7.10

Two successive displacements take place, which are denoted

i) $\underset{\sim}{x}' = \underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x})$

ii) $\underset{\sim}{x}'' = \underset{\sim}{x}' + \underset{\sim}{u}'(\underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x})) = \underset{\sim}{x}' + \underset{\sim}{u}'(\underset{\sim}{x}')$

We know that both displacements give rise to small strains, telling us that Cauchy's strain tensor is sufficient.

We now express $u'(\underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x}))$ using a first-order Taylor expansion:

$$\underset{\sim}{u}'(\underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x})) \simeq \underset{\sim}{u}'(\underset{\sim}{x}) + (\underset{\sim}{\nabla}\underset{\sim}{u}'(\underset{\sim}{x}))^T \cdot \underset{\sim}{u}(\underset{\sim}{x})$$

We know that the second term is much smaller than the first, because $(\underset{\sim}{\nabla}\underset{\sim}{u}'(\underset{\sim}{x}))^T$ is small. Thus:

$$\underset{\sim}{\nabla}\underset{\sim}{u}'(\underset{\sim}{x}') \simeq \underset{\sim}{\nabla}\left[\underset{\sim}{u}'(\underset{\sim}{x}) + (\underset{\sim}{\nabla}\underset{\sim}{u}'(\underset{\sim}{x}))^T \cdot \underset{\sim}{u}(\underset{\sim}{x})\right] \simeq \underset{\sim}{\nabla}\underset{\sim}{u}'(\underset{\sim}{x})$$

We define the total displacement $\underset{\sim}{u}''$ by

$$\underset{\sim}{x}'' = \underset{\sim}{x} + \underset{\sim}{u}''(\underset{\sim}{x})$$

$$\Rightarrow \underset{\sim}{u}''(\underset{\sim}{x}) = \underset{\sim}{u}(\underset{\sim}{x}) + \underset{\sim}{u}'(\underset{\sim}{x}') \simeq \underset{\sim}{u}(\underset{\sim}{x}) + \underset{\sim}{u}'(\underset{\sim}{x})$$

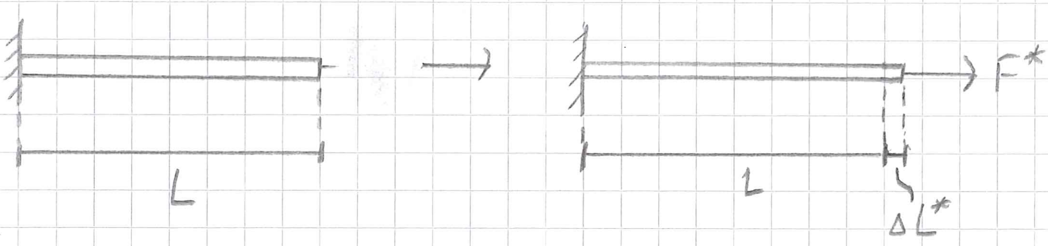
The final strain tensor then becomes

$$\underset{\sim}{\varepsilon} = \frac{1}{2}\left(\underset{\sim}{\nabla}\underset{\sim}{u}'' + \underset{\sim}{\nabla}\underset{\sim}{u}''^T\right)$$

$$= \frac{1}{2}\left(\underset{\sim}{\nabla}(\underset{\sim}{u} + \underset{\sim}{u}') + \underset{\sim}{\nabla}(\underset{\sim}{u} + \underset{\sim}{u}')^T\right)$$

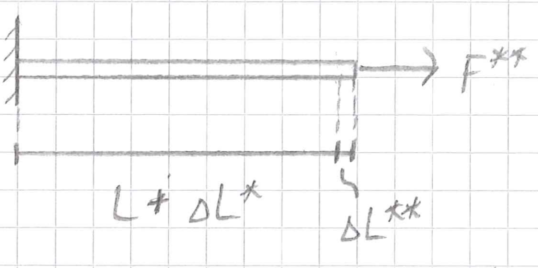
Physical motivation in one dimension:

Let us assume we have a rod of length L, which is subjected to a force F^*, giving the rod an extension ΔL^*.



We know that $\varepsilon^* = \dfrac{\Delta L^*}{L}$

We then increase the applied force, to get an additional extension ΔL^{**}



with strain

$$\varepsilon^{**} = \frac{\Delta L^{**}}{L + \Delta L^*} = \frac{\Delta L^{**}}{L} \cdot \frac{1}{1 + \frac{\Delta L^*}{L}}$$

$$\simeq \frac{\Delta L^{**}}{L}\left(1 - \frac{\Delta L^*}{L}\right)$$

$$\simeq \frac{\Delta L^{**}}{L}$$

So the total strain becomes

$$\varepsilon = \varepsilon^* + \varepsilon^{**} \simeq \frac{\Delta L^* + \Delta L^{**}}{L}$$