)dy$ and $c_2 = \int_0^1 yf(y)dy$. Thus need to find f different from the zero-function such that $\frac{1}{3}c_1 - c_2x = \alpha f(x)$, for all x.

Suppose first $\alpha \neq 0$. Then $f(x)$ has form $-c_2/\alpha x + c_1/3\alpha$. Inserting this expression for f in c_1 and c_2 gives

$$c_1 = \int_0^1 f(y)dy = \int_0^1 (-\frac{c_2}{\alpha}y)dy + \int_0^1 \frac{c_1}{3\alpha} = -\frac{c_2}{2\alpha} + \frac{c_1}{3\alpha}$$

and

$$c_2 = \int_0^1 yf(y)dy = -\frac{c_2}{3\alpha} + \frac{c_1}{6\alpha}.$$

We obtain a system of equations

$$c_1(1 - \frac{1}{3\alpha}) + c_2\frac{1}{2\alpha} = 0$$
$$c_1\frac{1}{6\alpha} - c_2(\frac{1}{3\alpha} + 1) = 0.$$

We want to find f different from the zero function so we look for non-trivial constants c_1, c_2. The homogeneous system above has a non-zero solution if and only if the determinant is 0. This condition gives $36\alpha^2 = 1$, so possible values of α that give non-zero f are $\alpha_1 = 1/6$ and $\alpha_2 = -1/6$.

The case $\alpha_1 = 1/6$ gives $c_1 = 3c_2$ so eigenvectors for α_1 are functions of the form $f(x) = 6c_2(-x+1)$ with $c_2 \neq 0$. The case $\alpha = -1/6$ gives $c_1 = c_2$ so eigenvectors are $f(x) = 2c_2(3x-1)$ with $c_1 \neq 0$.

Now suppose $\alpha = 0$. Then $Kf(x) = 0$ for all $x \in [0,1]$ implies $\frac{1}{3}c_1 - xc_2 = 0$ for all $x \in [0,1]$, so taking $x = 0$ then $x = 1$ gives $c_1 = c_2 = 0$. If we let $g(y) = y$ be the identity function on $[0,1]$, we have found that any non-zero f in the orthogonal complement of $\text{span}\{1, g\}$ is an eigenvector corresponding to 0.

For problem 2b, note that since 1 is not an eigenvalue for K, by the Fredholm alternative the equation $f = Kf + g$ has a unique solution for each $g \in H$.

Problem 3.

For problem 3a we multiply u on both sides of $Lu = \alpha u$. This gives the equation $uu'' + \alpha u^2 - qu^2 = 0$. We integrate this equation from 0 to 1 and use the assumption $u(1)u'(1) - u(0)u'(0) \leq 0$ to get exactly the claimed inequality. Since $q(x) \geq 0$ for all $x \in [0,1]$ it follows that $\alpha \geq 0$.

For 3b we have $Lu = -u''$ on $\mathcal{D}(L) = \{u \in C^2([0,1]) \mid u'(0) = 0, u'(1) = 0\} \subset L^2([0,1])$. A real number α is eigenvalue for L if $Lu = \alpha u$, equivalently $u'' + \alpha u = 0$. The characteristic equation is $r^2 + \alpha = 0$. The discriminant is -4α. By 3a, we know that $\alpha \geq 0$.

Case 1: $\alpha = 0$. Then $r = 0$, so the solutions are of form $u(x) = A + Bx$, and $u \in \mathcal{D}(L)$ implies $B = u'(0) = 0$. Hence $u(x) = 1$ is normalised eigenvector corresponding to $\alpha = 0$.

Case 2: $\alpha > 0$. The solutions of $r^2 + \alpha = 0$ are $i\sqrt{\alpha}$ and $-i\sqrt{\alpha}$, so $u(x) = A\cos\sqrt{\alpha}x + B\sin\sqrt{\alpha}x$. Then $u'(0) = 0$ gives $B = 0$, so $u(x) = A\cos\sqrt{\alpha}x$. We look for $u \neq 0$ so from $u'(1) = 0$ we get $\sin\sqrt{\alpha} = 0$. Thus $\sqrt{\alpha} \in \{n\pi : n > 0\}$. We obtain $\alpha_n = (n\pi)^2$, $n = 1, 2, \ldots$ as non-zero eigenvalues of L. A normalized eigenvector for α_n is $u_n(x) = \sqrt{2}\cos(n\pi x)$.

Problem 4.

Problem 4a: Since g is continuous on $(0,\infty)$ it is Borel measurable. For $a > 0$ write $(0,a] = \cup_{n=1}^{\infty}[\frac{1}{n}, a]$. The function $\chi_{[\frac{1}{n},a]}g$ for $n \geq 1$ is measurable and non-negative. The sequence $\chi_{[\frac{1}{n},a]}g$ is increasing with pointwise limit the function $g\chi_{(0,a]}$. By the monotone convergence theorem we have

$$\int_{(0,a]} g d\lambda = \lim_n \int \chi_{[\frac{1}{n},a]} g d\lambda = \lim_n \int_{\frac{1}{n}}^{a} g(t)dt = \lim_n (\frac{a^p}{p} - \frac{1}{pn^p}),$$

which is a^p/p, as wanted. We used that for the continuous function g on bounded intervals $[\frac{1}{n}, a]$, the Lebesgue integral is the same as the Riemann integral.

Problem 4b: In case $a = 0$, so that $f = 0$, we have $\phi_f(t) = \mu(\{x : 0 > t\}) = \mu(\emptyset) = 0$. Since also $\chi_{(0,a)} = \chi_\emptyset = 0$, the two claimed identities are valid. For $a > 0$ we have

$$\{x \in X : f(x) > t\} = \{x \in X : \chi_A(x) > \frac{t}{a}\} = \begin{cases} A & \text{if } 0 < t < a \\ 0 & \text{if } a \leq t. \end{cases}$$

This shows $\phi_f(t) = \mu(A)\chi_{(0,a)}$. The second identity follows from 4a and the computations

$$p \int_{(0,\infty)} g\phi_f d\lambda = p \int \chi_{(0,\infty)} g\mu(A)\chi_{(0,a)} d\lambda = p\mu(A) \int g\chi_{(0,a)} d\lambda$$
$$= p\mu(A) \int g\chi_{(0,a]} d\lambda = a^p \mu(A).$$

Problem 4c: If $f = 0$ then $\phi_f = 0$ as in problem 4b. Then $\int_{(0,\infty)} g\phi_f d\lambda = 0 = \int f^p d\mu$. Assume f is not the zero-function. Then

$$\{x \in X : f(x) > t\} = \{x : \sum_{j=1}^{m} a_j \chi_{A_j}(x) > t\} = \bigcup_{j=1}^{m} \{x \in X : a_j \chi_{A_j}(x) > t\}.$$

If we denote $B_j = \{x \in X : a_j\chi_{A_j}(x) > t\}$, then $B_j = \emptyset$ if $a_j = 0$ or $0 < a_j \leq t$ and $B_j = A_j$ if $0 < t < a_j$. In particular, $B_j \cap B_k = \emptyset$ if $j \neq k$. By problem 4b we have $\phi_{f_j} = 0$ if $a_j = 0$ and $\phi_{f_j} = \mu(A_j)\chi_{(0,a_j)}$ if $a_j > 0$. Since μ is additive, we get

$$\phi_f(t) = \mu(\bigcup_{j=1}^{m} B_j) = \sum_{j=1}^{m} \mu(B_j) = \sum_{j=1}^{m} \phi_{f_j}(t).$$

This shows that ϕ_f is Borel measurable (because sums of measurable functions are again measurable). By linearity of integral and part 4b we also get

$$p \int_{(0,\infty)} g\phi_f d\lambda = \sum_{j=1}^{m} p \int_{(0,\infty)} g\phi_{f_j} d\lambda = \sum_{j=1}^{m} a_j^p \mu(A_j),$$

which is exactly $\int_X f^p d\mu$ by the definition of the integral.

Problem 4d: Let $\{f_n\}_{n\geq 1}$ be a sequence of simple, nonnegative measurable functions such that $f_n \leq f_{n+1}$ for all n and $f_n \to f$. Then also f_n^p increases to f^p because $p \geq 1$. We claim that the sequence ϕ_{f_n} increases to ϕ_f. Since for every $t > 0$ we have

$$B_n = \{x \in X : f_n(x) > t\} \subset \{x \in X : f_{n+1}(x) > t\} = B_{n+1}$$

and μ is a measure, it follows that $\phi_{f_n}(t) \leq \phi_{f_{n+1}}(t)$ for all $n \geq 1$ and all t. Since also $f_n \leq f$ we get similarly that $\phi_{f_n} \leq \phi_f$ for all $n \geq 1$. Thus $\lim \phi_{f_n} \leq \phi_f$.

Let $t \in (0, \infty)$. Let $x \in X$ such that $f(x) > t$. Since $f_n(x) \to f(x)$ there must be an $n \geq 1$ such that $t < f_n(x) < f(x)$ (otherwise, $\lim_n f_n(x) \leq t$). So $x \in \cup_n B_n$. It follows that $\{x \in X : f(x) > t\} \subset \cup_n B_n$. Hence, by continuity of μ, we have

$$\phi_f(t) = \mu(\{x \in X : f(x) > t\}) \leq \mu(\cup_n B_n) = \lim_n \mu(B_n) = \lim \phi_{f_n}(t).$$

It follows that $\phi_f \leq \lim \phi_{f_n}$, so in fact $\phi_f = \lim \phi_{f_n}$. Since each ϕ_{f_n} is Borel measurable by 4c, so must ϕ_f be. Note that $g\phi_f = \lim g\phi_{f_n}$. Now (1) follows from two applications of the monotone convergence theorem (in two different measurable spaces) and 4c:

$$\begin{aligned}
\int_x f^p d\mu &= \lim_n \int_X f_n^p d\mu \\
&= \lim_n p \int_{(0,\infty)} g\phi_{f_n} d\lambda \\
&= p \int_{(0,\infty)} \lim_n g\phi_{f_n} d\lambda \\
&= p \int_{(0,\infty)} g\phi_f d\lambda.
\end{aligned}$$