1 Life and state-dependent insurance

1.1 Introduction

Life and pension insurance are arrangements for which payments streams are determined by *states* occupied by inviduals. These contracts, operated according to pre-determined rules, typically last for a long time, even up to half a century and beyond. A simple example is an agreement where an account first is built up and then harvested after a certain date. At first glance this is only a savings account, but *insurance* is put into it by including randomness due to how long people live. When such accounts are managed for many individuals simultaneously, it becomes possible to balance life cycles against one another so that short lifes (for which savings are not used up fully) partially finance long ones. There is much sense in this. In old age benefits do not stop after a certain agreed date, but may go on until the recepient dies.

Pension schemes like these come with an *active*, premium-paying state first and then the *retirement* where the savings are utilized. Term insurance makes use of the account for a one-time payment to beneficiaries upon the death of the policy holder. The two forms may be combined, and there are are also other variants. An important distinction is between **defined benefit** (DB) and **defined contributions** (DC) schemes. With the former the starting point is certain economic rights after retirement which determine how much money it takes to sustain them whereas it is the other way around with DC arrangements. Now the retirement benefit follows from the earlier build-up of the account. Note the difference in terms of inflation and financial risk. DB arrangenments set a *target* to be met (although it migth depend on inflation and on the future wage level) whereas members of DC schemes carry financial risk themselves. A more complex type of treaties is the disability scheme introduced in Section 6.6. Now the individuals may move back and forth between several states. Multi-state arrangements of this kind will be treated in Sections 12.5 and 12.6.

This chapter deals with the *traditional* part of the methodology where the so-called **technical** rate of interest r is treated as given and where benefits are not linked to economic variables like inflation or wage level. This creates a very orderly system of pricing and risk evaluation which must be mastered before the complications of the financial side are introduced into the set-up in Chapter 15.

1.2 The anatomy of state-dependent insurance

Introduction

All payments streams in this chapter run over a fixed time sequence $t_k = kh$ for $k = 0, 1, \ldots$, and there is a fixed rate of interest r adapted to h. An important, elementary concept is that of an **annuity**. This is a fixed and constant payment stream, due at each t_k. Suppose it is of size 1, run over K time steps and is **in arrears**. Its present value is then

$$v + v^2 + \ldots + v^K = v\frac{1 - v^K}{1 - v}. \tag{1.1}$$

Here $v = 1/(1 + r)$ is the discount factor, introduced to make formulae of this chapter neater. Schemes "in arrears" take payments at the termination of each period (none at $t_0 = 0$). The corresponding version **in advance** has present value

$$1 + v + \ldots + v^{K-1} = \frac{1 - v^K}{1 - v}. \tag{1.2}$$

Now amounts are received at the start of each period.

Such payment streams may be broken off by death which suggests that their value should be adjusted for mortality. That is carried out in Section 12.4. A more general viewpoint is to link payment to the underlying **state** of the individual. We then draw on the **process** $\{C_l\}$ of Section 6.6 which defines the state of the individual at age l as C_l. Suppose l_0 is the the age at the start of the scheme. Age after k time steps is then $l_0 + k$, and payment at that time is denoted

$$\zeta_k = s_l(C_l) \qquad \text{where} \qquad l = l_0 + k. \tag{1.3}$$

Here $s_l(i)$ is a **payment function** depending on l. For example, we might have

$$\underset{\textit{contributing premium}}{s_l(0) = -\pi} \qquad \underset{\textit{receiving benefit}}{s_l(0) = s} \qquad \underset{\textit{dead}}{s_l(1) = 0,}$$

contributions to the scheme being counted *negative*. This is an ordinary, individual pension treaty where premium π is contributed early and utilized as benefit s later. Such arrangements are dicussed in Section 12.4. What we need at this stage is the very idea that payments streams, influenced by the life cycle of the individual, are stochastic.

There is nothing unique in the representation (1.3). The pension arrangement above placed all individuals in state 0 while alive, but it also perfectly appropriate to pass them to a different state at retirement (that was done in Section 6.6). Of course, the function $s_l(i)$ mut be modified accordingly. The crucial point is that the stream of *payments* is fixed by the stream of *states*. This relationship is determined by the contract, and there is no uncertainty in it. Or so we assume throughout the present chapter. In reality benefits might be linked to future development of inflation or the wage level. This requires stochastic models from finance and economics and is discussed in Chapter 15.

Equivalence pricing

Suppose the contract is set up at time $t_0 = 0$. The present value of all the payments are then

$$\text{PV}_0 = \sum_{k=0}^{\infty} v^k \zeta_k \qquad \text{with expectation} \qquad E(\text{PV}_0) = \sum_{k=0}^{\infty} v^k E(\zeta_k) \qquad . \tag{1.4}$$

One way to plan the arrangement is to make

$$E(\text{PV}_0) = 0 \qquad \textbf{(equivalence condition)}. \tag{1.5}$$

This is known as the principle of equivalence. All payments into and out of the scheme are then balanced so that their present value is zero. If the insurer is a company, there is no profit in this (and no expenses are covered), but obviously that comes down to the specification of v (or r). A private company would expect surplus returns on the management of the assets.

Equivalence can be used to adjust premium (paid early) to match given benefits later. Consider a simple pension scheme, entered at age l_0. With π and s being premium and benefit, we have

$$\underset{\textit{while saving}}{E(\zeta_k) = -\pi\, {}_kp_{l_0}} \qquad \text{and} \qquad \underset{\textit{while drawing benefit}}{E(\zeta_k) = s\, {}_kp_{l_0},}$$

where ${}_kp_{l_0}$ is the probability of surviving the next k periods. These life table probabilitities were introduced in Section 6.2, and there will be more in Section 12.3 below. The expressions are intutive (see e.g. Section 12.4). Suppose retirement starts at l_r, lasting until the end of life. With payments in advance the expected present value is

$$E(\mathrm{PV}_0) = -\pi \sum_{k=0}^{l_r-l_0-1} v^k\, {}_kp_{l_0} + s \sum_{k=l_r-l_0}^{\infty} v^k\, {}_kp_{l_0},$$

and the equivalence condition yields an equation that can be solved for π. Arrangements of this kind are discussed in Section 12.4.

The reserve

Consider a contract k time units after it has been set up. The present value of the remaining payments immediately before payment k is

$$\mathrm{PV}_k = \sum_{i=0}^{\infty} v^i \zeta_{k+i} \qquad \text{with expectation} \qquad V_k = E(\mathrm{PV}_k | C_{l_0+k}). \tag{1.6}$$

Here the definition of V_k is very much the same as the expected present value in (1.5) above, except that it has been highlighted that it is *conditional* quantity. When t_k is reached, the underlying state C_{l_0+k} is known, and we may regard V_k as the **value** of the contract at that time. The way treaties are designed usually ensures that $V_k > 0$. A client renoncing all future rights in exchange for a lump sum, may be paid the amount V_k, and neither party would then lose on average. Of course, the value V_k is uncertain *prior to* t_k, since there are then several possibilitities for the state C_{l_0+k}[1].

The quantity V_k is also known as the **reserve** at time k and typically enters the balance sheets of the companies. Is there a connection to to the savings value of the *preceding* payments? Those have been $\zeta_0, \ldots, \zeta_{k-1}$, and with interest earnings added the account at time k becomes

$$V_k' = -\sum_{i=0}^{k-1} (1+r)^i \zeta_i. \tag{1.7}$$

Here the minus sign captures that the first payments are normally premium, counted negative (and making V_k' positive). The savings value fails to include risk. For example, suppose an individual dies early. In many schemes the company then simply takes over the account, and that typically makes

$$V_k' < V_k;$$

how will be seen in Section 12.4. But the savings values V_k' and risk-adjusted values V_k are under the equivalence condition equal on *average* i.e.

$$E(V_k') = E(V_k), \tag{1.8}$$

[1]This defintion of value is only valid when $\{C_l\}$ is a Markov chain; see Section 6.6. The point is that all information about future fluctatuations of the life cycle is then carried by the last state C_l. For more general defintions see the reading list at the end of the chapter All models used in this book are Markovian.

as is proved in Section 12.7.

The portfolio viewpoint
The reserve V_k in (1.6) is the company liability under the agreement and becomes the *portfolio* liability when added over all contracts. This is only an expectation; the present values of the actual payments will not be the same, but typically the discrepancy is too small to matter much; see Section 3.4. Relying on average values is standard in life insurance.

It is useful to examine how liabilities distribute over time so that they can be managed jointly with assets. Let $\{C_{jl}\}$ be the life cycle process of individual j, s_{jl} his payment function and l_j his present age. All those would be available on file. The net liability to be settled k time units ahead is then

$$\mathcal{X}_k = \sum_{j=1}^{J} E(\zeta_{jk}|C_{jl_j}) \tag{1.9}$$

where ζ_{jk} is the payment for policy holder j at time t_k, a stochastic quantity. What counts is the expected value *given* the state $\{C_{jl_j}\}$ today. For liability on *portfolio* level all those must be added. How this quantities are computed in practice will be demonstrated in several examples throughout this chapter. If all $\mathcal{X}_k$ are added and discounted, we end up with the portfolio reserve

$$\mathrm{PV}_0 = \sum_{k=0}^{\infty} v^k \mathcal{X}_k; \tag{1.10}$$

see also Exercise ?.

1.3 Survival modelling

Introduction
The state process $\{C_k\}$ must be described in probabilistic terms. This draws on survival modelling, a subject briefly touched in Section 6.2. All evaluations of insurance liabilities in this book are in discrete time, and we count how long people live as Lh, where L is an integer and $h > 0$ is fixed. The link to a continuous viewpoint is discussed below. The probability distribution of L is called a **life table**. We shall, in particular, be concerned with the **survival** probabilities

$${}_kp_l = \Pr(L \geq l + k|L \geq l)$$

introduced in Section 6.2. These define how likely it is for a person of age lh to live on for a period of at least kh more. The one-step version ${}_1p_l = p_l$ generates the entire life table since

$${}_{k+1}p_l = p_{l+k} \cdot {}_kp_l, \qquad k = 0, 1, \ldots \qquad \text{starting at} \qquad {}_0p_l = 1. \tag{1.11}$$

This recursion was proved in Section 6.2. An obvious extension is

$${}_{k+i}p_l = {}_ip_{l+k} \cdot {}_kp_l; \tag{1.12}$$

i.e the product of the probabilities of surviving periods of length k and i equals the probability of surviving both. A related quantity is the probability of dying during the k period. This is denoted

$${}_kq_l = \Pr(k + l - 1 \leq L \leq l + k|L \geq l)$$

and enjoys a simple relationship to the survival probabilities. Indeed,

$$ {}_kq_l = (1 - p_{l+k-1}) \cdot {}_{k-1}p_l, \qquad k = 1, 2, \ldots \tag{1.13} $$

which is obvious upon the definitions.

The purpose of this section is to introduce issues relating to the modelling and use of life tables. Women live longer than men, and we need separate tables.

Modelling through intensities

How long people live is a continuous random variable Y and is often described mathematically through the so-called **hazard rate**

$$ \mu(y) = \frac{f(y)}{1 - F(y)}. $$

Here $f(y)$ and $F(y)$ are density and distribution function of Y. Hazard rates define $\mu(y)h$ as the probability of dying during a small time interval of length h. This is actually the time-hetereogenous Poisson process of Section 8.2, except for the incident (death) now occurring only once (!). But this does not alter the probability of *no event* which must be as stated in Section 8.2. Indeed, no event in the interval from y_0 to y_1 means that an individual of age y_0 lives longer than y_1, and we may from (**??**) conclude that

$$ \Pr(Y \geq y_1 | Y \geq y_0) = \exp\left(-\int_{y_0}^{y_1} \mu(s)\, ds\right) \qquad y_1 > y_0.. \tag{1.14} $$

Alternatively, note that

$$ \mu(y) = -\frac{d\log\{1 - F(y)\}}{dy} $$

since $f(y)$ is the derivative of $F(y)$. Hence

$$ 1 - F(y) = \exp\left(-\int_{y}^{\infty} \mu(s)\, ds\right) \tag{1.15} $$

from which (1.14) can be deduced; see Exercise 12.?. The life table follows by inserting $y_0 = lh$ and $y_1 = (l + k)h$ in (1.14), which yields

$$ {}_kp_l = \exp\left(-\int_{lh}^{(l+k)h} \mu(y)\, dy\right), \qquad k = 0, 1, 2, \ldots \tag{1.16} $$

This tells us how life tables come from intensity functions, a line that will be pursued below. Extensions to general state processes will be given during the last part of the chapter.

The Gomperz-Makeham model

The most popular mathematical description of mortality intensities goes back to 1860 (!) and bears the name of Gomperz and Makeham who made separate contributions. This is a parametric curve of the form

$$ \mu(y) = \theta_0 + \theta_1 \exp(\theta_2 y), \tag{1.17} $$

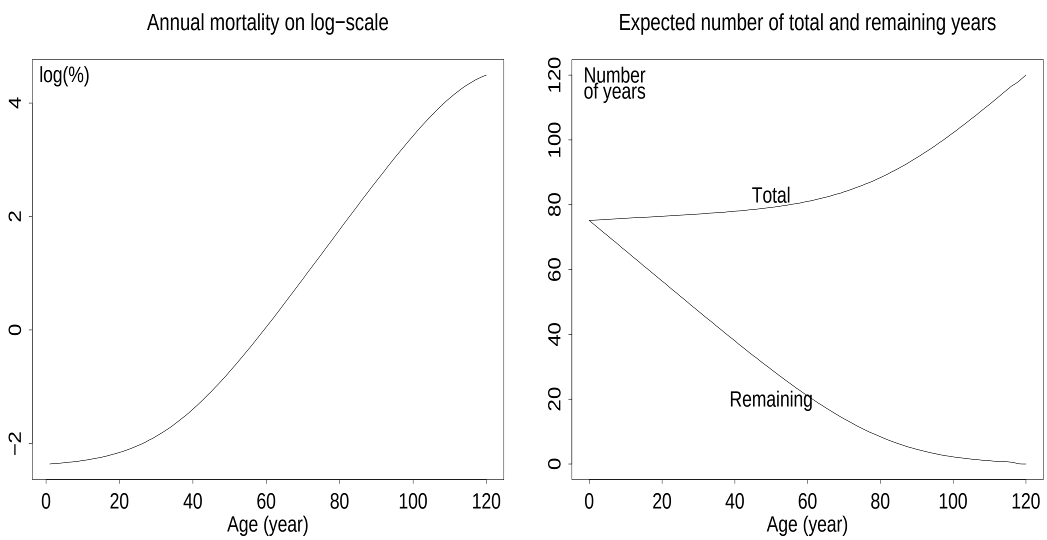


Figure 12.1 A Gomperz-Makeham model (parameters in text). On the left: **Logarithm** *of annual mortalities in %. On the right: Expected total and remaining number of years given the age.*

where θ_0, θ_1 and θ_2 are parameters. By integrating out the exponential in (1.16) this yields for the survival probabilites

$$_kp_l = \exp\left(-\theta_0 kh - \frac{\theta_1}{\theta_2}(e^{\theta_2(k+l)h} - e^{\theta_2 lh})\right) \tag{1.18}$$

The one-year mortalities $q_l = 1 - p_l$ have been ploted in in Figure 12.1 left for

$$\theta_0 = 0.0009, \qquad \theta_1 = 0.000044, \qquad \theta_2 = 0.09761$$

The average length of life is then about 75 years, plausible for males in many developed countries. Note that the quantity on the vertical axis is the *logarithm* of q_l (otherwise the resolution would have been very poor). The curve crosses zero at about 60 years. This means that the likelihood of a 60 year old male dying the coming year is about 1%, and the probabilities rise rapidly afterwards. This model will be used for laboratory experiments later[2].

Expected survival

The parameters of a Gomperz-Makeham model tell us little about how long people live on average or, more generally, about how many expected *remaining* years they have. A mathematical expression (Exercise ?) is

$$E(Y - y_0 | Y \geq y_0) = \frac{\int_{y_0}^{\infty}\{1 - F(y)\}\,dy}{1 - F(y_0)}, \tag{1.19}$$

which usually must be evaluated numerically. A discret version (Exercise 6.2.6) is

$$E(L - l_0 | L \geq l_0) = \sum_{k=1}^{\infty} {}_kp_{l_0}, \tag{1.20}$$

[2]It has been in use by the insurance industry in Norway since 1963 (!) and still is (2005).

which again takes a computer to calculate. The following recursion combines (1.11) and (1.20).

Algorithm 12.1 Expected remaining survival

0 Input: $\{p_l\}$, l_0, $k^{\max}$, h

1 $P \leftarrow 1$ and $E \leftarrow 0$ %*Initial survival probability and expectation*

2 For $k = 1, 2, \ldots, k^{\max}$ do %*For example, choose* $k^{max} = 120/h - l_0$

3 $\quad P \leftarrow Pp_l \quad$ and $\quad E \leftarrow E + P$ %*Here* P *equals* ${}_k p_{l_0}$

4 Return $E \leftarrow Eh$ %*Expected remaing life length*

The algorithm returns an approximation to the expected number of remaining years of an individual who has reached $l_0 h$. Lowering the time incement h increases accuracy. How this quantity behaves under the Gomperz-Makeham model above is shown in Figure 12.1 right, using $h = 0.01$ years. At 60 an indvidual has a lttle more than 20 years left on average. Of course, as a person grows older, the expected *total* number of years goes up.

Using historical data

Life tables must be estimated from historical data. That can be made be quite complicated, since most developed countries have seen a persistent trend towards people living longer. Section 15.2 offers a sketchy treatment. Here the issue is ignored. Historical data are then the number of people of age l at some point during the period of observation. A person of age l in the beginning then contributes to n_l the first year, to n_{l+1} the second year[3] and so on. When counted over all individuals and all years we obtain a statistical record for which

$$n_l \geq n_{l+1} \geq n_{l+1} \geq \ldots$$

and where n_{l+1} is conditionally binomial given n_l. The procedure has to be run separately for men and women, since their survival probabiltites deviate strongly (higher for women everywhere).

Suppose $p_l = p_l(\theta_0, \theta_1, \theta_2)$ is some parametric model like Gomperz-Makeham. Estimation is then the question of determining the three parameters θ_0, $\theta_1$and θ_2, and the natural way is through maximum likelihood. To derive the log likelihood function note that n_{l+1} given n_l has density function

$$f(n_{l+1}|n_l) = \left(\begin{array}{c} n_{l+1} \\ n_l \end{array} \right) p_l^{n_{l+1}} (1 - p_l)^{n_l - n_{l+1}};$$

i.e. a binomial density, as pointed out above. Taking logarithms and adding over all l yields

$$\mathcal{L}(\theta_0, \theta_1, \theta_2) = \sum_l \{n_{l+1} \log(p_l) + (n_l - n_{l+1}) \log(1 - p_l)\} \quad \text{where} \quad p_l = p_l(\theta_0, \theta_1, \theta_2),$$

where constant factors have been removed. Maximization with respect to θ_0, θ_1 and θ_2 must usually be done numerically (see Section 7.3).

If no model has been imposed the expression for l can be maximaized with respect to the survival probabilities p_l themselves. This yields the elementary, **non-parametric** estimate

$$\hat{p}_l = \frac{n_{l+1}}{n_l}. \tag{1.21}$$

[3]The same would (of course) apply to other time increments.

We simply count the number of people who have survived to the next time step. Unlike the mortalities plotted in Figure 12.1 those obtained from (1.21) will wriggle in a way that is not quite smooth; see Section 15.2 where numerical examples are provided.

Monte Carlo algorithms
We shall for the bootstrap experiments in Section 12.6 need to simulate survival data. Sampling from a life table is non-uniform multinomial samplimg. It was discussed in Section 4.2 how it is done. The fast algorithms 4.2a,b are best, but if speed isn't important, the following ineffecient method is possible.

Algorithm 12.2 Survival data
0 Input: p_l for $l = l_0, \dots l_e$. *% For example* $l_e = 120/h$.
1 For $l = l_0, \dots, l_e$ do
2 Draw $U^* \sim$ uniform
3 If $(U^* > p_l)$ then *%Life terminated.*
4 Return $L^* = l$ and stop

This inefficient method is easy to implement and is sufficient for many applications and was used with the studies in Chapter 15.2.

1.4 Single life arrangements

Introduction
Insurance products for which mortality risk is the only source of uncertainty are among the most common ones. The standard types are reviewed below. They are expressed through mortality-adjusted values of annuities for which actuarial science has indigeneous mathematical notation. A simple version of that theory is presented in this section along with the main ideas behind elementary life insurance.

Valuation under mortality risk
Consider a pre-determined sequence of payments $\{\zeta_k\}$ ceasing at the time of death. The discounted value of payment k is $v^k\zeta_k$, and it takes place as long as $L \geq l_0 + k$ where l_0 is the age of the individual at the time the contract is drawn up. Since

$$\Pr(L \geq l_0 + k | L \geq l_0) = {}_k p_{l_0},$$

it follows that *expected*, discounted value of payment k must be $v^k \zeta_k \, {}_k p_{l_0}$. Adding over all k yields the expected present value

$$E(\mathrm{PV}_0) = \sum_{k=0}^{\infty} v^k \zeta_k \, {}_k p_{l_0} \qquad \textbf{(payments while alive)} \qquad (1.22)$$

of the entire payments stream $\{\zeta_k\}$.

A second basic scheme are benefits released by the death of the indvidual. Now there is a payment at time k if $L = l_0 + k$ (the policy holder then died during the preceding period). The probability of that event at time $t_0 = 0$ is ${}_k q_{l_0}$; see (1.13), and the expected present value becomes

$$E(\mathrm{PV}_0) = \sum_{k=0}^{\infty} v^k s_k \, {}_k q_{l_0} \qquad \textbf{(payment at the time of death)} \qquad (1.23)$$

These two valuations are the central ones in single life insurance.

Mortality adjustment for pension plans
A pension plan is pre-determined payments $\{zeta_k\}$ of the former type that is halted at the time of death. In practice there are *contributions* (counted negative) in the beginning and *withdrawals* (counted positive) at the end, but that is of no concern for the mathematics. The value of insurance contracts was in Section 12.2 defined as the expected present value of all remaing transactions under the agreement. For a policy holder who is alive at time k this amounts to the left hand side of (1.24) below. If the arrangement satisfies the condition of equivalence (1.5), there is a second valuation as well. Indeed, it is verified in Section 12.7 that if $E(\mathrm{PV}_0) = 0$ in (1.22), then

$$\underbrace{\sum_{i=0}^{\infty} \zeta_{k+i}\, v^i\, {}_ip_{l_0+k}}_{\textit{prospective reserve}} \quad = \quad \underbrace{-\sum_{i=0}^{k-1} \zeta_i \frac{(1+r)^{k-i}}{{}_{k-i}p_{l_0+i}}}_{\textit{retrospective reserve}}. \tag{1.24}$$

If $k = 0$ is inserted on the left, we are back to the the expected present value (1.22). On the right the past contributions $-\zeta_0, \ldots, -\zeta_{k-1}$ are added with interest, but $-\zeta_i$ is at time k attributed a *higher* value than the ordinary accumulated saving $V_i = (-\zeta_i)(1+r)^{k-i}$. Why is that? Why should the account be credited V_i/p_i rather than just V_i where $p_i = {}_{k-i}p_{l_0+i}$? Because the *expected* value of the contribution then becomes

$$\underbrace{\frac{V}{p_i} \times p_i}_{\textit{insured alive}} \quad + \quad \underbrace{0 \times (1-p_i)}_{\textit{insured dies}} \quad = \quad V_i,$$

coinciding with the savings value. The extra is compensation for the insurer pocketing the money if the insured dies early.

Life insurance notation
In practice insurance products often come as fixed annuities, and actuarial science has erected special notation to express such situations mathematically. Consider constant payments of one money unit over K time periods which ceases when the individual dies. The *expected*, discounted value of the k'th payment must be $v^k\,{}_kp_{l_0}$. With payment **in arrears** this yields

$$\ddot{a}_{l_0:\overline{K}|} = \sum_{k=1}^{K} v^k\,{}_kp_{l_0} \qquad \text{(payable in arrears while alive)}, \tag{1.25}$$

a special case of (1.22) above. This is the expected present value of an annuity, paid (or received) as long as the indvidual is alive. The notation signifies that it lasts K periods and that the age is l_0 at the start. Note that all transfers take place at the end of the periods (none at $t_0 = 0$). There is a similar quantity for payment **in advance**; i.e.

$$a_{l_0:\overline{K}|} = \sum_{k=0}^{K-1} v^k\,{}_kp_{l_0} \qquad \text{(payable in advance while alive)}. \tag{1.26}$$

Now there is nothing at $t_K = Kh$. The two variants are related to each other through the obvious relationships

$$a_{l_0:\overline{K}|} = 1 + \ddot{a}_{l_0:\overline{K-1}|} \qquad \text{or} \qquad \ddot{a}_{l_0:\overline{K}|} = a_{l_0:\overline{K+1}|} - 1$$

There is special notation for annuities of infinite extension; i.e

$$\ddot{a}_{l_0} = \ddot{a}_{l_0:\overline{\infty}|} \qquad \text{and} \qquad a_{l_0} = a_{l_0:\overline{\infty}|}\,. \tag{1.27}$$

Here payment goes on until the recepient (or payer) dies.

There are also **deferred** versions where the annuity starts to run at some point in the future. Defined benefit schemes with rewards from an retirement age l_r many years ahead is an example. The expected present value of such an annuity over K periods is

$$\underbrace{v^{l_r-l_0}\cdot{}_{l_r-l_0}p_{l_0}\cdot a_{l_r:\overline{K}|}}_{\textit{in advance}} \qquad \text{or} \qquad \underbrace{v^{l_r-l_0}\cdot{}_{l_r-l_0}p_{l_0}\cdot \ddot{a}_{l_r:\overline{K}|}}_{\textit{in arrears}}\,, \tag{1.28}$$

where there is a discount over the period $l_r - l_0$ and a second factor accounting for the individual being staying alive until the payment stream begins.

Similar arguments can be advanced with one-time payments compensating death. Suppose one money unit is awarded when $L = l_0 + k$. The probability of this is ${}_kq_{l_0}$, leading to the expected present value

$$\ddot{A}_{l_0:\overline{K}|} = \sum_{k=1}^{K} v^k\, {}_kq_{l_0}, \tag{1.29}$$

which is a special case of (1.23).

Common insurance arrangements
The following examples show how the mortality-adjusted coefficients are put to work. It is assumed that premia are calculated through equivalence. Most payments are in advance (but in arrears would be almost the same). Others examples can be found among the exercises.

Life annuitites This is the simplest way conceivable of smoothing mortality risk. Suppose an individual has amassed savings Π at the age of retirement l_r and purchases an annuity that lasts for K time units or until his death. The pension s received each period is then

$$s\cdot a_{l_r:\overline{K}|} = \Pi \qquad \text{or} \qquad s = \frac{\Pi}{a_{l_r:\overline{K}|}}. \tag{1.30}$$

Note that the amount received is increased by the company being allowed to take the remainder of the of the funds in case of an early death.

Pension plans The preceding example was a contributed benefit scheme, the pension being determined from the savings. Defined benefit is the other way around. Now savings are built up to support a given pension. Suppose the individual enters at age l_0 and retires at l_r to receive a fixed pension s during the ensuing K time periods (or until his death). With all payments in advance the present value of the scheme at the time it is drawn up is

$$\underbrace{-\pi\cdot a_{l_0:\overline{l_r-l_0}|}}_{\textit{contributing stage}} \qquad + \qquad \underbrace{s\cdot v^{l_r-l_0}\cdot{}_{l_r-l_0}p_{l_0}\cdot a_{l_r:\overline{K}|}}_{\textit{benefit stage}}\,,$$

and the equivalence premium becomes

$$\pi = s\cdot v^{l_r-l_0}\cdot{}_{l_r-l_0}p_{l_0}\cdot\frac{a_{l_r:\overline{K}|}}{a_{l_0:\overline{l_r-l_0}|}}. \tag{1.31}$$

The agreement that the insurer is allowed to take over surplus in case of an early death make π smaller than it would have been otherwise.

Pure endowments These are savings contracts where the policy holder is paid a lump sum at a future date if alive, merely a pension plan with one single benefit (i.e. $K = 1$). The contribution necessary to sustain such an arrangement is

$$\pi = s \cdot v^{l_r - l_0} \cdot \frac{{}_{l_r - l_0}p_{l_0}}{a_{l_0:\overline{l_r - l_0}|}}, \tag{1.32}$$

which follows form (1.31) by noting that $a_{l_r:\overline{1}|} = 1$; see (1.26).

Term insurance The policy holder has now set up the contact to reward certain beneficiaries a lump sum upon his death. An arrangement lasting of this type lasting for K time units has at time zero the present value

$$\underbrace{-\pi \cdot a_{l_0:\overline{K}|}}_{\textit{premium}} \quad + \quad \underbrace{s \cdot \ddot{A}_{l_0:\overline{K}|}}_{\textit{benefit}} \, .$$

where $\ddot{A}$ was defined in (1.29). This becomes zero if

$$\pi = s \cdot \frac{\ddot{A}_{l_0:\overline{K}|}}{a_{l_0:\overline{K}|}}. \tag{1.33}$$

Like pension schemes such contracts have positive values at any point in time, and if broken off, the insured is entitled to a compensation. The reason is that death risk is *smaller* in the beginning whereas the fixed premium scheme smooths it out; see Exercise ? where the mathematics is carried out.

Endowments These are arrangements where there are one-time pay-outs following the death of the policy holder and also a final endowment if the insurer is alive. The present value at time zero is now

$$\underbrace{-\pi \cdot a_{l_0:\overline{K}|}}_{\textit{premium}} \quad + \quad \underbrace{s_1 \cdot \ddot{A}_{l_0:\overline{K}|}}_{\textit{term part}} \quad + \quad \underbrace{s_2 \cdot v^K \cdot {}_Kp_{l_0} \cdot a_{l_0:\overline{K}|}}_{\textit{endowment part}} \, .$$

and we can again determine the premium from equivalence. Now

$$\pi = s_1 \cdot \frac{\ddot{A}_{l_0:\overline{K}|}}{a_{l_0:\overline{K}|}} + s_2 \cdot v^K \cdot \frac{{}_Kp_{l_0}}{a_{l_0:\overline{K}|}}. \tag{1.34}$$

Note that the benefits are unequal in the two cases.

Portfolio liabiltities

Evaluation on the portfolio level can be carried out by adding reserves over all individual accounts, but it is also of interest to examine how liabilities distribute over time. A chief motivation is the management of assets and liabilities jointly, an exercise drawing more and more interest. An important example is pension insurance for which the details are as follows. Available on file for policy j is

$$\underbrace{l_j}_{\textit{current age}} \qquad \underbrace{l_{rj}}_{\textit{retirment age}} \qquad \underbrace{\pi_j}_{\textit{premium}} \qquad \underbrace{s_j}_{\textit{pension}}.$$

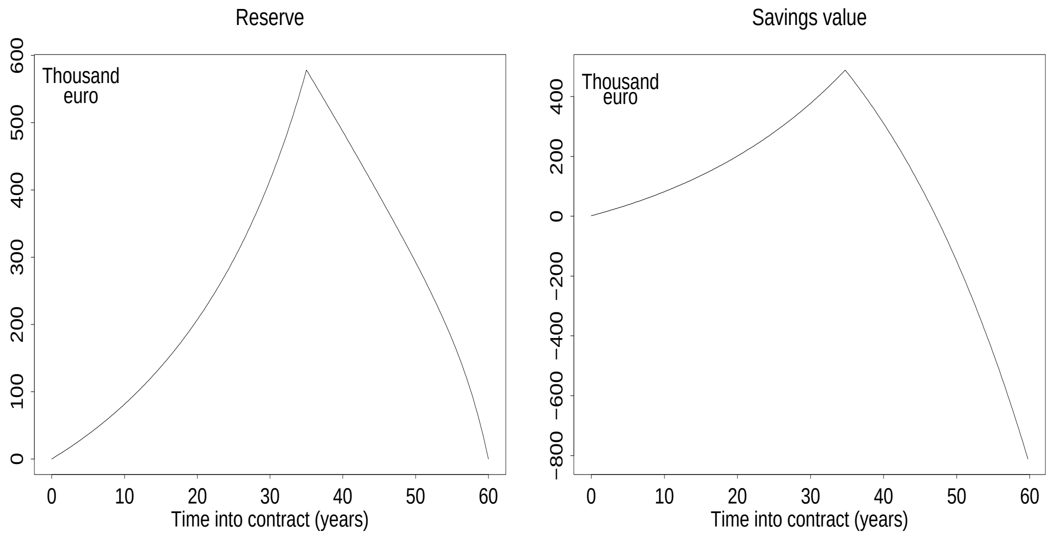


Figure 12.2 Reserve and savings value of a pension insurance under conditions described in the text.

With payment in advance it is easily seen that the expected pay-off k time units ahead is

$$\mathcal{X}_k = -\sum_{l_j+k<l_{rj}} \pi_j \, {}_kp_{l_j} + \sum_{l_j+k\geq l_{rj}} s_j \, {}_kp_{l_j}$$

where the two sums are those paying premium and those receiveing benefit. The present vaue (current reserve) is

$$\mathcal{PV}_0 = \sum_{k=0}^{\infty} v^k \mathcal{X}_k$$

which could have been found by examining mortality-adjusted annuities for each policy holder and added those. That is a bit faster computationally, but fails to address the distribution in liabilities over time; for a numerical example, see Section 12.6.

A numerical example
How reserve and savings value in ordinary pension insurance evolve is illustrated in Figure 12.2. The contract was entered at the age of 30 years with retirement at 65. After that date an annual pension of 50000 euro was received quarterly and in advance up to the age of 90. The survival model was the Gomperz-Makeham one in Section 12.2. Premium, also quarterly and in advance, was calculated from the equivalence condition. That gave the as the amounts (quarterly and in thousand euro)

Premium (quarterly)	1.634	2.189	2.902.
Interest (annual)	4%	3%	2%

depending on the *annual* technical rate of interest shown. Clearly contributions are quite sensitively dependent on that quantity.

The reserve when the insured stays alive is shown in Figure 12.2 left. A top is reached at the retirement age (65). After that there is a decline to zero at 90 where the scheme terminates. It is interesting to compare with the the savings value of the account, likewise calculated on the assumption that the individual stays alive. Let V_k is the book value after k quarters. Then V_k develops according to the recursion

$$\underset{\textit{before retirement}}{V_k = (1+r)(V_{k-1}+\pi)} \qquad \underset{\textit{after retirement}}{V_k = (1+r)(V_{k-1}-s)}$$

starting at $V_0 = 0$. All quantities are *quarterly* (included the rate of interest r). Note that the book value is *smaller* than the reserve during the contributing stage, because it fails to adjust for the survival risk. It becomes strongly negative at the end for for very long lifes, causing a loss for the insurer, but he has compensation for *short* lives where the account is positive.

1.5 Multi-state insurance I: Modelling

Introduction

The general outline of multi-state insurance was introduced in Section 12.2, but nothing was said about the detailed modelling of the state process $\{C_l\}$ and the detailed computation of premium and liabilities. These are vast subjects with a huge literature behind them. Much of it is in continuous time, but a lot can be achieved by simple means through recursions running in *discrete* time. We then need the transition probabilites

$$p_l(i|j) = \Pr(C_{l+1} = i|C_l = j)$$

introduced in Section 6.6 and their k-step extension

$$_k p_l(i|j) = \Pr(C_{l+k} = i|C_l = j). \qquad (1.35)$$

All evaluations of risk will be carried out in terms of those when we get around to it in the next section.

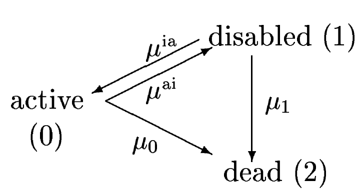
In reality the sequence $\{C_l\}$, is an extract of a process $C(y)$ running in continuous time; i.e.

$$C_l = C(y_l) \qquad \text{with} \qquad y_l = lh.$$

That viewpoint was used with survival modelling in Section 12.3 and is at least as important here. The framework imposed is that of **Markov processes**. All information on the future course of $C(y)$ then rests with the current $C(y_0)$; what has happened before is immaterial. We met the same definition in discrete time in Section 6.6, and indeed any sequence $C_l = C(lh)$ in discrete time inherits the Markov property from its parent $C(y)$. One of the central themes of this section is how models on the two time scales are linked probabilistically.

Disability modelling

Continuous time modelling is carried out in tems of **intensities** $\mu_{i|j}(y)$ of passing between states j and i. The chief issues can be introduced through the disability scheme from Section 6.6, which was of the form



with three states, now labelled 0, 1 and 2. Here $\mu^{\rm ai}(y)$ and $\mu^{\rm ia}(y)$ describe how often an individual may move between active and disabled. Their mortalities $\mu_0(y)$ and $\mu_1(y)$ are often unequal, usually with $\mu_1(y) \geq \mu_0(y)$.

A special case is

$$\mu_0(y) = \mu_1(y) = \mu(y) \qquad \text{and} \qquad \mu^{\rm ia}(y) = 0.$$

Rehabilitation (going back to active state after having been disabled) is then excluded[4] and the mortality is the same in both groups. This simplification corresponds to the Danish disability model below and is a rare case of the transition probabilities (1.35) being available in closed form. Indeed, an indvidual may leave the active state 0 due to both disability and death. During a small time increment of length h the probabality of one of these events is (approximately)[5]

$$\mu(y)h + \mu^{\rm ai}(y)h = \{\mu(y) + \mu^{\rm ai}(y)\}h$$

which shows (as was indeed seen in Chapter 8 too) that intensities due to different sources are *added*. Since no one enters state 0 after having left it, we may from (1.16) conclude that

$$_kp_l(0|0) = \exp\left(-\int_{lh}^{(l+k)h} \{\mu(y) + \mu^{\rm ai}(y)\}\,dy\right), \tag{1.36}$$

and in the same way

$$_kp_l(1|1) = \exp\left(-\int_{lh}^{(l+k)h} \mu(y)\,dy\right). \tag{1.37}$$

Moreover,

$$_kp_l(2|1) = 1 - {}_kp_l(1|1) \tag{1.38}$$
$$_kp_l(2|0) = {}_kp_l(2|1) \tag{1.39}$$
$$_kp_l(1|0) = 1 - {}_kp_l(0|0) - {}_kp_l(2|0), \tag{1.40}$$

where the first identity is due to rehabilitation being excluded and the second one to death intensities being the same in both groups. Other transitions probabilities are zero.

Example: The Danish disability model

Disability models for public schemes, reviewed in Habermann and Pitacco (1998), vary between countries. The Danish model is one of the very simplest, ignoring rehabilitation and unequal mortality between active and disabled. Drivers of the scheme are then the intensities $\mu(y)$ and $\mu^{\rm ai}(y)$, for which Gomperz-Makeham intensities are supplied. Thus

$$\mu(y) = \theta_0 + \theta_1 \exp(\theta_2 y) \qquad \text{and} \qquad \mu^{\rm ai}(y) = \eta_0 + \eta_1 \exp(\eta_2 y).$$

[4]As this book is being written that is in many countries not a very strong prossibility, despite the authorities being keen on promoting it.

[5]We may ignore the possibility of first becoming disable and then dying since the combined probability is of order h^2. It was seen in Section 8.10 that this is too small to matter as $h \to 0$.

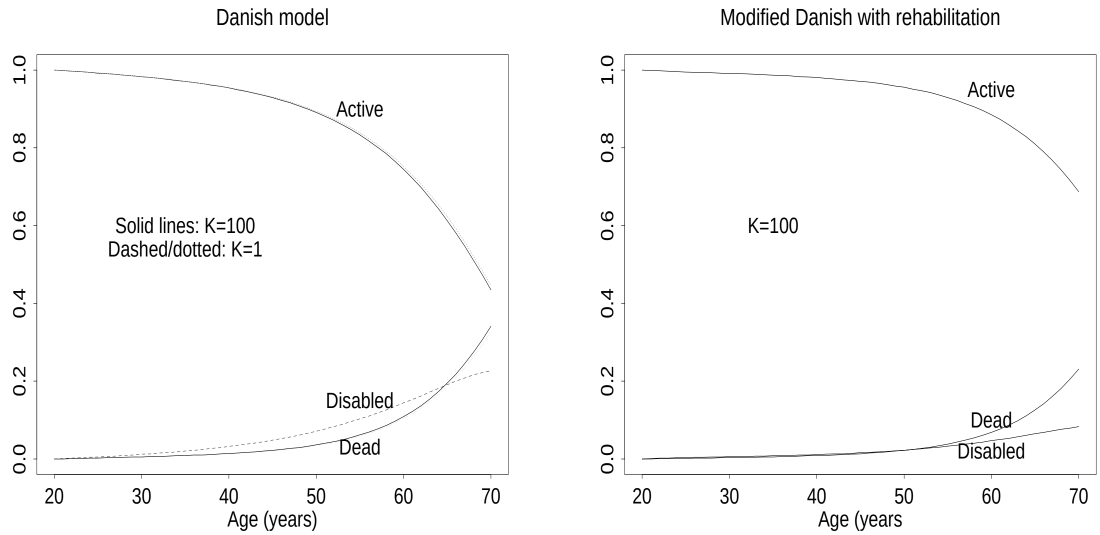


Figure 12.3 Distributions among states in disability schemes under the conditions described in the text.

with parameters[6]

$$\theta_0 = 0.0004, \qquad \theta_1 = 0.00000347, \qquad \theta_2 = 0.1382,$$

and

$$\eta_0 = 0.0005, \qquad \eta_1 = 0.0000759, \qquad \eta_2 = 0.08750.$$

The survival model is quite different from the one in Section 12.3 and corresponds to an average life of 71.4 years, four years less than there. It is possible to find closed mathematical expressions for the transition probabiltities. They become similar to those in (1.18); see Exerice ??, and have been plotted in Figure 12.3 left showing how a population divide between the three states when everybody enter the scheme at age 20 (the meaning of the parameter K is explained below).

Intensity modelling: General
The general case makes use of intensities $\mu_{i|j}(y)$ for passing from state j to i. For the disabilitiy model above these are on matrix form

$$\begin{array}{lll} & \mu_{1|0}(y) = \mu^{\text{ai}}(y) & \mu_{2|0}(y) = \mu_0(y) \\ \mu_{0|1}(y) = \mu^{\text{ia}}(y) & & \mu_{2|1}(y) = \mu_1(y) \\ \mu_{0|2}(y) = 0 & \mu_{1|2}(y) = 0, & \end{array}$$

where the diagonal elements $\mu_{ii}(y)$ do not matter. Systems like these with general, age-dependent intensities $\mu_{j|i}(y)$ is a powerful modelling tool, as will above all emerge in the next section.

As a simple example, consider a modification of the Danish disability model where the first

[6]They are adapted to the present notation; different values are found in Haberman and Pitacco's book (p. 100).

line of the matrix is altered to cater for rehabilitation and higher expected length of life within the active group, specifically

$$\mu_{0|1} = \mu^{\text{ia}}(y) = 0.2 \qquad \text{and} \qquad \mu_{2|0} = 0.6 \times \mu(y).$$

This specifies a strong potential for conquering disability. People who have become disabled now have a 20% chance of recovering within a year, independent of age. At the time of writing (2005) this is hardly realism, but its effect is evident in Figure 12.3 where the distribution among the three states has been plotted for the ordinary Danish model on the left and the new one on the right. There is no difference at all in the disability frequency, but on the right people recover, leading to much fewer disabled individuals. The lower death rate in the majority group (the active one) is evident too.

From one-step to k-step transitions

We digress a moment to comment on how k-step transition probabilitities are connected to one-step ones. Consider the following path starting at $y_0 = l_0 h$:

$$\begin{array}{lccccc} & i_0 & \overset{{}_k p_{l_0}(j|i_0)}{\longrightarrow} & j & \overset{p_{l_0+k}(i|j)}{\longrightarrow} & i. \\ \textit{age} & l_0 h & & (l_0+k)h & & (l_0+k+1)h \end{array}$$

The probabilities of the two transitions are those marked. They must be multiplied for the probabililty of the entire path, and when added over all j, this gives us the likelihood of ending at state i. In other words,

$$_{k+l_0}p_{l_0}(i|i_0) = \sum_j p_{l_0+k}(i|j)\ {}_k p_{l_0}(j|i_0), \qquad k = 0, 1, \ldots, \tag{1.41}$$

which is a recursion in k, starting at

$$_0 p_l(i|i_0) = \begin{array}{ll} 1 & i = i_0 \\ 0 & i \neq i_0. \end{array} \tag{1.42}$$

It is easy to implement this it in the computer, and it will below be utilized on several occasions. The prerequisite is that $\{C_l\}$ follows a model of the Markov type[7].

From intensities to transition probabilities

It is for actuarial calculations necessary to connect the intensities to the transition probabilities (1.35) on the time scale used. The standard way is through the **Chapman-Kolmogorov** system of differential equations. Many textbooks in in actuarial science reviews this approach; see the reading list at the end of the chapter. Closed form solutions are rare (the Danish disability model is an exception), but the differential equations can be solved numerically. An alternative way is to use the recursion (1.41). That is less efficient computationally than the highly polished subroutines for solving differential equations in commercial software, but it holds the advantage that you can implement it on your own without much knowledge of numerical mathematics.

Start at $y_0 = l_0 h$, select a suitably *large* integer K and consider the sequence

$$y_k = y_0 + k\frac{h}{K}, \qquad k = 1, 2, \ldots, \tag{1.43}$$

[7] The multiplication of the two probalilities of the path is not valid otherwise.

much more finely meshed than the sequence $\{kh\}$. The idea is to utilize that on small time scale transition probabilities and intensities are virtually the same. Define

$$\tilde{p}_{l_0+k}(i|j) = \mu_{i|j}(y_k)\frac{h}{K} \quad i \neq j \qquad \text{and} \qquad \tilde{p}_{l_0+k}(j|j) = 1 - \sum_{i \neq j} \tilde{p}_{l_0+k}(i|j),$$

which is an ordinary set of transition probabilities[8]. Suppose (1.41) are run kK steps with these quantities as one-step probabilities. The entire time period is then

$$(kK) \times (h/K) \quad = \quad kh;$$

i.e precisely the one sought, and the output approximates the k-step probabilities on the *original* time scale.

The detailed mathematics is given in Section 12.7 where it is shown that the computations are conveniently organized through the following recursion. For given i_0 let

$$a_0(i) = \begin{matrix} 1 & i = i_0 \\ 0 & i \neq i_0, \end{matrix} \tag{1.44}$$

and for $k = 0, 1, \ldots$ run

$$a_{k+1}(i) = a_k(i) + \frac{h}{K} \sum_{j \neq i} \{\mu_{i|j}(y_k)a_k(j) - \mu_{j|i}(y_k)a_k(i)\} \qquad \text{for all } i. \tag{1.45}$$

Here $\{y_k\}$ is the age sequence in (1.43). Then *approximately*

$${}_kp_{l_0}(i|i_0) \doteq a_{kK}(i), \tag{1.46}$$

which becomes exact as $K \to \infty$. The calculations must be repeated for all l_0 and all i_0 of interest. Algorithm 12.3 in the next section shows how commands are organized into a computer program.

An obvious question is how large K must be selected for the accuarcy to be satisfactory. There isn't really any need to keep K low (since the computations are completed rapidly in any case). Yet intensity functions are always smooth functions of age y, and $K = 100$ or $K = 1000$ should be enough for most purposes. An illustration for the Danish disability model is given in Figure 12.3 left. Even such a crude choice as $K = 1$ does not lead to results deviating too much from those under $K = 100$, and a monthly time scale ($K = 12$) was virtually undistinguishable from $K = 100$.

Fitting from historical data
With extensive historical experience these probabilies may be estimated direcly. Let ${}_kn_l(j,i)$ be the number of times in the historical material an individual is in state j at age l *and* in state i at age $l + k$. In practice such counts are easily inferred from the record of all policy holders. The estimates are then

$${}_k\hat{p}_l(i|j) = \frac{{}_kn_l(j,i)}{n_l(j)} \qquad \text{where} \qquad n_l(j) = \sum_i {}_kn_l(j,i). \tag{1.47}$$

[8]Provided K is large enough.

Here $n_l(j)$ is the number of individuals in state j at age l. Note that the estimated model is a viable set of transition probabilitities. Firstly

$$\sum_i {}_k\hat{p}_l(i|j) = 1,$$

and it is also easily verified that the recursion (1.41) is satisfied. But the method requires plenty of historical data. If that isn't available, intensity modelling is probably better.

1.6 Multi-state insurance II: Evaluation

Introduction

The fairly general system of multi-state insurance introduced above has a *stochastic* part represented by the intensities $\mu_{i|j}(y)$ for movements between states and a *contract* part defined by payments $s_l(i)$ that apply in state i at age l. This is a powerful framework, enabling us to express many arrangements found in practice in a convenient mathematical form; see below. To complete the narrative we must show how payment streams *adjusted for risk* are evaluated. That is essentially a question of hanging money transfers drawn up in advance on the probabilistic schemes of the preceding section. We shall below demonstrate how this is done, including a simple, *general* recepy for computer implementation. The schemes in Section 12.4 are all special cases of this.

Equivalence premium

In practice one (or more) of the states is a contributing one. Suppose a fixed contribution π is made in state 0 up to age l_r. The present, expected value of such a payment stream for an individual joining at age l_0 is then

$$E(\mathrm{PV}_0) = -\pi \sum_{k=0}^{l_r-l_0-1} v^k\,{}_kp_{l_0}(0|0) + \sum_{k=l_r-l_0}^{\infty} v^k s_k(0)\,{}_kp_{l_0}(0|0) + \sum_{k=1}^{\infty} v^k \sum_{i>0} s_{l_0+k}(i)\,{}_kp_{l_0}(i|0). \tag{1.48}$$

where payments in state 0 are allowed even after the retirement age l_r. There is a common structure in all three sums. Payments are discounted (through v^k) and then multiplied with probabilities that they are actually made. That yields the *expected* present value. Solving the equation $E(\mathrm{PV}_0) = 0$ for π determines the equivalence premium.

As a simple example consider a pension scheme with special compensation for a widow outliving the policy holder. Following his death she is during the rest of her life payed a fixed sum s_2 each period. Before that they both receive (a possibly different sum) s_1 after his retirment at l_r. There are three states:

state 0	state 1	state 2,
he is alive	*he is dead, she is alive*	*both are dead*

with payment functions

$$\begin{array}{ll} s_l(0) = -\pi, & l < l_r \\ s_l(0) = s_1, & l \geq l_r \end{array} \qquad \text{and} \qquad s_l(1) = s_2.$$

and nothing in state 2. Let the survival probabilities be ${}_kp_{l_0}$ (for him) and ${}_k\tilde{p}_{\tilde{l}_0}$ (for her; her age was $\tilde{l}_0$ when the contract was drawn up). The expected present value (1.47) of the entire scheme becomes

$$E(\mathrm{PV}_0) = -\pi \sum_{k=0}^{l_r-l_0-1} v^k\, {}_kp_{l_0} + s_1 \sum_{k=l_r-l_0}^{\infty} v^k\, {}_kp_{l_0} + s_2 \sum_{k=1}^{\infty} v^k (1 - {}_kp_{l_0}) {}_k\tilde{p}_{\tilde{l}_0}.$$

Here the first two sums can be written in terms of annuity coefficients, as in Section 12.4. For the "widow" part to materialize he must be dead and she alive. Probabilities of these events are $1 - {}_kp_{l_0}$ and ${}_k\tilde{p}_{\tilde{l}_0}$ respectively, and when multiplied (independent events) define the probability that s_2 is payed.

Disability and retirement in combination

As a second example consider a disability arrangement where there is a fixed benefit s_1 per period when disabled. At retirement l_r all individuals (whether disabled ot not) are transferred to an ordinary pension s_2, possibly different from s_1. A simple mathematical description makes use of the following three states:

state 0	state 1	state 2,
active/retired	*disabled/retired*	*dead*

with payment functions

$$\begin{array}{ll} s_l(0) = -\pi, & l < l_r \\ s_l(0) = s_2, & l \geq l_r \end{array} \quad \text{and} \quad \begin{array}{ll} s_l(1) = s_1, & l < l_r \\ s_l(1) = s_2, & l \geq l_r \end{array}$$

Possible models for the life cycle fluctuations are those of Section 12.5.

The equivalence premium is determined from the equation

$$-\pi \sum_{k=0}^{l_r-l_0-1} v^k\, {}_kp_{l_0}(0|0) + s_1 \sum_{k=1}^{l_r-l_0-1} v^k\, {}_kp_{l_0}(1|0) + s_2 \sum_{k=l_r-l_0}^{\infty} v^k \{{}_kp_{l_0}(0|0)) + {}_kp_{l_0}(1|0))\} = 0,$$

where the pension after retirement age is received whether in state 0 and 1. Figure 12.4 left shows the annual premium when

$$s_1 = 30 \qquad s_2 = 20 \qquad l_r = 65 \qquad r = 4\%,$$

s_1 and s_2 being in thousand Euro. The disability models were those plotted in Figure 12.3. Premium is a little above 3000 euro annually with entry at 30 and increase rapidly when the arrangement is started at high age.

Portfolio liabiltities

The portfolio reserve can be computed by adding outstanding payments for all members of the scheme, as pointed out in Section 12.2. For planning purposes it is often useful to follow the different course of examining the liability distribution over time. We shall now devise a simple scheme that evaluates the sequence of net liabiltities $\mathcal{X}_k$ at all times t_k. Start by introducing $\mathcal{N}_{l_0}(i_0)$ as the number of individuals of age l_0 in state i_0 at time $t_0 = 0$. These are counts observed. At time t_k the expected number in state i among them will be

$${}_kp_l(i|i_0)\, \mathcal{N}_{l_0}(i_0).$$

Adding over all i_0 yields

$$ {}_k\mathcal{N}_{l_0}(i) = \sum_{i_0} {}_kp_l(i|i_0)\,\mathcal{N}_{l_0}(i_0); \tag{1.49} $$

as the average number of people of age l_0 today *and* in state i after k time units. Note the mathematical notation which is the same as the one used for probabilitities.

Net liabilities are the number of people occupying the various states multiplied with the amount they pay; i.e.

$$ \mathcal{X}_k = \sum_{l_0} \sum_i s_{l_0+k}(i)\,{}_k\mathcal{N}_{l_0}(i), \tag{1.50} $$

from which the present value of the entire sequence $\{\mathcal{X}_k\}$ can be obtained by ordinary discounting. If the payment function $s_l(i)$ vary over the policies, supply their average.

Numerical scheme

How do we get hold of the expected counts ${}_k\mathcal{N}_{l_0}(i)$ if the life cycle model has been set up on continuous scale? Answer: Through the earlier recursion (1.45) for the transition *probabilitities* in Section 12.5. In fact, the *very* same procedure may be used, provided it is started differently. Thus, let

$$ a_0(i) = \mathcal{N}_{l_0}(i) \qquad \text{(all } i\text{)} \tag{1.51} $$

and

$$ a_{k+1}(i) = a_k(i) + \frac{h}{K} \sum_{j \neq i} \{\mu_{i|j}(y_k)a_k(j) - \mu_{j|i}(y_k)a_k(i)\} \qquad \text{(all } i\text{)}, \tag{1.52} $$

where $\{y_k\}$ is the age sequence in (1.43). Then

$$ {}_k\mathcal{N}_{l_0}(i) \doteq a_{kK}(i), \tag{1.53} $$

which again becomes exact as $K \to \infty$. The scheme is justified in Section 12.7. The operation must be repeated for all l_0. The following procedure summarizes all steps necessary to combine this scheme with the payments:

Algorithm 12.3 Liabilities in state-dependent insurance

Input: $\{\mu_{i|j}(y)\}$, $\{s_l(i)\}$, $\{\mathcal{N}_{l_0}(i_0)\}$ and K — *%K as in (1.51)*

$\forall k$: $\mathcal{X}_k \leftarrow 0$

Repeat $\forall l_0$: — *%Contributions from %all l_0 must be added*

    $\forall i$: $\mathcal{N}(i) \leftarrow \mathcal{N}_{l_0}(i)$ and $y \leftarrow l_0 h$

    For $k = 0, 1, \ldots$ do

        Repeat $\forall i$: $\mathcal{X}_k \leftarrow \mathcal{X}_k + s_{l_0+k}(i)\mathcal{N}(i)$ — *%Update liabilities*

        Repeat K times — *%Inner loop*

            $y \leftarrow y + h/K$ — *%Update age*

            $\forall i$: $D(i) \leftarrow 0$

            Repeat $\forall (i,j)$: $D(i) \leftarrow D(i) + \mu_{i|j}(y)\mathcal{N}(j) - \mu_{j|i}(y)\mathcal{N}(i)$

            Repeat $\forall i$: $\mathcal{N}(i) \leftarrow \mathcal{N}(i) + (h/K)D(i)$

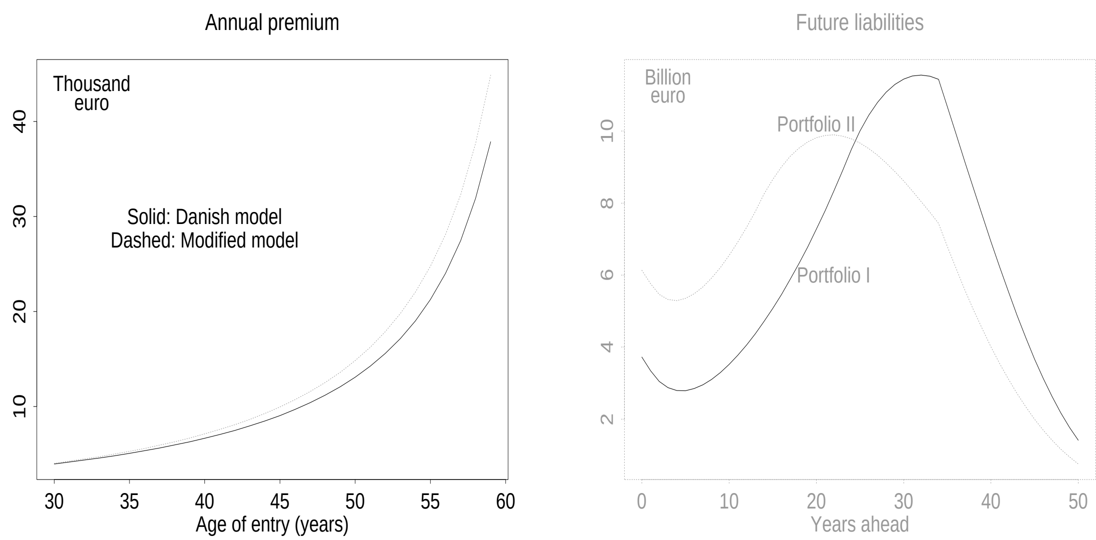


Figure 12.4 The combined disability/pension scheme described in the text. **Left:** *Premia against age of entry.* **Right:** *Distribution of liabilities under the modified Danish model.*

%Inner loop completed; counts re-computed

11 Return $\mathcal{X}_0, \mathcal{X}_1, \ldots$.

The algorithm keeps track on the counts for each k and update liabiltities (Line 5). When that is repeated for all l_0, on output $\mathcal{X}_0, \mathcal{X}_1, \ldots$ contain the liabilties. The sequence is easily combined with assesments of financial risk; see Chapter 15.

Example: Disabilitiy and retirement continued
Liabilities for the combined pension and disability arrangement defined above have been plotted in Figure 12.4 right. There were one million policies, owned by people between 30 and 89 years. Age distributions at time $t_0 = 0$ were

Portfolio I:	$\mathcal{N}_{l_0} = c\exp(-0.05\lvert l_0 - 40\rvert)$	$l_0 = 30, \ldots, 89$
Portfolio II:	$\mathcal{N}_{l_0} = c\exp(-0.05\lvert l_0 - 50\rvert),$	$l_0 = 30, \ldots, 89$

with 90% active and 10% disabled in each age group. The constant c ensured that the number of policies was one million exactly[9] We are entering at an arbitrary time of the portfolio. Some are paying premium; others are receiving benefit.

The two portfolios differ. A much higher share of old members is found in Portfolio II, and the bulk of the payments are now a decade earlier. Such information is relevant for planning investments; see Chapter 15.

1.7 Mathematical arguments

Section 12.2
Book value and risk-adjusted value We shall prove that the book value V_k' defined in (1.7)

[9]This is achieved by $c = 10^6/(\mathcal{N}_{30} + \ldots \mathcal{N}_{89})$.

is equal in expctation to the risk-adjusted value V_k in (1.6). For the former

$$E(V_k') = E\left(-\sum_{i=0}^{k-1}(1+r)^{k-i}\zeta_i\right) = -(1+r)^k\sum_{i=1}^{k-1} v^i E(\zeta_i)$$

whereas for the latter

$$E(V_k) = E\{E(\text{PV}_k|C_k)\} = E(\text{PV}_k)$$

by the rule of double expectation. Inserting (1.6) for $E(\text{PV}_k)$ yields

$$E(V_k) = \sum_{i=0}^{\infty} v^i E(\zeta_{k+i})$$

which implies that

$$\begin{aligned}-E(V_k') + E(V_k) &= (1+r)^k \sum_{i=0}^{k-1} v^i E(\zeta_i) + \sum_{i=0}^{\infty} v^i E(\zeta_{k+i}) \\ &= (1+r)^k\left(\sum_{i=0}^{k-1} v^i E(\zeta_i) + \sum_{i=0}^{\infty} v^{k+i} E(\zeta_{k+i})\right) = (1+r)^k \sum_{i=0}^{\infty} v^i E(\zeta_i) = 0\end{aligned}$$

by the condition of equivalence (1.4)

Section 12.3
Fitting the Gomperz-Makeham model The natural way is through likelihood estimation. We must then utilize that

$$n_{l+1}|n_l \quad \sim \quad \text{binomial}(n_l, p_l)$$

from which it follows that its (conditional) density function is

$$\binom{n_l}{n_{l+1}} p_l^{n_{l+1}}(1-p_l)^{n_l - n_{l+1}}.$$

Taking logarithms and adding over all l yields the likelihood function

$$\mathcal{L}(\theta_0, \theta_1, \theta_2) = \sum_{l=l_0}^{l_e-1}\{n_l \log(p_l) + (n_l - n_{l+1}\log(1-p_l)\} \qquad (1.54)$$

where p_l is given by (1.18) as

$${}_k p_l = \exp\left(-kh - \frac{\theta_1}{\theta_2} e^{lh}\frac{e^{kh}-1}{h}\right).$$

Maximization takes optimization software. A primitive way (though perfectly feasible) is to compute the likelihood surface over a combination of discrete values for the three parameters and select the triple that returns the maximum value.

Estimation must be carried out separately for men and women. Multi-step versions are easil found by plugging the estimates into the recursion formula (1.11) and even (1.13).

Section 12.4
The reserve formula (1.24) The expected present value of a pension plan $\{s_k\}$ may be written

$$E(\mathrm{PV}_0) = \sum_{i=0}^{\infty} s_i v^i {}_i p_{l_0} \quad = \quad \sum_{i=0}^{k-1} s_i v^i {}_i p_{l_0} + \sum_{i=0}^{\infty} s_{k+i} v^{k+i} {}_{k+i} p_{l_0},$$

where the the sum on the very right runs over time points *after* k. Note that

$$v^{k+i} = v^i \times v^k, \qquad \text{and} \qquad {}_{k+i}p_{l_0} = {}_i p_{l_0+k} \times {}_k p_{l_0}$$

where the factorization on the right is (**??**). Inserting into the second sum for $E(\mathrm{PV}_0)$ yields

$$E(\mathrm{PV}_0) = \sum_{i=0}^{k-1} s_i v^i {}_i p_{l_0} + v^k {}_k p_{l_0} \sum_{i=0}^{\infty} s_{k+i} v^i {}_i p_{l_0+k},$$

which should be zero by the equivalence condition (1.7). For the second sum this implies that

$$\sum_{i=0}^{\infty} s_{k+i} v^i {}_i p_{l_0+k} = -\sum_{i=0}^{k-1} s_i \frac{v^i}{v^k} \frac{{}_i p_{l_0}}{{}_k p_{l_0}} = -\sum_{i=0}^{k-1} s_i (1+r)^{k-i} \frac{1}{{}_{k-i} p_{l_0+i}}$$

after inserting $v = 1/(1+r)$ and noting that

$$\frac{{}_i p_{l_0}}{{}_k p_{l_0}} = \frac{\Pr(L \geq l_0 + k | L \geq l_0)}{\Pr(L \geq l_0 + i | L \geq l_0)} = \Pr(L \geq l_0 + k | L \geq l_0 + i) = {}_{k-i} p_{l_0+i}$$

This is the link (1.24) between the retrospective and prospective reserve.

Section 12.5
Recursion (1.41) Note that

$$\Pr(C_{l+k+1} = i | C_l = i_0) = \sum_j \Pr(C_{l+k+1} = i, C_{l+k} = j | C_l = j)$$

so that from the Markov assumption

$$\begin{aligned}\Pr(C_{l+k+1} = i | C_l = i_0) &= \sum_j \Pr(C_{l+k+1} = i | C_{l+k} = j, C_l = j) \Pr(C_{l+k} = j | C_l = i_0) \\ &= \sum_j \Pr(C_{l+k+1} = i | C_{l+k} = j) \Pr(C_{l+k} = j | C_l = i_0).\end{aligned}$$

In the mathematical notation used this can be written

$${}_{k+l_0} p_{l_0}(i|i_0) = \sum_j p_{l_0+k}(i|j) \; {}_k p_{l_0}(j|i_0),$$

which is (1.41).

This relationship may be multiplied on both sides with $\mathcal{N}_{l_0}(i_0)$, which is the number of individuals in state i_0 at time $t_0 = 0$. Then

$$\{{}_{k+l_0} p_{l_0}(i|i_0) \mathcal{N}_{l_0}(i_0)\} \quad = \quad \sum_j p_{l_0+k}(i|j) \; \cdot \; \{{}_k p_{l_0}(j|i_0) \mathcal{N}_{l_0}(i_0)\},$$

and add this over i_0 on both sides. Then

$$_{k+1}\mathcal{N}_{l_0}(i|i_0) = \sum_j p_{l_0+k}(i|j)_{k+1}\mathcal{N}_{l_0}(j|i_0)$$

where

$$_k\mathcal{N}_{l_0}(j|i_0) = \sum_{i_0} {_kp_{l_0}}(j|i_0)\mathcal{N}_{l_0}(i_0)\},$$

is the expetced number of people of age l_0 today who is in state j at time t_k. This is the same recursion as for probabilities and implies that the ensuing argument also proves the recursion (1.52) in section 12.6

Re-write (1.41) as

$$_{k+l_0}p_{l_0}(i|i_0) = p_{l_0+k}(i|i)\ {_kp_{l_0}}(i|i_0) + \sum_{j\neq i} p_{l_0+k}(i|j)\ {_kp_{l_0}}(j|i_0),$$

and insert

$$p_{l_0+k}(i|i) = 1 - \sum_{j\neq i} p_{l_0+k}(j|i).$$

Then

$$_{k+l_0}p_{l_0}(i|i_0) = {_kp_{l_0}}(i|i_0) + \sum_{j\neq i}\{p_{l_0+k}(i|j)\ {_kp_{l_0}}(j|i_0) - p_{l_0+k}(j|i)\ {_kp_{l_0}}(i|i_0)\},$$

almost (1.45). It only remains to replace

$$p_{l_0+k}(i|j) \quad \text{by} \quad \mu_{i|j}(y_k)\frac{h}{K} \qquad \text{and} \qquad {_kp_{l_0}}(j|i_0) \quad \text{by} \quad a_k(i_0).$$

The error in this is proportional to $(h/K)^2$ (as in Section 8.10) and when carried out kK times becomes or order of magnitude

$$kK \times \left(\frac{h}{K}\right)^2 = \frac{kh^2}{K},$$

which becomes zero as $K \to \infty$.

Section 12.6
The recursions (1.52)

1.8 Futher reading

Life insurance mathematics goes far back. In fact, much of its early history was long before the computers. The needs of that era lead to an extensive algebraic notation, indigeneous to the subject, but nothing has been made of that here. A good summary is provided by Gerber (1990), including the common viewpoint of *continuous payment streams*. You'll find much of that in almost all modern academic literature on life insurance.

For a broad perspective the book to read is presumably Booth, Chadburn, Cooper, Haberman and James (1999), which contains, among other things, extensive lists of references prior to its date of publication. Many of the more practical issues are dealt with and also the modern *contributing* life insurance schemes which are discussed in chapter 14. The interplay with mathematical finance is not covered. For that you may consult Panjer (1998); see, e.g., also the next three chapters below. A specialist book dealing with multi-state models is Haberman and Pitacco (1999), again with many references on its topic. There you will find the more sophisticated techniques mentioned in section 11.4 for identifying transition probabilites in general Markov chains.

Booth, P., Chadburn, R., Cooper, D., Haberman, S. and James, D. (1999). Modern Actuarial Theory and Practice. London: Chapman & Hall.

Gerber, H. U. (1990). Life Insurance Mathematics. Berlin: Springer-Verlag.

Haberman, S,. and Pitacco, E. (1998). Actuarial Models for Disability Insurance. London: Chapman & Hall.

Panjer, H. (1998) (ed). Financial Economics. The Actuarial Foundation.

1.9 Exercises