Mathematical Finance
UiO-MAT4730/9730
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Assignment

The assignment is mandatory to take part to the final examination and it is a good exercise to test ones knowledge. Each student must hand in the assignment paper within deadline at the administration office of the Department of Mathematics, University of Oslo, 7th floor, Niels Henrik Abel building.

Deadline: *October 27, 2016 at 14:00.*

Task 1

Consider a complete, standard financial market $\mathcal{M} = (r(\cdot), b(\cdot), \sigma(\cdot), S(0))$, governed by the stochastic differential equation

$$dS_0(t) = r(t) S_0(t) dt, \quad S_0(0) = 1,$$
$$dS_n(t) = b(t) S_n(t) dt + \sum_{d=1}^{D} \sigma_{nd}(t) S_n(t) dW^{(d)}(t), \quad S_1(0) > 0,$$

for $n = 1, ..., N$. Recall that given an initial endowment $x \geq 0$, we say that a consumption/investment pair (c, π) is admissible at x if the corresponding wealth process

$$X^{x,c,\pi}(t) \geq 0, \qquad P - a.s,$$

for all $t \in [0, T]$ and we denote it by $(c, \pi) \in \mathcal{A}(x)$. Prove that a necessary condition for $(c, \pi) \in \mathcal{A}(x)$ is that

$$\mathbb{E}\left[H_0(T) X^{x,c,\pi}(T) + \int_0^T H_0(t) c(t) dt\right] \leq x$$

where $H_0 = \{H_0(t)\}_{t\in[0,T]}$ is the state-price density process. Give a detailed proof.

Task 2

Consider the market model in Task 1. Assume the existence of a risk-neutral measure Q and consider self-financing investments with null consumption. Show that the following two definitions are equivalent.

(1) **Definition 1.** A contingent claim F is replicable on the financial market if there exists an admissible portfolio π such that the final value $X^{x,\pi}(T) = F, P$-*a.s.* . Where $x = \mathbb{E}_Q\left[S_0^{-1}(T) F\right]$ and S_0 is the money market account used as a numeraire.

(2) **Definition 1.** A contingent claim F is replicable on the financial market if there exists a self-financing portfolio π such that the final value $X^{x,\pi}(T) = F, P$-*a.s.* and the discounted value process

$$\bar{X}^{x,\pi}(t) = \frac{X^{x,\pi}(t)}{S_0(t)}$$

is a Q-martingale.

Task 3

Consider a market model of two assets on the time horizon $[0,T]$ with $T>0$:

$$dS_0(t) = rS_0(t)\,dt, \quad S_0(0) = 1,$$
$$dS_1(t) = bS_1(t)\,dt + \sigma S_1(t)\,dW(t), \quad S_1(0) > 0,$$

where r, b and $\sigma > 0$ are real constants constants and W is a standard Brownian motion. Find the optimal consumption/investment scheme $(c^*, \pi^*) \in \mathcal{A}(x)$ with $x > 0$ such that

$$V(x) := \sup_{(c,\pi)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^T \frac{1}{\gamma} e^{-\beta t} c(t)^\gamma\,dt\right] = \mathbb{E}\left[\int_0^T \frac{1}{\gamma} e^{-\beta t} c^*(t)^\gamma\,dt\right],$$

where $\beta > 0$ and $0 < \gamma < 1$. Recall that $\mathcal{A}(x)$ represents the set of admissible schemes.