1 Evaluating risk: A primer

1.1 Introduction

The hardest part of quantitative risk analysis is to find the stochastic models and judge their realism. This is discussed later. What is now addressed is how models are used once they are in place. Only a handful of distributions have been introduced, and yet a good deal can be achieved already. The present chapter is a *primer* introducing main arenas and their first treatment computationally. We start with property insurance where core issues can be reached with very simple modelling. That applies to liabiltities in life insurance too, but now the mean feature is different. Once the stochastic model is given, there is little risk left. Of course, this doesn't rule out much uncertainty in the model itself (see Chapter 15). When we go to financial risk there are once again random events with strong impact. Target of this chapter is the general line. Many interesting points (demanding heavier modelling) are left out and treated later.

A unifying theme is Monte Carlo as a problem solver. By this is *not* meant computational technique which was treated in the preceding chapter (and in the next one too). What is on the agenda is the art of making the computer work for a purpose, how we arrange for it to chew away on computational obstacles and how it is utlized to get a feel for numbers. Monte Carlo is also an efficient way of handling the myriad of details in practical problems. Feed them into the computer and let simulation take over. Implementation is often straightforward, and existing programs might be re-used with minor variations. The potential here is endless. Is there a new contract clause, an exception from the exception say? With mathematics you need new expressions and may have to work them out anew, almost from scratch. If you use simulations there is perhaps no more than an additional statement in the computer code. And often the mathematics becomes too unwieldy to be of much merit at all.

1.2 General insurance: Opening look

Introduction

Risk variables in general insurance are amounts paid as compensations for damages or accidents, say X to a policy holder and $\mathcal{X}$ to cover the entire portfolio. Consider the representations

$$\underset{\textit{policy level}}{X = Z_1 + \ldots + Z_N} \qquad \text{and} \qquad \underset{\textit{portfolio level, identical risks}}{\mathcal{X} = Z_1 + \ldots + Z_{\mathcal{N}}} \tag{1.1}$$

where N and $\mathcal{N}$ are the number of insurance incidents and $Z_1, Z_2, \ldots$ what it costs to settle them. If N (or $\mathcal{N}$) is zero, the corresponding sum X (or $\mathcal{X}$) is zero too. The underlying period of time T (often one year) influence the models for N and $\mathcal{N}$.

For these descriptions to make sense all of $Z_1, Z_2, \ldots$ must follow the same probability distribution. That is plausible when dealing with a single policy. Surely an unlikely second event isn't on average any different from the first? However, for portfolios claims depend on the object insured and the sum it is insured for, and we must keep track on where it comes from by going through the entire list of policies. If there are J of them with claims $X_1, \ldots, X_J$, then

$$\mathcal{X} = X_1 + \ldots + X_J \qquad \text{where} \qquad X_j = Z_{j1} + \ldots Z_{jN_j}, \tag{1.2}$$

and claim numbers N_j and losses $Z_{j1}, Z_{j2}, \ldots$ depend on the policy index j. If all Z_{ji} have common distribution, $\mathcal{X}$ in (1.2) collapses to $\mathcal{X}$ in (1.1) by taking $\mathcal{N} = N_1 + \ldots + N_J$.

Enter contracts and their clauses

The preceding representations do not take into account the distinction between what an incident costs and what a policy holder receives, a common thing. If the actual compensation is some function $H(z)$ of the total replacement value z, then (1.1) left changes to

$$X = H(Z_1) + \ldots + H(Z_N) \qquad \text{where} \qquad 0 \leq H(z) \leq z. \tag{1.3}$$

Note that $H(z)$ can't exceed total cost z. A common type of contract is

$$H(z) = \begin{array}{cl} 0, & z \leq a \\ z-a, & a < z \leq a+b \\ b & z > a+b. \end{array} \tag{1.4}$$

Here a is a **deductible** subtracted z (no reimbursement below it) whereas b is a maximum insured sum per claim. These quantities typically vary over the portfolio so that $a = a_j$ and $b = b_j$ for policy j.

Re-insurance, introduced in Section 1.2, is handled mathematically in much the same way. The original risk is now shared between cedent and re-insurer through contracts that may apply to both individual policies/events *and* to the portfolio aggregate $\mathcal{X}$. Typical representations of the re-insurer part are

$$\underset{\textit{re-insurance per event}}{X^{\text{re}} = H(Z_1) + \ldots + H(Z_N)} \qquad \text{or} \qquad \underset{\textit{re-insurance on portfolio level}}{\mathcal{X}^{\text{re}} = H(\mathcal{X}),} \tag{1.5}$$

where $H(z)$ and $H(x)$ define the contracts, satisfying $0 \leq H(z) \leq z$ and $0 \leq H(x) \leq x$ as before. The example (1.4) is prominent in re-insurance too. It is now called a **layer** $a \times b$ contract and a is the **retention** limit of the cedent (who keeps all risk below it).

Re-insurance means cedent net responsibility falling to $X^{\text{ce}} = X - X^{\text{re}}$ or $\mathcal{X}^{\text{ce}} = \mathcal{X} - \mathcal{X}^{\text{re}}$ instead of X and $\mathcal{X}$. Whether the point of view is cedent or re-insurer, there is a common structure in how we proceed. Claim numbers and losses are described by stochastic models, allowing us to generate Monte Carlo realisations that are feed them into contracts and clauses for the final payments. That will be the agenda in Section 3.3.

Stochastic modelling

The critical part of risk evaluation in general insurance is the uncertainty of the original claims. Much will be said on that issue in Part II, yet a number of problems can be attacked right away through the following introductory observations. Claim numbers, whether N (for policies) or $\mathcal{N}$ (portfolios) are often well described by the Poisson distribution. The parameters are then

$$\underset{\textit{policy level}}{\lambda = \mu T} \qquad \text{and} \qquad \underset{\textit{portfolio level}}{\lambda = J\mu T,} \tag{1.6}$$

where μ is the expected number of claims per policy per time unit. For example, in automobile insurance $\mu = 5\%$ annually might be plausible (incident for one car in twenty). This central parameter (known as an **intensity**) will be explored thoroughly in Chapter 8 and used as vehicle for more advanced modelling.

Then there is the loss Z per event, usually taken to be independent of N. Unlike the case with claim numbers there is now little theoretical support, and modelling is almost always a question of pure experience. Common choices of distributions are Gamma, log-normal and Pareto, all introduced in the preceding chapter. How Z is modelled is discussed in Chapter 9.

Risk diversification

The core idea of insurance is risk spread on many units. Insight into this issue can be obtained through very simple means if policy risks $X_1, \ldots, X_J$ are **stochastically independent**. The portfolio aggregate has then mean and variance

$$E(\mathcal{X}) = \xi_1 + \ldots + \xi_J \qquad \text{and} \qquad \text{var}(\mathcal{X}) = \sigma_1^2 + \ldots + \sigma_J^2,$$

where $\xi_j = E(X_j)$ and $\sigma_j = \text{sd}(X_j)$. Independence is the prerequisite for the variance formula. Introduce

$$\bar{\xi} = \frac{1}{J}(\xi_1 + \ldots + \xi_J) \qquad \text{and} \qquad \bar{\sigma}^2 = \frac{1}{J}(\sigma_1^2 + \ldots + \sigma_J^2)$$

which is the *average* expectation and variance. Then

$$E(\mathcal{X}) = J\bar{\xi} \quad \text{and} \quad \text{sd}(\mathcal{X}) = \sqrt{J}\bar{\sigma}, \qquad \text{so that} \qquad \frac{\text{sd}(\mathcal{X})}{E(\mathcal{X})} = \frac{\bar{\sigma}/\bar{\xi}}{\sqrt{J}}, \tag{1.7}$$

which are formulae of merit.

What they tell us is that portfolio means grow with J and their standard deviations with $\sqrt{J}$, a much smaller number. As the number of policies goes up the unpredictable part represented by the standard deviation loses in relative importance and can be ignored eventually. A precise argument rests on the law of large numbers in probability theory (Appendix A.4). As $J \to \infty$, both $\bar{\xi}$ and $\bar{\sigma}$ tend to their population means, and the ratio $\text{sd}(\mathcal{X})/E(\mathcal{X})$ in (1.7) approaches 0. In other words, *insurance risk can be diversified away through size.*

Are large portfolios, therefore, risk-free? That's actually how we operate in life insurance and pensions (Section 3.4), but it is never that simple. There is always uncertainty in the underlying model, and risks may well be dependent as we shall see in Section 6.3. The big insurers and re-insurers of this world handle hundreds of thousands of policies. Their portfolio risk isn't zero!

1.3 How Monte Carlo is put to work

Introduction

This section is a first demonstration of Monte Carlo as a problem solver. The arena is general insurance where many problems relating to pricing and control can be worked out from simulated samples $\mathcal{X}_1^*, \ldots, \mathcal{X}_m^*$ of the portfolio pay-out $\mathcal{X}$. How they are generated is shown below through skeleton algorithms. Ideas for validation are included. These things are best learned by example, and we shall illustrate on the following concrete case:

$J = 100$	$\mu T = 10\%$	$Z \sim \text{Pareto}(\alpha, \beta)$ with $\alpha = 3,\ \beta = 2,$
number of policies	*annual claim frequency*	*claim size distribution*

which is a small portfolio, potentially with very large claims. The average number is $J\mu T = 10$ incidents annually, and expected loss and standard deviation per event are $\xi_z = E(Z)$ and $\sigma_z = \mathrm{sd}(Z)$ which become $\xi_z = 1$ and $\sigma_z = \sqrt{3}$; see formulae in (**??**) and (**??**) in Section 2.5. The set-up is reminiscent of industrial installations insured through a small company such as a **captive**[1]. The uncertainty from one year to the next one is huge which Monte Carlo handles better than most other computational methods.

Skeleton algorithms

Monte Carlo implementation of portfolio risk in general insurance does not differ much from Algorithm 1.1. When risks are identical:

Algorithm 3.1. Portfolio risk, identical policies

0 Input: $\lambda = J\mu T$, distribution of Z
1 $\mathcal{X}^* \leftarrow 0$
2 Generate $\mathcal{N}^*$ *%Often Poisson(λ) by means of Algorithm 2.11, alternative model in Chapter 8*
3 For $i = 1, \ldots \mathcal{N}^*$ do
4   Draw Z^* *%Algorithms in Section 2.5, additional ones in Chapter 9*
5      $\mathcal{X}^* \leftarrow \mathcal{X}^* + Z^*$ *%Extension: Add $H(Z^*)$ instead; see Algorithm 3.3*
6 Return $\mathcal{X}^*$ *%Re-insurance in terms of $\mathcal{X}$: Return $H(\mathcal{X}^*)$ instead*

The logic is straightforward. Start by drawing the number of claims $\mathcal{N}^*$ and add the $\mathcal{N}^*$ losses incurred, drawing each one randomly. On Lines 2 and 4 sub-algorithms must be inserted. The study below employs Algorithm 2.11 (for Poisson distributions) and Algorithm 2.9 (Pareto). The number of commands are *not* high. Note that the algorithm also applies to individual policies by taking $J = 1$. A second loop around the preceding commands then yields a version where policy risks vary:

Algorithm 3.2. Portfolio risk; heterogeneous case

0 Input: Information on all policies
1 $\mathcal{X}^* \leftarrow 0$
2 For $j = 1, \ldots, J$ do
3   Draw X_j^* *%Algorithm 3.1 for* **single policies**, *information on j'th policy read from file*
4      $\mathcal{X}^* \leftarrow \mathcal{X}^* + X_j^*$
5 Return $\mathcal{X}^*$

This second algorithm takes us through the entire portfolio, which could be a long loop, but that doesn't matter much. Modern computational facilities are up to it, and often most of the computer work is to draw the claims anyway (which is the same amount of work with both algorithms). How the set-up is modified to deal with re-insurance is explained below.

Checking the program

[1] A captive is an insurance company set up by a mother firm to handle its insurance, often for reasons of taxation.

	95% reserve					99% reserve				
m	Repeated experiments					Repeated experiments				
1000	20.8	21.0	21.6	19.9	19.9	27.9	27.5	31.3	29.6	31.3
10000	20.7	20.7	21.4	20.8	20.8	30.2	30.1	30.5	31.0	31.1

Table 3.1 Reserve for the Section 3.3 portfolio. No limit on responsibility

Does the program work as intended? Anyone who has attempted computer programming, knows how easily errors creep in. Bug detection technique may belong to computer science, but we shouldn't ignore it, and many situations offer tricks that can be used for control. In the present instant one way is to utilize that

$$E(\mathcal{X}) = J\mu T\xi_z \qquad \text{and} \qquad \text{sd}(\mathcal{X}) = \sqrt{J\mu T(\xi_z^2 + \sigma_z^2)},$$

which is proved in Section 6.3. Though the simulations may have been meant for the percentiles of $\mathcal{X}$, we can always compute their average and standard deviation and compare them to the the exact ones. Most programming errors will then materialize.

For the portfolio described in Section 3.1 a test based on 1000 simulations gave the following results:

Exact premium	Monte Carlo $\bar{X}^*$	Exact standard deviation	Monte Carlo s^*.
10	9.84 (0.22)	6.32	6.84 (0.77)

The number of simulations is not large, and there are discrepancies between the exact values and the Monte Carlo approximations. Are they within a plausible range? If not, the reason can only be programming error. Here both Monte Carlo assessments are within ± two standard deviations (in parenthesis), and there is no sign of anything being wrong. Method: Estimated standard deviations are

$$\frac{s^*}{\sqrt{m}} \quad (\text{for } \bar{X}^*) \qquad \text{and} \qquad \frac{s^*}{\sqrt{2m}}\sqrt{1+\kappa^*/2} \quad (\text{for } s^*);$$

see (**??**) and (**??**). The kurtosis κ^* (value: 48.6 here) can be taken from the simulations $\mathcal{X}_1^*, \dots, \mathcal{X}_m^*$ as explained in Exercise 2.2.8. Insert $s^* = 6.84$, $\kappa^* = 48.6$ and $m = 1000$ and you get the values above.

Computing the reserve

Simulations produce simple assessments of the reserve (no theoretical expressions now!). They must then be ranked, say in in descending order as

$$\mathcal{X}_{(1)}^* \geq \mathcal{X}_{(2)}^* \geq \dots \geq \mathcal{X}_{(m)}^*,$$

and $\mathcal{X}_{(m\varepsilon)}^*$ is used for the ε-percentile. The Monte Carlo variability of such assessments have been indicated in Table 3.1 which lists results from five repeated experiments. The uncertainty appears uncomfortably high when $m = 1000$, but is much dampened for $m = 10000$. Are your standards so strict that even the latter isn't enough? Arguably even $m = 1000$ suffices in a case like the present one where model parameters are almost certain to be hugely in error; see Chapter 7.

It seems to be a growing trend towards $\epsilon = 1\%$ as international standard. If adopted here, a company has to set aside around $30 - 31$ (say million) as guarantee for its solvency, about three

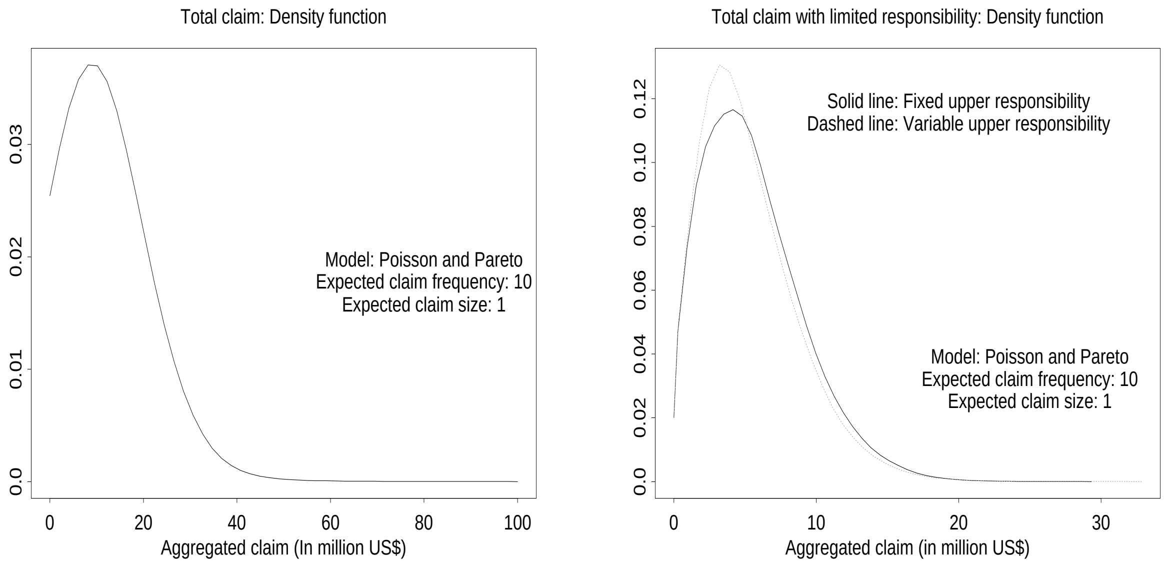

Figure 3.1 *Density function of portfolio in Section 3.3 without (***left***) and with (***right***) limit on responsibility.* **Note:** *Scales on axes differ.*

times as much as the average loss of the portfolio. But expenses *could* go higher. In Figure 3.1 left the probability density function of $\mathcal{X}$ has been estimated (using 100000 simulations). Skewness is very pronounced, and variation stretches all the way up to 100 million. These high values are so rare that the solvency criterion does not capture them.

When responsibility is limited

The modification when compensations are $H(Z)$ instead of Z was indicated in Algorithm 3.1; see comment on Line 5. Instead of the command $\mathcal{X}^* \leftarrow \mathcal{X}^* + Z^*$ take

$$\mathcal{X}^* \leftarrow \mathcal{X}^* + H^* \qquad \text{where} \qquad H^* \leftarrow H(Z^*).$$

A sub-algorithm is needed to compute the Monte Carlo compensation H^* to the insured. It runs as follows when $H(z)$ is the payment function (1.4):

Algorithm 3.3 Deductible and maximum responsibility

0 Input Z^* and limits a, b
1 $H^* \leftarrow Z^* - a$
2 If($H^* < 0$) then $H^* \leftarrow 0$
    else if $H^* > b$ then $H^* \leftarrow b$
3 Return H^*

The logic is easy. Start by subtracting the deductible a. If we now are *below* zero or *above* the upper limit, output is modified accordingly, and the original claim Z^* has been changed to the amount H^* actually reimbursed.

The effect on the reserve has been indicated by means of the following experiment. Let a be a common deductible and allow the upper limits of responsibility $b_1, \dots, b_{100}$ to vary over portfolio.

Scenarios examined were

$$a = 0.5,\ b_1 = \ldots = b_{100} = 3.5 \qquad\qquad a = 0.5,\ b_1 = \ldots = b_{50} = 1.5,\ b_{51} = \ldots = b_{100} = 5.5.$$

Fixed limit scenario — *Variable limit scenario*

On the left all contracts are equal with maximum responsibility 3.5 times the average claim, and on the right the upper limit was either 1.5 or 5.5, still with 3.5 as the average.

Running $m = 100000$ simulations under the two regimes lead to the following assessments of the reserve under variation of the solvency level:

90%	95%	99%	99.97%
10.2	12.0	15.6	21.9

Upper limits: Fixed

90%	95%	99%	99.97%
9.7	11.5	15.2	21.7

Upper limits: Variable

Values at 99% (about 15 now for both regimes) has been halved compared to the unlimited case, and it doesn't matter too much that the upper limit depends on policy. Estimated density functions of the portfolio claims are plotted in Figure 3.1 right. The distribution is still skew, but no longer with those super-heavy tails you find on the left.

Dealing with re-insurance

Re-insurance in terms of single events is computationally very much the same as in the preceding example, but contracts that apply to the aggregated claim $\mathcal{X}$ are different. Now the re-insurer share is $\mathcal{X}^{\mathrm{re}} = H(\mathcal{X})$ which leads to the pure re-insurance premium

$$\pi^{\mathrm{re}} = E\{H(\mathcal{X})\} \qquad \text{approximated by} \qquad \pi^{\mathrm{re}*} = \frac{1}{m}\sum_{i=1}^{m} H_i^* \quad \text{where} \quad H_i^* = H(\mathcal{X}_i^*).$$

Cedent net responsibility is $\mathcal{X}^{\mathrm{ce}} = \mathcal{X} - H(\mathcal{X})$. Let $C = \mathcal{X}^{\mathrm{ce}}$ to ease notation. Then cedent net reserve becomes

$$C^*_{(m\varepsilon)} \quad \text{where} \quad C^*_{(1)} \geq C^*_2 \geq \cdots \geq C^*_m, \quad \text{sorted from} \quad C^*_i = \mathcal{X}^*_i - H(\mathcal{X}^*_i),\ \ i = 1, \ldots, m.$$

Re-insurer share H_i^* is computed by applying Algorithm 3.3 to the output from Algorithm 3.1 or 3.2; see comments in Algorithm 3.1.

The portfolio of the preceding section with Pareto distributed claims is used as illustration in Table 3.2. Original cedent responsibility was unlimited (not common in practice), and the re-insurer part $\mathcal{X}^{\mathrm{re}} = \max(\mathcal{X} - a, 0)$. This makes re-insurance coverage unlimited too (again uncommon). Contracts of this particular type is known as **stop loss**. Table 3.2 shows re-insurance premium and cedent net reserve (99%) as the retention limit a is varied. Note how the reserve of around 30 million is cut down to one third (10 million) when the re-insurer covers all obligations above $a = 10$ million. The cost is 2.2 million in pure premium (and usually more in premium paid). In practice companies taylor the amount of re-insurance by balancing capital saved against extra cost.

1.4 Life insurance: A different story

Introduction

Liabilities in life and pension insurance are rarely handled by the approach of the preceding section *m = 1000000 simulations*

Retention limit (a)	0	10	20	30	40	50
99% cedent reserve	0	10	20	30	30.7	30.7
Re-insurance pure premium	10	2.23	0.39	0.11	0.041	0.021

Table 3.2 *Cedent reserve and pure re-insurance premium for arrangement described in the text. Unlimited re-insurance coverage*

and for good reason too. This section indicates what lies behind by examining common arrangements such as **life annuities** where individuals receive fixed benefits until they die and **term insurance** where a lump sum is released upon the death of the policy holder. The uncertainty is due to how long people live. This is expressed mathematically through **life tables** which are probabilities ${}_kp_l$ of an individual of age $y_l = lh$ living k periods longer. The time increment h may for example be a year or a month; see Chapter 12 for more on these quantities.

A simple calculation

Uncertainty in life insurance can be understood through a simplified portfolio where all individuals are of the same age $y = lh$ and of the same sex so that the survival probabilities ${}_kp_l$ are the same for everybody. Consider a pension scheme of J policy holders collecting each period benefits $s_1, \dots, s_J$ until they die. The amount X_{jk} received by individual j at time t_k is s_j or zero, according to the probabilities

$$\underset{\textit{policy holder dead at } t_k}{\Pr(X_{jk} = 0) = 1 - {}_kp_l} \qquad \text{and} \qquad \underset{\textit{policy holder alive at } t_k}{\Pr(X_{jk} = s_j) = {}_kp_l.}$$

Hence

$$E(X_{jk}) = s_j\,{}_kp_l = \xi_{jk} \qquad \text{and} \qquad \text{var}(X_{jk}) = s_j^2\,{}_kp_l(1 - {}_kp_l) = \sigma_{jk}^2,$$

almost mean and variance for ordinary Bernoulli variables, see also Exercise 3.2.1. Our target (as in Section 3.2) is the portfolio aggregate $\mathcal{X}_k = X_{1k} + \ldots + X_{Jk}$. Introduce

$$\bar{s} = \frac{1}{J}\sum_{j=1}^{J} s_j \qquad \text{and} \qquad \bar{\sigma}_s^2 = \frac{1}{J}\sum_{j=1}^{J}(s_j - \bar{s})^2 = \frac{1}{J}\left(\sum_{j=1}^{J} s_j^2\right) - \bar{s}^2,$$

where you should verify the identity on the very right yourself if it is unfamiliar. It follows that

$$\bar{\xi}_k = \frac{1}{J}\sum_{j=1}^{J}\xi_{jk} = \left(\frac{1}{J}\sum_{j=1}^{J} s_j\right){}_kp_l = \bar{s}\,{}_kp_l$$

and that

$$\bar{\sigma}_k^2 = \frac{1}{J}\sum_{j=1}^{J}\sigma_{jk}^2 = \left(\frac{1}{J}\sum_{j=1}^{J} s_j^2\right){}_kp_l(1 - {}_kp_l) = (\bar{\sigma}_s^2 + \bar{s}^2){}_kp_l(1 - {}_kp_l).$$

These expressions may be inserted into (1.7) which yields after some manipulations

$$\frac{\text{sd}(\mathcal{X}_k)}{E(\mathcal{X}_k)} = \frac{\bar{\sigma}_k/\bar{\xi}_k}{\sqrt{J}} = \left(\frac{(1/{}_kp_l - 1)(1 + (\bar{\sigma}_s/\bar{s})^2)}{J}\right)^{1/2}. \tag{1.8}$$

How much is this? Try ${}_kp_l = 0.98$, $\bar{\sigma}_s = \bar{s}$ and $J = 100$ and you get 0.02; i.e. standard deviation is no more that 2% of the expected value even for a portfolio of micro-size, and goes further down to 0.2% for ten thousand policies and 0.02% for one million. At portfolio level randomness doesn't amount to much, and usually only expectations $E(\mathcal{X}_k)$ are reported.

Term insurance isn't quite the same. Now pay-off follows death and not survival, and in (1.8) ${}_kp_k$ must be replaced by $1 - {}_kp_l$. This increases uncertainty considerably although still usually ignored by tradition; consult also the term insurance portfolio simulated in Section 1.5.

Simulating pension schemes

The preceding argument suggests that simulation doesn't play one of the leading roles in life insurance. That is true and yet a bit premature. Monte Carlo *is* a highly relevant tool with other aspects; see Chapter 15. But even in the present context where random effects are largely unimportant, it isn't such a bad idea to build simulation models to visualize what happens. Traditional pension arrangements (known as **defined benefit** schemes) has a build-up stage where premium π is contributed. After retirement at age l_r a pension ζ is drawn. Such cash flows will be denoted $\{\zeta_l\}$. In the present case

$$\zeta_l = -\pi \quad \text{if} \quad l < l_r \qquad \text{and} \qquad \zeta_l = \zeta \quad \text{if} \quad l \geq l_r,$$

where premium is counted *negative*. Uncertainty as to how long the policy holder lives converts the *fixed* payment stream $\{\zeta_l\}$ into a *random* one $\{X_k\}$ which in turn defines a *random* present value

$$\text{PV}_0 = \sum_{k=0}^{\infty} d_k X_k \qquad \text{where} \qquad d_k = \frac{1}{(1+r)^k}. \tag{1.9}$$

Simulations are organized as follows:

Algorithm 3.4 Pension cash flow for single individual

0 Input: $\{{}_1p_l\}$, $\{\zeta_l\}$, initial age l
1 Initial: $\text{PV}_0^* \leftarrow 0$, $d \leftarrow 1$
2 For $k = 0, 1, \ldots, K$ do — *%Present value including K periods ahead*
3 $\quad \text{PV}_0^* \leftarrow \text{PV}_0^* + \zeta_l d$ — *%Payment in advance*
4 $\quad l \leftarrow l + 1$ and $d \leftarrow d/(1+r)$ — *%Update and discount*
5 $\quad$ Draw $U^* \sim$ uniform
6 $\quad$ If $(U^* > {}_1p_l)$ **stop** and return PV_0^* — *%Policy holder has died*

The algorithm goes through the life of the policy holder up to K years ahead and tests (on Line 6) whether he (or she) stays alive. If so, the discounted benefit is added to the present value the next time. The cash flow $\{\zeta_l\}$ must be stored on input and is arbitrary. Payments in Algorithm 3.4 are **in advance**, and the set-up requires a slight change if they are made **in arrears**; i.e. at the termination of each period; see Chapter 12. To simulate the entire portfolio run the algorithm once for each policy holder and add the output.

Example: A run-off portfolio

Run-off status means a portfolio where existing obligations are being liquidated. There is no premium income and no new recrutes. Such situations occur in all forms of insurance. Here a

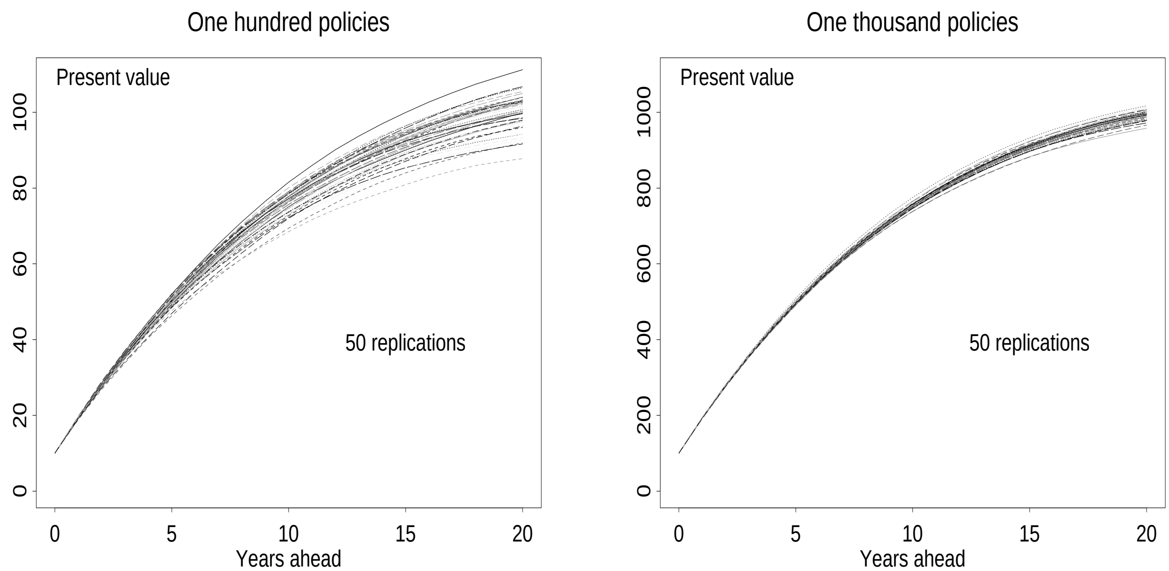

Figure 3.2 *Simulated present values for portfolios of pension liabilities. Conditions outlined in the text.*

pension scheme with members past the retirement age is being considered. The question is how much money it takes to support the benefit they receive until they die.

Such evaluations are based on specific life tables. The one selected is of the **Gomperz-Makeham** type. An invidual of age l survives until next year with probability

$$\log({}_1p_l) = -0.0009 - 0.0000462e^{0.090767 \times l}$$

where parameters are for males in a developed country and correspond to a life expectancy of 75 years; see Section 12.3. All individuals were past 60 (drawing pension) with 93 as the oldest. The age distribution between these extremes were laid out as explained in Section 15.2. Additional assumptions were

$$\zeta = 0.1, \quad l_r = 60, \quad r = 3\%, \quad J = 100 \text{ or } J = 1000,$$

where the money unit could be million US$ (correspond to an annual pension of 100000).

The portfolios are small since the idea is to indicate that the uncertainty due to how long people live isn't very important. In Figure 3.2 the present values of payments up to K years ahead have been computed and plotted against K. The downward curvature is caused by the discounting (and also by mortality increasing with age), but the main thing is random variation. It is quite small on the right, somewhat larger when $J = 100$ on the left.

1.5 Financial risk: Derivatives as safety

Introduction

Financial risk can be reduced by spreading investments on different assets. Another way is through financial **derivatives** or **options** which protect against market movements that generate loss.

Numerous types of arrangements are in practical use (with the academic literature containing still more). The so-called **European** contracts (the only ones to be considered) release at a future date the amount

$$X = H(\mathcal{R})v_0, \tag{1.10}$$

where $\mathcal{R}$ is the return of a financial investment at that time, $H(r)$ a function containing the detailed payment clauses and v_0 the original value of the investment. Derivatives are secondary risk products, *derived* from primary ones, much like re-insurance being secondary products in the insurance world.

The right to future compensation against unwelcome market news has to be paid for. This section explains how such fees (called **option premia**) are calculated. The underlying mathematical argument (which is complicated) is deferred to Chapter 14, and equity is the only type of asset considered. Derivatives in the money market is at least as important, but you have to consult Chapter 14 for those.

Equity puts and calls
There are no limits on the number of functions $H(r)$ you could use in (1.10). Two of the most widely used ones are

$$\underbrace{X = \max(r_g - \mathcal{R}, 0)v_0}_{\textit{put option}} \qquad \text{and} \qquad \underbrace{X = \max(\mathcal{R} - r_g, 0)v_0}_{\textit{call option}}, \tag{1.11}$$

where r_g is a given rate of interest and v_0 the initial value of the underlying asset (assumed to be equity). The **put** option on the left releases compensation whenever the return $\mathcal{R}$ falls below the floor r_g. This provides a **guaranteed return** in the sense that

$$(1 + \mathcal{R})v_0 + X = (1 + \mathcal{R})v_0 + \max(r_g - \mathcal{R}, 0)v_0 \geq (1 + r_g)v_0$$

and r_g is a minimum return on the investment (but only when the fee is not counted). Many life insurance products are drawn up with such clauses. **Call** options are the opposite. Now there is extra money if $\mathcal{R}$ exceeds r_g; see (1.11) right, and a borrower is protected against high financial cost.

How equity options are valued
Derivatives are paid for up-front when contracts are drawn up. If that occurs at $t_0 = 0$ and the expiry is at T, possible valuations could be

$$\underbrace{\pi = e^{-rT}E(X)}_{\textit{actuarial pricing}} \qquad \text{or} \qquad \underbrace{\pi = e^{-rT}E_Q(X)}_{\textit{risk neutral pricing}}, \tag{1.12}$$

where e^{-rT} is an ordinary discount. The central feature is the expected pay-off. Why on earth shouldn't we use ordinary **actuarial** pricing on the left? That's how we operate in insurance and re-insurance, but financial derivatives are different in that they can be **hedged** in a way ordinary insurance can not. Consider call options. Sellers of such contracts lose in a rising market when $\mathcal{R} > r_g$ but they may hold the underlying stock on the side and in this way off-set the loss (at least partially). Their real risk must then be smaller than X itself, and $E(X)$ isn't a break-even price as in insurance.

The issue requires a big dose of mathematics. A series of fine-tuned hedging operations will in Chapter 14 lead to the so-called **risk-neutral** price in (1.12) right. It doesn't look that different from the other! Yet there *is* a crucial difference. The expectation is now calculated with respect to a special valuation model (usually denoted Q) which is as follows. Suppose the option $X = H(R)$ applies to a single equity asset with return R following the usual log-normal model. Mean and variance are then proportional to T (you'll see why in Section 5.7) so that $R = \exp(\xi T + \sigma\sqrt{T}\varepsilon) - 1$ where ε is $N(0,1)$. The Q-model then turns out to be

$$R = \exp(\xi_q T + \sigma\sqrt{T}\varepsilon) - 1 \qquad \text{where} \qquad \underset{Q\text{-}model}{\xi_q = r - \tfrac{1}{2}\sigma^2}, \tag{1.13}$$

the same as the other one with the exception of the different ξ. Now

$$E_Q(R) = \exp(\xi_q T + \sigma^2 T/2) - 1 = \exp(rT) - 1$$

and the expected return of holding stock coincides with what you get from a bank account. That's not much to get from risky shares, but then it is for valuation only. What lies behind is a hedging so perfect that all risk for the option seller (in theory) disappears(!). Consequence: Derivatives are valued as if equity risk does not matter.

The Black-Scholes formula

The put option in (1.11) has premium

$$\pi(v_0) = E_Q\{\max(r_g - R, 0)\}v_0 \qquad \text{where} \qquad R = \exp(\xi_q T + \sigma\sqrt{T}\varepsilon) - 1$$

for which a closed formula is available.. Indeed,

$$\pi(v_0) = \{(1 + r_g)e^{-rT}\Phi(a) - \Phi(a - \sigma\sqrt{T})\}v_0, \quad a = \frac{\log(1 + r_g) - rT + \sigma^2 T/2}{\sigma\sqrt{T}}, \tag{1.14}$$

where $\Phi(x)$ is the standard normal integral. The result, verified in Section 3.7, is known as the **Black-Scholes** formula and is one of the most prominent results in modern finance. If you take the trouble of differentiating it with respect to σ, you will discover that

$$\frac{\partial \pi(v_0)}{\partial \sigma} = \varphi(a - \sigma)v_0 \qquad \text{where} \qquad \varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

This is always positive which means that higher uncertainty makes put options more expensive. That seems plausible, but it isn't a general result, and other derivatives are different. A similar pricing formula for calls will be detailed in Chapter 14.

Options may be traded, and we need their value at time t_k. This is an easy matter. In the Black-Scholes formula replace v_0 with the value of the underlying stock at t_k and T with the remaining time $T - t_k$ to expiry.

Options on portfolios

Valuation for derivatives that apply to several assets jointly is a direct extension of the one-asset case. Now the risk-neutral model for J equity returns $R_1, \ldots, R_J$ is

$$R_j = \exp(\xi_{qj} T + \sigma_j\sqrt{T}\varepsilon_{qj}) - 1, \qquad \underset{Q\text{-}model}{\xi_{qj} = r - \tfrac{1}{2}\sigma_j^2}, \qquad \text{for} \quad j = 1, \ldots, J, \tag{1.15}$$

where $\varepsilon_1, \ldots, \varepsilon_J$ are $N(0,1)$ and usually *correlated*. Correlations and volatilities are inherited from the real model whereas the original expectations ξ_j are replaced by the risk-neutral version $\xi_{qj} = r - \sigma_j^2/2$. What is *not* the same as in the one-asset case is computation. Closed pricing formulae do not exist, and Monte Carlo is the usual method. The algorithm runs as follows:

Algorithm 3.5. Simulating equity options

0 Input: r, v_0 volatilities and correlations, asset weights $w_1, \ldots, w_J$

1 Draw $\varepsilon_1^*, \ldots, \varepsilon_J^*$ *%All $N(0,1)$ and correlated, Algorithm 2.4 or 5.2*

2 $\mathcal{R}^* \leftarrow 0$

3 Repeat for $j = 1, \ldots, J$

4 $\quad R^* \leftarrow \exp(rT - \sigma_j^2 T/2 + \sigma_j \sqrt{T}\varepsilon^*) - 1$ *%Return j'th asset*

5 $\quad \mathcal{R}^* \leftarrow \mathcal{R}^* + w_j R^*$ *%Updating portfolio return*

6 $X^* \leftarrow H(\mathcal{R}^*)v_0$ *%For put options:* *If $\mathcal{R}^* \geq r_g$ then $X^* \leftarrow 0$ else $X^* \leftarrow (r_g - \mathcal{R}^*)v_0$*

7 Return X^*.

The program converts a correlated sample of standard normal Monte Carlo variables $\varepsilon_1^*, \ldots, \varepsilon_J^*$ into the portfolio return $\mathcal{R}^*$ and a pay-off X^*. From m replications $X_1^*, \ldots, X_m^*$ we may compute the discounted average

$$\pi^* = \frac{e^{-rT}}{m} \sum_{i=1}^{m} X_i^*$$

and use that as an *approximate* option premium. This kind of simulation will also be needed in Chapter 15 when financial risk is examined from a broader perspective.

Are equity options expensive?

The minimum return of a put option cited above was *after* premium has been paid, and the **effective** minimum is lower. When this expense is drawn from the original capital v_0, the balance sheet becomes

$$\underbrace{\frac{v_0}{1+\pi(1)}}_{\text{equity protected}} \quad + \quad \underbrace{\pi(1)\frac{v_0}{1+\pi(1)}}_{\text{option premium}} \quad = \quad \underbrace{v_0}_{\text{original capital}}$$

where value of the equity is reduced to $v_0/\{1+\pi(1)\}$ after the option premium has been subtracted. At expiry the investor is guaranteed

$$\frac{v_0}{1+\pi(1)}(1+r_g) = (1+r_g')v_0 \qquad \text{where} \qquad r_g' = \frac{r_g - \pi(1)}{1+\pi(1)} < r_g - \pi(1),$$

and the effective minimum return is a little lower than $r_g - \pi(1)$.

How much is the option premium $\pi(1)$ eating up? It depends on the circumstances. Here is an example with $J = 4$ risky assets with equal weights $w_1 = \ldots w_4 = 0.25$. Their model is assumed log-normal and equi-correlated (as in Section 2.3) and with common volatility. The annual guarantee $r_g = 7\%$ and the risk-free rate $r = 4\%$ lead through Algorithm 3.5 to the prices in Figure 3.3 left where $\pi(1)$ is plotted (in per cent) against asset volatility for two different values of the correlation.

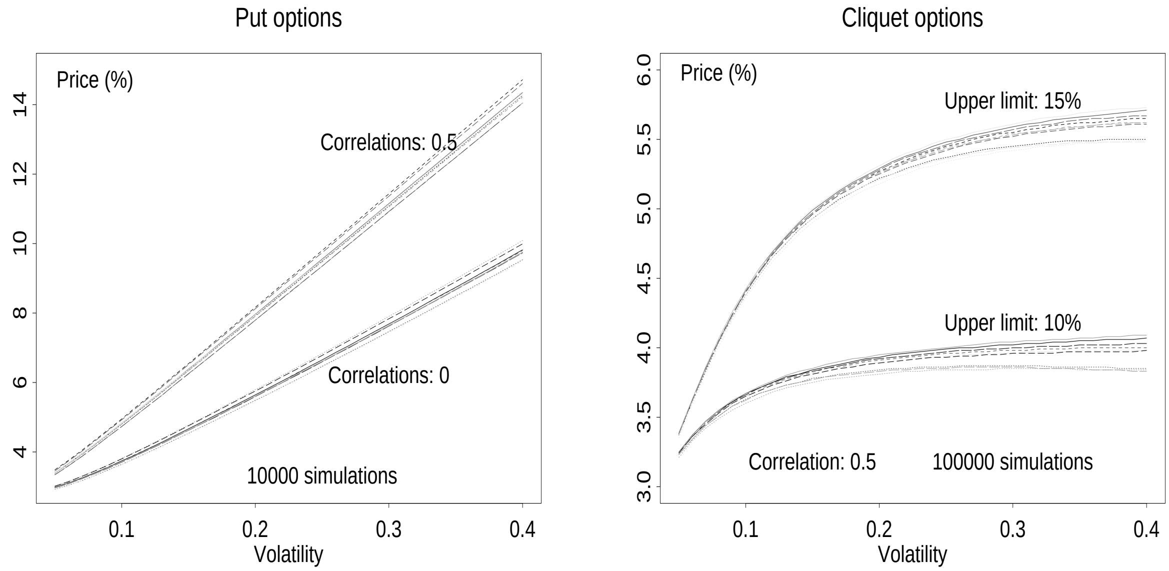

Figure 3.3 *Prices of put options (left) and cliquet options (right), quoted in per cent of the original holding. Conditions given in the text.*

Options are expensive! Annual volatilities of 25% (not unrealistic at all) would lead to a cost of $6-10\%$ depending on the correlation. High volatility and high correlation increase the uncertainty and make the price higher. Ten replications, each based on $m = 10000$ simulations, are plotted jointly[2] and indicate a Monte Carlo uncertainty that might in practice be found unacceptable for fixing the price.

One way to lower the cost is to allow the option seller to keep the top of the return. Such instruments, sometimes called **cliquet** options, have the pay-off

$$H(\mathcal{R}) = \begin{array}{ll} r_g - \mathcal{R}, & \mathcal{R} \leq r_g \\ 0, & r_g < \mathcal{R} \leq r_c \\ -(\mathcal{R} - r_c), & \mathcal{R} > r_c, \end{array} \tag{1.16}$$

where the third line signals that return above a ceiling r_c is kept by the option seller. The guarantee is still r_g, but this trick makes the instrument cheaper; see Figure 3.3 right where the underlying conditions are as before (correlation between assets are 0.5) When the ceiling is $r_c = 15\%$ and the volatility 25%, the price of the cliquet is close to half of that of the put.

1.6 Risk over long

Introduction

Life insurance in Section 3.4 looked many years ahead and investments of such funds must too. Even general insurance (often on an annual basis) may benefit from a long-term view, for example for planning capital requirements. We are then dealing with recursions like

$$\mathcal{Y}_k = \mathcal{Y}_{k-1} + \underbrace{\mathcal{R}_k\mathcal{Y}_{k-1}}_{\textit{financial income}} + \underbrace{\Pi_k}_{\textit{premium}} - \underbrace{\mathcal{O}_k}_{\textit{overhead}} - \underbrace{\mathcal{X}_k}_{\textit{liabilities}}, \tag{1.17}$$

[2]Smoothness of the curves was achieved by using common random numbers; see Section 4.3.

for $k = 1, 2 \ldots$. Here financial income ($\mathcal{R}_k\mathcal{Y}_{k-1}$), premium income ($\Pi_k$), overhead cost ($\mathcal{O}_k$) and liabilitites ($\mathcal{X}_k$) are integrated into summaries that yield **net assets** $\{\mathcal{Y}_k\}$. There could be other contributions as well, and both liabilities and financial return could be complex affairs with many sub-contributions; see the extensions in Chapter 11 (general insurance) and Chapter 15 (life insurance). Here details are kept very simple.

The ruin problem

The recursion (1.17) starts at $\mathcal{Y}_0 = v_0$. How the the inital capital v_0 should be determined under a long-term view is a classic of actuarial science. A commonly used criterion is that net assets should be positive at all times up to some terminating t_K. In mathematical terms the event $\mathcal{Y}_1, \ldots, \mathcal{Y}_K > 0$ must occur with high probability or the opposite that the **ruin probability**

$$p^{\text{ru}}(v_0) = \Pr(\underline{\mathcal{Y}} < 0 | Y_0 = v_0) \qquad \text{where} \qquad \underline{\mathcal{Y}} = \min(\mathcal{Y}_1, \ldots, \mathcal{Y}_K) \tag{1.18}$$

is small. If $\underline{\mathcal{Y}} < 0$, the portfolio at some point is out of money, and this outcome should be a remote one.

The phrase 'ruin' is not to be taken too literally as companies (and regulators) are supposed to intervene long before that happens. Ruin probabilities are for planning and to indicate a suitable amount of capital. A standard yardstick is the equation

$$p^{\text{ru}}(v_0) = \epsilon \qquad \text{with solution} \qquad v_0 = v_{0\varepsilon}, \tag{1.19}$$

and if $v_{0\varepsilon}$ is put up in the beginning, the chance of net assets falling below zero during the next K periods is no more than ϵ. Solutions can in special cases be approximated by mathematical formulae; see Section 3.8. Monte Carlo is usually easier and often more accurate too.

A skeleton algorithm

Computer simulations of net asset values $\{\mathcal{Y}_k\}$ from the recursion (1.17) can be organized as follows:

Algorithm 3.6 Integrating assets and liabilities

0 Input: Models for $\mathcal{R}_k$ and $\mathcal{X}_k$, sequences $\{\Pi_k\}$ and $\{\mathcal{O}_k\}$

1 $\mathcal{Y}_0^* \leftarrow v_0$ and $\underline{\mathcal{Y}}^* \leftarrow$ large value   *% Intial reserve and initial minmum*

2 For $k = 1, \ldots, K$ do

3   Generate $\mathcal{X}^*$ *%Liability in period k, could be life or non-life, simple possibilities: Algorithms 3.1 or 3.2*

4   Generate $\mathcal{R}^*$ *%Financial return in period k, simple possibility: Algorithm 2.4*

5   $\mathcal{Y}_k^* \leftarrow (1 + \mathcal{R}^*)\mathcal{Y}_{k-1}^* + (\Pi_k - \mathcal{O}_k) - \mathcal{X}^*$

6   If $\mathcal{Y}_k < \underline{\mathcal{Y}}^*$ then $\underline{\mathcal{Y}}^* \leftarrow \mathcal{Y}_k$ *%Updating the minimum*

7 Return $\mathcal{Y}_0^*, \ldots, \mathcal{Y}_K^*$ and $\underline{\mathcal{Y}}^*$

The scheme is a loop over k with procedures generating liabilities (Line 3) and financial return (Line 4). Both may be the end product of complex simulations with countless sub-components, and liabiltities may be both life and non-life. Only simple versions drawing on earlier algorithms are considered below, and premium income Π_k and overhead $\mathcal{O}_k$ are fixed, therefore not *-marked (but there will be stochastic versions in Chapter 11). It is implicit in Algorithm 3.6 that liabilities and financial return are unrelated and generated independently of each other. Does that appear obvious? There are in real life many situations (especially in life insurance) where economic factors influence both and create links between them. Such issues are discussed in Chapter 15.

The algorithm returns a minimum value $\underline{\mathcal{Y}}^*$ over K periods. This has been built into the recursion (Line 6) by updating the preceding minimum if the current asset $\mathcal{Y}_k^*$ is smaller. With m replications $\underline{\mathcal{Y}}_1^*, \dots, \underline{\mathcal{Y}}_m^*$ the ruin probability is approximated by

$$p^{\text{ru}*}(v_0) = \frac{1}{m}(I_1^* + \dots + I_m^*) \qquad \text{where} \qquad \begin{aligned} I_i^* &= 0 \quad \text{if } \underline{\mathcal{Y}}_i^* > 0 \\ &= 1 \quad \text{otherwise,} \end{aligned} \tag{1.20}$$

which is simply a count how many times the net assets at some point has become negative. We would like to solve the equation

$$p^{\text{ru}*}(v_{0\epsilon}^*) = \epsilon$$

so that $v_{0\epsilon}^*$ can be used as an approximation to the exact value $v_{0\epsilon}$ in (1.19). There is in general no simple way to do this, and the ususal method is by trial and error; see below.

Underwriter risk

Underwriting is the insurance part of the business with the financial side ignored. Many actuarial evaluations are of this type. Algorithm 3.6 still applies if you take $\mathcal{R}^* = 0$ on Line 5. Unlike the general case there *is* now a smart way to determine the approximate solution of the ruin problem. Start at $v_0 = 0$ with *no* initial capital. The account will often go into minus (economically this means that money is borrowed from somewhere), but we may still run it and generate $\underline{\mathcal{Y}}_1^*, \dots, \underline{\mathcal{Y}}_m^*$ as m realisations of the minimum. Rank them in ascending order as $\underline{\mathcal{Y}}_{(1)}^* \leq \dots \leq \underline{\mathcal{Y}}_{(m)}^*$. An approximation of the exact $v_{0\epsilon}$ is then

$$v_{0\epsilon}^* = -\underline{\mathcal{Y}}_{(\epsilon m)}^* \qquad \text{where} \qquad v_{0\epsilon}^* \to v_{0\epsilon} \text{ as } m \to \infty; \tag{1.21}$$

see Section 3.7 where the result is proved.

As an example consider the large claims portfolio of Section 3.3. There were on average ten claims per year of the heavy-tailed Pareto type with limited responsibility (same for all portfolios). Simulated scenarios (100 replications) have been plotted jointly in Figure 3.4 left. All were started from $v_0 = 15$ million (the 99% *annual* reserve for this portfolio, see Section 3.3) The net premium $\Pi_k - \mathcal{O}_k$ was fixed at 6.0 exceeding the pure premium (5.4) by about 10%, which accounts for a slight average drift upwards, barely discernable and over-shaddowed by the enormous uncertainty. Earnings are sometimes huge, (up to 100% and more over ten years), but losses may be severe too (despite coverage being limited). These matters have now been learned in advance and may be used as input to our business strategy.

Evaluations of ruin probabilities over five years are shown in Figure 3.4 right under variation of the initial capital v_0. Recall that the *annual* 99% reserve for this portfolio was 15 million, but over five years this corresponds to no more than $92 - 93\%$, and it must now be doubled to reach 99%. There are two versions in Figure 3.4 right. The smooth, solid line is based on the same sequence of random numbers for each v_0. This approach, known as **common random numbers**, leads to smooth pictures, well suited for presentation. The alternative dotted line took different random sequences for each v_0 plotted. That leads to annoying bumps due to randomness and is an

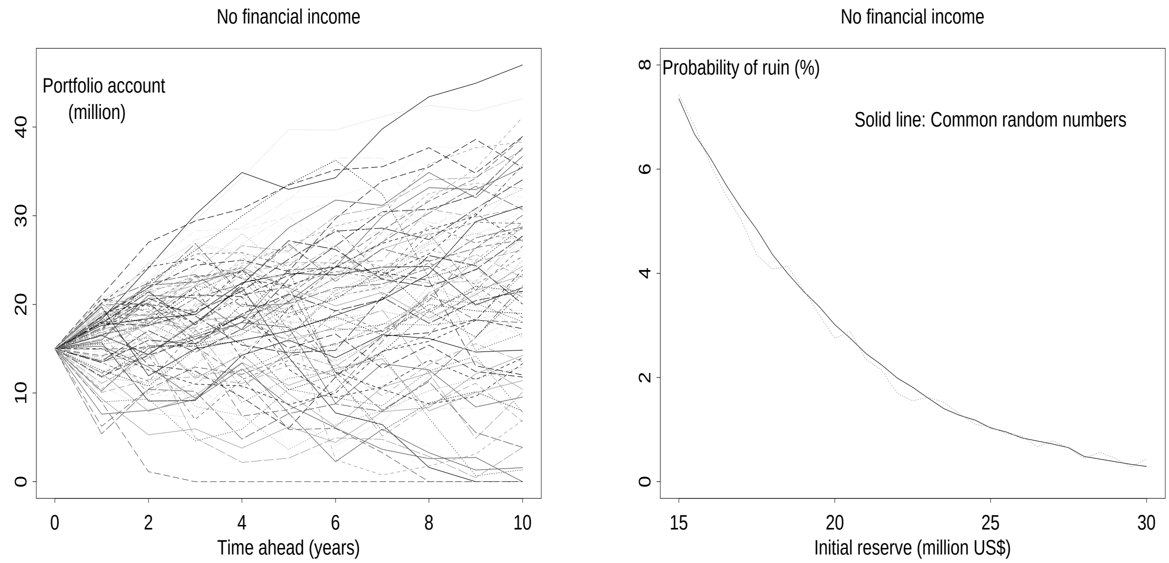

Figure 3.4 *Underwriter results for the portfolio of Section 3.3.* **Left**: *One hundred simulated scenarios.* **Right:** *Ruin probabilities (5 years) against initial capital from* $m = 10000$ *simulations*

inferior simulation strategy for other reasons too; see Chapter 4.

Financial income added

How much are underwriter results changed when financial earnings are included? A first example is given in Figure 3.5 left where a fixed annual return $\mathcal{R} = 5\%$ of capital has been added the simulations in Figure 3.4 left. There is now a noticeable lift upwards, yet the dominant force is still insurance uncertainty. If the original capital is placed at 5% annually in a bank, it has after ten years grown to $15 \times 1.05^{10} \doteq 24.4$, right in the middle of the heap. In practice there is not only financial income, but also financial risk. If investments are in equity, the portfolio returns $\mathcal{R}^*$ can be simulated through the commands on Lines 1-5 in Algorithm 3.5 and inserted on Line 4 in Algorithm 3.6.

As an example, consider a financial portfolio with $J = 4$ equally weighted, risky assets and a risk-less bank account earning a fixed rate of interest r. This leads to the portfolio return

$$\mathcal{R}_k = w_0 r + \sum_{j=1}^{J} w R_{jk}, \qquad \text{where} \qquad w_0 + Jw = 1.$$

The weights w_0 and w are kept constant at all times during simulations. It was assumed that equity returns $R_{1k}, \ldots, R_{Jk}$ followed a log-normal model with common annual drift ξ, and their volatilities $\sigma = 15.97\%$ and correlations $\rho = 0.5$. were common too. The model scenarios were varied as follows:

Model scenario	r	w_0	w	ξ	Expected portfolio return
I (no financial risk)	5%	1	0	5%	5%
II (low yield)	5%	0.4	0.15	0.0360	5%
III (high yield)	5%	0.4	0.15	0.1006	12%.

Ruin probabilities are plotted in in Figure 3.5 right. They are all much lower than those in Figure 3.4 where financial income wasn't taken into account. Note how the low-yield, risky financial

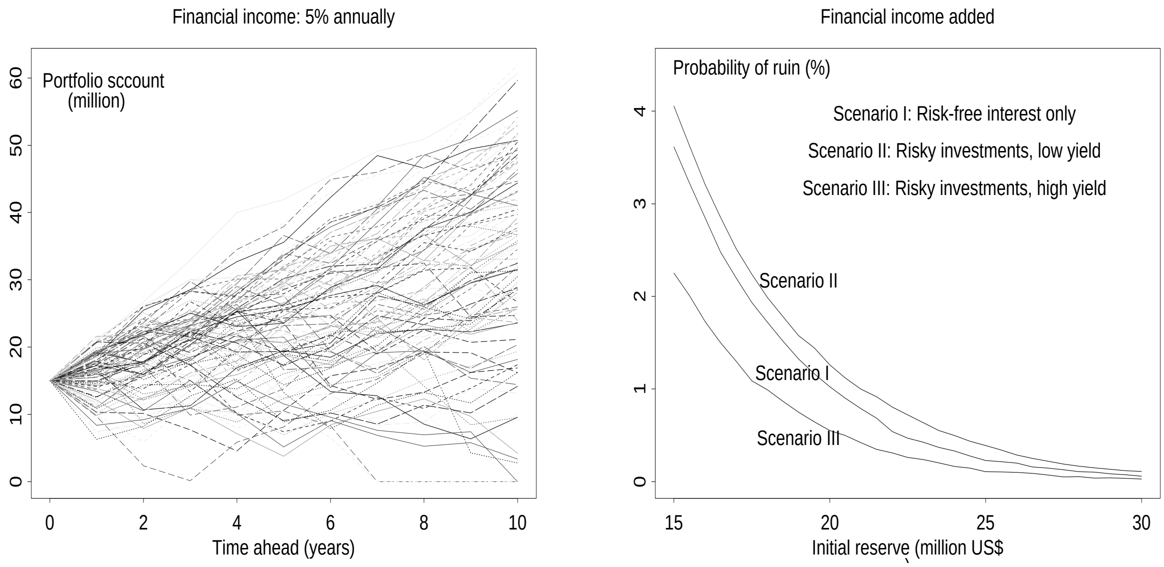

Figure 3.5 Left*: Simulated portfolio returns (*100 *replications) with* 5% *fixed annual financial return added those in Figure 3.4 left.* **Right:** *Ruin probabilities (over five years) against initial reserve under the conditions described in the text.*

scenario II raises the curve compared to the risk-less scenario I. Lowest risk of ruin is scenario III where the high expected yield pushes the assessments down despite the financial risk carried. There are no simple criteria like (1.21) to approximate $v_{0\epsilon}$, but it is possible to read the required percentiles off from the plots.

1.7 Mathematical arguments

Section 3.5

The Black-Scholes formula Premium for the put options in terms of single assets is

$$\pi(v_0) = e^{-rT} E_Q\{\max(r_g - R, 0)\} v_0 \qquad \text{where} \qquad R = \exp(\xi_q T + \sigma\sqrt{T}\varepsilon) - 1,$$

and where $\varepsilon \sim N(0,1)$. There is a positive pay-off if

$$R < r_g \qquad \text{or equivalently if} \qquad \varepsilon \le a = \frac{\log(1+r_g) - \xi_q T}{\sigma\sqrt{T}},$$

and the option premium becomes

$$\pi(v_0) = e^{-rT} v_0 \int_{-\infty}^{a} (1 + r_g - e^{\xi_q T + \sigma\sqrt{T}x})\varphi(x)dx$$

where $\varphi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$. Splitting the integrand yields

$$\pi(v_0) = e^{-rT} v_0 \left(1 + r_g)\Phi(a) - e^{\xi_q T} \int_{-\infty}^{a} e^{\sigma\sqrt{T}x}\varphi(x)dx\right),$$

where the integral on the right equals

$$\int_{-\infty}^{a} e^{\sigma\sqrt{T}x}(2\pi)^{-1/2}e^{-x^2/2}dx = e^{\sigma^2 T/2}\int_{-\infty}^{a}(2\pi)^{-1/2}e^{-(x-\sigma\sqrt{T})^2/2} = e^{\sigma^2 T/2}\Phi(a - \sigma\sqrt{T})$$

so that

$$\pi(v_0) = e^{-rT} v_0\{(1 + r_g)\Phi(a) - e^{\xi_q T + \sigma^2 T/2}\Phi(a - \sigma\sqrt{T})\}.$$

Inserting $\xi_q = r - \sigma^2/2$ from the Q-model on the right in (1.13) yields (1.14).

Section 3.6
Solvency without financial earning. We shall prove (1.21) which applies to the recursion (1.17) without financial income. Let $\underline{\mathcal{Y}}(v_0)$ signify that the initial capital is v_0. Then

$$\underline{\mathcal{Y}}(v_0) = \underline{\mathcal{Y}}(0) + v_0;$$

i.e. the effect of adding capital that does not earn interest, is to lift all simulations a fixed amount v_0. Note that $\underline{\mathcal{Y}}^*_{(\epsilon m)}$ is approximately the ϵ-percentile of $\underline{\mathcal{Y}}(0)$. Hence

$$\epsilon \doteq \Pr(\underline{\mathcal{Y}}(0) \leq \underline{\mathcal{Y}}^*_{(\epsilon m)}) = \Pr(\underline{\mathcal{Y}}(v_0) - v_0 \leq \underline{\mathcal{Y}}^*_{(\epsilon m)})$$

and

$$\epsilon \doteq \Pr(\underline{\mathcal{Y}}(v_0) \leq 0) \qquad \text{if} \qquad v_0 = -\underline{\mathcal{Y}}^*_{(\epsilon m)},$$

as was to be proved.

1.8 Bibliographical notes

General references Property insurance, life insurance and financial derivatives are all treated in later parts of this book. If you seek simple mathematical introductions to these subjects right away, try Straub (1998), Mikosch (2004) or Boland (2007) (property insurance), Gerber (1990) (life insurance) and Roman (2003) or Benth (2004) (financial derivatives).

Monte Carlo and implementation The main theme of the present chapter has been Monte Carlo as a problem solver. Introductory books emphasizing the role of the computer are scarce in insurance (Daykin, Pentikäinen and Pesonen (1994) is an exception), but there are more of them in finance, for example Shaw (1998), Benninga (2001) and Evans and Olson (2002). Coding and implementation is a fairly young scientific discipline, but old enough for reviews on how it's done having started to appear in computer science. Hanson (1999) discusses program *re-use* and Baier and Katoen (2008) treats program *verification* with stochastic models included; see also Kaner, Falk and Ngyuen (1999). These are themes that may may merit more attention that has been provided here. Section 3.3 gave examples where output was tested against known mathematical formulae for special cases. This is often a helpful type of technique.

Other numerical methods Compound distributions of portfolio liabilities (Section 3.3) and ruin probabilities (Section 3.6) have often been tackled by methods other than Monte Carlo. Simple approximations coming from the central limit theorem and its Cornish-Fisher extension will be presented in Section 10.2. So-called saddlepoint approximations is another possibility (see Jensen, 1995). Very popular in certain quarters is the **Panjer** recursion which works with discrete claim size distributions. A continuous distribution is always *approximately* discrete (see Section 4.2), and the discretization is no limitation on practical use. The approach was popularized by Panjer (1981) following earlier work by Adelson (1966). Later contributions have extended the original idea to cover ruin too; see Dickson (2005) for a review, but financial risk is *not* included, and Panjer recursions lack the versatility of Monte Carlo. A simple outline of the ruin problem from a mathematical point of view is provided by Dickson (2005); see also Section 10.8 for more references on this issue.

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1.9 Exercises

Section 3.2
Exercise 3.2.1 Let B be a *Bernoulli* variable, i.e.

$$\Pr(B = 0) = 1 - p \qquad \text{and} \qquad \Pr(B = 1) = p,$$

and define $X = sB$ where s is fixed. This model covers both *term* insurance (a one-time payment in case of death) and *pension* insurance (payment if the policy holder is alive). Here X may be an obligation for the coming time period. **a)** Explain the model. **b)** Show that

$$E(X) = ps \quad \text{and} \quad \text{var}(X) = p(1-p)s^2 \qquad \text{so that} \qquad \frac{\text{sd}(X)}{E(X)} = \sqrt{\frac{1}{p} - 1}.$$

c) When is randomness connected to survival/death most important, in term insurance or pension insurance? Insert suitable values for p.

Exercise 3.2.2 Let X be the total claim from a policy holder, as defined in (1.1). Suppose the claim frquency is Poisson distributed. Then (proof in Exercise 6.3.1)

$$\underbrace{\pi = E(X) = \mu T\xi_z}_{\textit{pure premium}} \qquad \text{and} \qquad \text{var}(X) = \mu T(\sigma_z^2 + \xi_z^2)$$

where $\xi_z = E(Z)$ and $\sigma_z = \text{sd}(Z)$. **a)** Use these formulas to verify that

$$\frac{\text{sd}(X)}{E(X)} = \frac{1}{\sqrt{\pi}}\left(\frac{\sigma_z^2}{\xi_z} + \frac{\pi}{\mu T}\right).$$

b) Deduce from this expression that the insurance of rare events with (possibly) very large claims contain much higher *relative* uncertainty than when claims are more frequent and smaller [Hint: *Both* terms on the right contribute to the conclusion, more in the next exercise].

Exercise 3.2.3 Suppose the claim size of the preceding exercise is log-normal. Then $Z = \beta\exp(\tau\varepsilon)$, where $\beta, \tau > 0$. We shall need for formulae for ξ_z and σ_z. Those are given in the introduction to the Section 3.3 exercises below (and also in Section 2.3). Consider two sets of parameters (μ_1, τ_1) and (μ_2, τ_2) for which

$$\mu_2 = \mu_1/5 \qquad \text{and} \qquad \tau_2 = \sqrt{\log(25) + \tau_1^2}.$$

a) Verify that the pure premium $E(X)$ is the same in both cases. **b)** Show that the ratio $\text{sd}(X)/E(X)$ is $5\sqrt{5} \doteq 11$ times larger under (μ_2, τ_2). **c)** What does this tell you about risk in different branches of property insurance?

Exercise 3.2.4 *Proportional* re-insurance means that the claim is split in two fixed fractions between cedent and re-insurer so that $H(z) = \gamma z$ is the re-insurer obligation. Here γ is a fixed by the contract $(0 < \gamma < 1)$. **a)** Show that the re-insurer responsibility is $X = \gamma(Z_1 + \ldots + Z_N)$ where N is the number of claims **b)** Derive the pure premium of the re-insurance when $Z = \beta\exp(\tau\varepsilon)$ as in the preceding exercise.

Exercise 3.2.5 It is quite common that a re-insurer parts with some of his risk to a second re-insurer. The situation is then:

$$\underset{\textit{cedent}}{Z} \quad \longrightarrow \quad \underset{\textit{first reinsurer}}{Z_1 = H_1(Z)} \quad \longrightarrow \quad \underset{\textit{second re-insurer}}{Z_2 = H_2(Z_1)}$$

Suppose the first re-insurance $H_1(z)$ is the $a \times b$ contract (1.4) and the second the proportional one $H_2(z_1)$ in the preceding exercise. Express the risk Z_2 carried by the second re-insurer in terms of Z.

Section 3.3
All exercises in this section assumes the log-normal as claim distribution; i.e.

$$Z = \beta\exp(\tau\varepsilon) \qquad \text{where} \qquad \varepsilon \sim N(0, 1),$$

and where β and τ are positive parameters. From (**??**) and (**??**) in Section 6.3

$$\xi_z = E(Z) = \beta\exp(\tau^2/2) \qquad \text{and} \qquad \sigma_z = \text{sd}(Z) = \xi_z\sqrt{\exp(\tau^2) - 1}.$$

Exercise 3.3.1 a) Implement Algorithm 3.3. when Z is lognormal. **b)** Check the program by taking $a = 0$ and $b =$ large value. If you simulate 10000 times and compute mean and standard deviation they should match the theoretical ones given above. Carry out this test when $\beta = 1$ and $\tau = 1$ and use these values for the rest of the exercise. **c)** Suppose $\mu = 5\%$ annually and that $a = 0.5$. Determine (by simulation) the pure premium for a re-insurance of one policy when $b = 2$, 3 and 5. **d)** How many simulations are necessary to prevent Monte Carlo error to be less than 0.1%?

Exercise 3.3.2 Do you believe it simpler to compute the pure premium of the preceding exercise through mathematics? Formulas can be worked out for which you only need the normal integral. But consider the case where there is a second re-insurer taking a part of the risk of the first, as depicted in Exercise 3.2.5. **a)** Justify the following command sequence for the claim against the second re-insurer:

$$\text{Generate } Z^*, \qquad Z_1^* \leftarrow H_1(Z^*), \qquad Z_2^* \leftarrow H_2(Z_1^*).$$

b) How do you proceed when both re-insurance contracts are of the $a \times b$ type detailed in (1.4) [Answer: You use Algorithm 3.3 twice.]. **c)** Determine the pure premium for the second re-insurance when

$\mu = 5\%,\ \beta = 1,\ \tau = 1,$	$a_1 = 0.5,\ b_1 = 3$	$a_2 = 2,\ b_2 = 3$
annual	*first re-insurance*	*second re-insurance*

d) Suppose the second contract is a maverick one where $H_2(z) = z/(1+z)$. Determine the pure premium of the re-insurance now.

Exercise 3.3.3 a) Implement Algorithm 3.1 for the total claim $\mathcal{X}$ against the company (i.e. the cedent). **b)** Explain how the program can be tested against the formulas in Exercise 3.2.2. [Hint: You use the output $\mathcal{X}_1^*, \ldots, \mathcal{X}_m^*$ from m runs and compute $\bar{X}^*$ and s^*.]. **c)** Carry out the test when

$\mu = 5\%,\ \beta = 1,\ \tau = 1,$	$J = 1000$
annual	*number of policies*

d) If the test works well, run so many simulations that a plot of the density function of $\mathcal{X}$ can be made. **e)** Determine the 1% upper percentile (Value-at-Risk) for the portfolio. Use $m = 10000$ and repeat five times so that you get a feeling for the simulation variability.

Exercise 3.3.4 a) Repeat the density plot in d) of the preceding exercise when $J\mu = 1000$ (instead of 50). **b)** Compare with the normal distribution, for example through a Q-Q plot (see e.g. Exercise 2.2.3). Comments? **c)** Repeat d) of Exercise 3.3.3 one more time, now use $J\mu = 50$, but change to $\tau = 0.5$. Closer to the normal than when $\tau = 1$?

Exercise 3.3.5 Suppose the portfolio is the same as in Exercise 3.3.3, but that now the cedent is protected by a re-insurance treaty of the $a \times b$ type that applies to single events. **a)** Compute the 1% percentiles of the cedent net responsibility when $a = 0.5$ and $b = 2, 3, 5$. **b)** The same exercise for the re-insurer.

Exercise 3.3.6 Suppose a cedent carries responsibility for the portfolio in Exercise 3.3.3c). and has protected risk through an $a \times b$ re-insurance treaty that applies to the total claim $\mathcal{X}$. Compute the re-insurance pure premium when $a = 40$ and $b = 60$, 80 and 100.

Section 3.4

Exercise 3.4.1 Consider a portfolio of term insurance where the sums insured are $s_1, \ldots, s_J$ and where each policy holder has the the same probability p of dying during the coming year. **a)** Calculate the sd-to-mean ratio (1.9) when $p = 1\%$ and the standard deviation s of the sums insured equal half their mean $\bar{\zeta}$. **b)** How large must portfolio size J for the ratio to be below 1%? Compare with pension insurance treated in the text.

Exercise 3.4.2 Consider J_a individuals of age a years entering a pension or life insurance portfolio. Such a group is sometimes called a **cohort**. The diagram shows how it evolves up to some final, high age b, in practice many decades after.

$$\underset{\textit{fixed}}{J_a} \overset{p_a}{\longrightarrow} \underset{\textit{first year}}{J_{a+1}} \overset{p_{a+1}}{\longrightarrow} \underset{\textit{second year}}{J_{a+2}} \overset{p_{a+2}}{\longrightarrow} \cdots \overset{p_{b-1}}{\longrightarrow} \underset{\textit{last year}}{J_b}$$

Here, $p_l = {}_1p_l$ is the survival probability, and J_l is the number individuals still alive at age l. **a)** Argue that

$$E(J_{a+1}) = p_a J_a, \qquad E(J_{a+2}) = (p_{a+1} p_a) \times J_a,$$

b) and in general

$$E(J_l) = (p_{l-1} \cdots p_a) \times J_a, \qquad \text{or} \qquad E(J_l) = \left(\prod_{k=a}^{l-1} p_k \right) \times J_a.$$

[Hint: For example, interprete J_l as a binomial random variable.]

Exercise 3.4.3 The purpose of this execise is to define a portfolio to run experiments in life and pension insurance on. One way is to imagine that during a long period of time J_a individuals at age a enter the portfolio each year. Suppose they stay until they die, and that there is no other recrutement. When we take responsibility for the portfolio, there will be a mixture of all age groups. **a)** Use the preceding exercise to argue the number of persons of age l might be approximately

$$J_l = (p_{l-1} \cdots p_a) \times J_a \qquad l = a+1, a+2, \ldots$$

b) Often we want to work with a given portfolio size J. Explain why we achive this by determining J_a from the equation

$$J_a \{1 + p_a + (p_{a+1} p_a) + \ldots + (p_{b-1} p_{b-2} \cdots p_a)\} = J.$$

c) Verify that the following algorithm lays out the portfolio:

Algorithm 3.9 Creating a life insurance portfolio

Input: a, b and $p_a, \ldots, p_b$ *%p_l is a survival probability*
$q_0 \leftarrow 1, \quad s \leftarrow 0$
For $l = a+1, \ldots, b$ do
    $q_l \leftarrow p_{l-1} q_{l-1}$ *%q_l is also denoted ${}_l p_a$*
    $s \leftarrow s + q_k$
% Loop terminated
For $l = a, \ldots, b$ do
    $J_l \leftarrow q_l (J/s)$
Return $J_a, \ldots, J_b$

The output are not integers. A simple way to deal with that is to round off to the nearest one. The portfolio is to be used for experimentation. Details in its inception do not matter.

Exercise 3.4.4 a) Lay out a portfolio of 1000 policies using the algorithm of the preceding exercise. Use the survival probabilities in Section 3.4 with $a = 30$ and $b = 90$ years. **b)** Implement Algorithm 3.2 with policy information drawn from this portfolio. **c)** Run the algorithm under the experimental conditons in Section 3.4. and investigate the variation in output.

Exercise 3.4.5 Consider a simplified pension scheme where all members have the same contract, each receiving a net payment ζ_l (if he is alive) during the period at age l. It is useful to count ζ_l *negative* if there is a contribution (premium) from the member to the scheme. In practice ζ_l shifts from negative to positive when retirement age is reached. Suppose there are to-day J_l members of age l. Future recrutement into the scheme is not taken into account, and members only leaves when they die. **a)** Show that the expected payment k years from now is

$$E(\mathcal{X}_k) = \sum_{l=a}^{b} J_l \, {}_k p_l \, \zeta_{l+k}$$

b) Computations can be organized as follows:

Algorithm 3.10 Expected net life insurance payment in k years

Input: k, J_l, p_l, and ζ_l
$e_k \leftarrow 0$ and $q \leftarrow 1$ %*Here* $e_k = E(\mathcal{X}_k)$.
For $l = a, \ldots, b$ do
    $q \leftarrow qp_l$ %*Here* q *is also denoted* ${}_lp_a$.
    $e_k \leftarrow e_k + J_l q \zeta_{l+k}$ %*Adding the contributions to* e_k.
Return $e_k = E(\mathcal{X}_k)$.

c) Justify the algorithm. It will be used with and without financial risk.

Exercise 3.4.6. Let PV_k be the present value of all payments into and out of the scheme up to (and including) period k. **a)** Show that its expectation can be computed according to the recursion

$$E(\mathrm{PV}_k) = E(\mathrm{PV}_{k-1}) + \frac{E(\mathcal{X}_K)}{(1+r)^k}, \qquad k = 1, 2, \ldots,$$

starting at $E(\mathrm{PV}_0) = 0$. **b)** How is Algorithm 3.10 put to use to compute the present value of all payments?

Exercise 3.4.7. This is a follow-up of Exercise 3.4.5. A common payment function is

$$\begin{aligned} \zeta_l &= -\pi \quad \text{if} \quad a \leq l < c \qquad && (\pi \text{ is premium per period}) \\ &= s \quad \text{if} \quad l \geq c && (s \text{ is pension per period}) \end{aligned}$$

where c is the retirement age. **a)** Explain how it is incorporated in Algorithm 3.10 of Exercise 3.4.5. **b)** Apply it with the portfolio laid out in Exercise 3.4.4 and the survival probabilities used there. The insurance contracts are defined by

$$\pi = 0.1365, \quad \zeta = 1, \quad c = 65.$$

Plot the output as a function of k up to $k = 90$. **c)**. Compute and plot the expected present value $E(\mathrm{PV}_k)$ when $r = 4\%$. **d)** Redo c) for a portfolio of 10000 individuals, all of age 30. Follow the portfolio 70 years ahead. Any comments?

Exercise 3.4.8 Suppose the situation is the same as in Exercise 3.4.5, but that we now are dealing with term insurance with a one-time payment ζ at the end of the period the policy holder dies. **a**) Modify Algorithm 3.10 so that it delas with this situation. **b)** Run it under the circumstances described in Exercise 3.4.7.

Section 3.5
Exercise 3.5.1 What is the difference between a risk-neutral and an ordinary model?

Exercise 3.5.2 Suppose $r = 4\%$ and $r_g = 7\%$ in the Black-Scholes formula (1.14). Use it to compute the put option premium (in per cent) for a single-asset option when $\sigma = 5\%$, 15%, 25% 35%. Comment on the variation in price.

Exercise 3.5.3 Suppose the time to maturity of a Black-Scholes put option is T. Since we are dealing with continously compounded rates, we may let rT be the risk-free and r_gT the guranteed rate of interest over a period of length T. **a**) Explain that the volatility up to T is $\sigma\sqrt{T}$. **b**) Rewrite the Black-Scholes formula (1.14) so that it covers a *general* time to expiry T. .

Exercise 3.5.4 Consider a single-asset equity option where $r = 4\%$ and $r_g = 6\%$ and $\sigma = 25\%$, all quoted as annual values. **a)** Compute the option premium in percent when $T = 1$ (a year), $T = 1/12$ (a month) and ($T = 1/52$) a week. **b)** Any problem with the model as T becomes small? More on that in Section 5.5.

Exercise 3.5.5 Suppose K is invested in equities according to a cautious strategy where a put option is purchased, guaranteeing minimum return r_g on the remaining capital v_0 *after* the premium $\pi(v_0)$ has been subtracted. **a)** Explain why v_0 is determined by the two equations

$$v_0 + \pi(v_0) = K \qquad \text{and} \qquad \pi(v_0) = \pi(1)v_0;$$

see (1.14). **b)** Let K_1 be the capital at the end of period one and $R_1 = (K_1/K) - 1$ the return under the strategy adopted. Show that if the standard log-normal model is assumed, then

$$K_1 = \max(e^{\xi+\sigma\varepsilon} - 1, r_g) \times \frac{K}{1+\pi(1)} \qquad \text{so that} \qquad R_1 = \frac{\max(e^{\xi+\sigma\varepsilon} - 1, r_g)}{1+\pi(1)} - 1.$$

c) Write down a simple algorithm that simulates the return under this strategy.

Exercise 3.5.6 We shall in this exercise experiment with the program of the preceding exercise, assuming that $r = 4\%$, $r_g = 6\%$ and $\sigma = 25\%$. The drift parameter ξ will be varied. All options run over an entire year. **a)** Use the formulas for mean and standard deviation of log-normal variables to deduce that as $r_g \to -\infty$

$$E(R_1) = \frac{e^{\xi+\sigma^2/2}}{1+\pi(1)} - 1 \qquad \text{and} \qquad \text{sd}(R_1) = \frac{e^{\xi+\sigma^2/2}}{1+\pi(1)}\sqrt{e^{\sigma^2} - 1}.$$

b) Check that the program of c) of Exercise 3.7.5 is correct by running it 10000 times with some small value of r_g, say $r_g = -100\%$. Compute the mean and standard deviation of the simulations and verify that they match the theoretical values. **c)** Sample the returns 10000 times when $\xi = 5\%$, 10% and 15%. Plot estimated density functions and compute means, lower and upper 5% percentiles.

Exercise 3.5.7 This is a continuation of the previous exercise. Suppose no protection is bought at all so that the entire capital is used to buy equity. **a)** Derive mean return and lower and upper 5% percentiles under this strategy. **a)** Compare with what we got under the first strategy. Comments?

Exercise 3.5.8 Let X^P and X^C be the pay-off functions in (1.11) and (1.12) for put and call options, and suppose they are based on the same guranteed return r_g. **a)** Show that

$$X^C - X^P = (R - r_g)v_0.$$

b) Use this to deduce that their option premia, now written $\pi^P(v_0)$ and $\pi^C(v_0)$, are linked through

$$\pi^C(v_0) - \pi^P(v_0) = e^{-r}(E_Q(R) - r^g)v_0$$

which implies that

$$\pi^C(v_0) = \pi^P(v_0) + \{1 - e^{-r}(1 + r_g)\}v_0.$$

This is known as a **parity** relation. **c)** use the Black-Scholes put option formula (1.14) to prove that

$$\pi^C(v_0) = \pi^C(1)v_0 \qquad \text{where} \qquad \pi^C(1) = \Phi(-a+\sigma) - (1+r_g)e^{-r}\Phi(-a).$$

Here a is defined in (1.14). [Hint: Use that $\Phi(x) = 1 - \Phi(-x)$ for the normal integral $\Phi(x)$.]

Exercise 3.5.9 Let $X^P(r_g)$ and $X^C(r_g)$ be the pay-off functions for put and calls, the guranteed return now being visible in the mathematica notation. **a)** Show that the pay-off function (**??**) for cliquet options can be written

$$X = X^P(r_g) - X^C(r_c)$$

and **b)** so that the option premium for the cliquet becomes

$$\pi = \pi^P(r_g) - \pi^C(r_c).$$

c) Compute the cliquet option premium when $r = 4\%$, $r_g = 6\%$, $\sigma = 25\%$ and $r_c = 9\%$, 12%, 15% and 20%. Comment?

Section 3.6
All exercises for this section make use of Algorithm 3.5. We shall be dealing with a property insurance portfolio for which there in period k are J_k policies. Claim frequency per policy and period is μ and expected claim size $E(Z) = 1$ per incident. The pure premium is then μ and portfolio premium income in period k

$$\Pi_k = J_k(1+\gamma)\mu.$$

where γ is the loading; see Section 1.3. Claim severities are of the form

$$Z = \exp(-\tau^2/2) + \tau\varepsilon),$$

which makes $E(Z) = 1$ and $\mathrm{var}(Z) = \exp(\tau^2) - 1$. We shall consider the value $\mathcal{V}_k$ of the account in period k, which propagates according to

$$\mathcal{V}_k = \mathcal{V}_{k-1} + \Pi_k - \mathcal{O}_k - \mathcal{X}_k, \qquad k = 1, 2, \ldots \qquad\qquad \mathcal{V}_0 = v_0;$$

see (**??**). Until Exercise 3.5.6 overrhead costs will be ignored (i.e $\mathcal{O}_k = 0$), and portfolio size $J_k = J$ will be constant.

Exercise 3.6.1 a) Use formulas in the Introduction to Exercises for Section 3.3 to show that

$$E(\mathcal{X}_k) = J\mu \qquad \text{and} \qquad \mathrm{var}(\mathcal{X}_k) = J\mu\exp(\tau^2).$$

b) Suppose $\mathcal{V}_0 = v_0$. Verify that

$$E(\mathcal{V}_k) = v_0 + k\gamma J\mu \qquad \text{and} \qquad \mathrm{var}(\mathcal{V}_k) = kJ\mu\exp(\tau^2).$$

Exercise 3.6.2 a) Implement Algorithm 3.5 under the assumptions stated above. **b)** Check the program against the formulas in Exercise 3.5.1 by running the program it 10000 times up to $K = 5$ when

$$v_0 = 10 \quad J\mu = 50, \quad \tau = 0.5, \quad \gamma = 10\%.$$

Exercise 3.6.3 a) Run the program 50 times up to $K = 20$ years under the assumptions in Exercise 3.5.2 and plot these 50 scenarios against time, as in Figure 3.3 left. **b)** Repeat the simulations when $\tau = 1$. Any comments? **c)** Run the program 1000 times and plot the estimated density functions of $\mathcal{X}_5$ and $\mathcal{X}_{20}$ both when $\tau = 0.5$ and when $\tau = 1$. Comments now. **d)** Determine the 1% Value-at-Risk for all cases in c) (now use 10000 simulations).

Exercise 3.6.4 Determine the probability of ruin after 5,10 and 20 years both when $\tau = 0.5$ and $\tau = 1$ when $v_0 =$?? is reserved initially.

Exercise 3.6.5 Determine the solvency requirement over 5,10 and 20 years under the same conditions as in the preceding exercise [Hint: use (1.21)].

Exercise 3.6.6 In the present (and the next) exercise future portfolio size J_k and overhead cost $\mathcal{O}_k$ will be allowed to vary with k. Assume that

$$J_{k+1} = (1+\delta_k)J_k, \qquad k = 1, 2, \ldots \qquad \text{where} \qquad \delta_k = \delta_1\exp\{\alpha(k-1)\}.$$

This is model of *progressive* or *regressive* growth depending on whether α is positive or negative. **a)** Plot it when $\alpha = -0.2$ and $J_1 = 100$. Overhead cost are assmued related to the size of the portfolios through

$$\frac{\mathcal{O}_k}{J_k} = \nu_k c_1 + (1 - \nu_k)c_\infty, \qquad \text{where} \qquad \nu_k = \exp\{\beta(J_k - J_1)\},$$

for $k = 1, 2, \ldots$. Here c_1 and c_∞ is overhead cost per policy when there are J_1 and infinitely many policies respectively. The third parameter β defines how fast the cost moves. **b)** Plot both the relative (i.e $\mathcal{O}_k/J_k$) and total cost $\mathcal{O}_k$ as a function of k when

$$c_1 = 50\%, \quad c_\infty = 10\%, \quad \beta = -0.0005 \qquad\qquad (\text{and } \alpha = -0.2 \text{ and } J_1 = 100 \text{ as before})$$

Exercise 3.6.7 We shall analyse the capital requirement for a newly formed company which expects to grow using the growth function and cost structure of the previous exercise. **a)** Implement these functions in Algorithm 3.5. **b)** Simulate 10 years ahead and plot 20 repeated scenarios with the parameters given in the preceding exercise and in addition

$$\mu = 5\% \quad \tau = 0.5 \text{ or } \tau = 1, \quad \gamma = 10\%$$

for claim frequency, size and premium loading. **c)** What is the initial capital needed to cover all liabiltities 10 years ahead when premium income is incorporated?

The exercises in this section redo some of those for Sections 3.4 and 3.5 with financial earnings added. Financial portfolios will be a mixture of the risk-free rate r and returns on equitites. Portfolio return is then

$$\mathcal{R} = (1 - w)r + wR, \qquad \text{where} \qquad R = \exp(\xi + \sigma\varepsilon) - 1,$$

using the standard log-normal model for equity returns. Here w is the weight placed on equity. The portfolio is rebalanced all the time to keep the weight fixed.

Exercise 3.6.8 Recall from Section 1.4 that the k-step return $\mathcal{R}_{0:k}$ evolves accoding to

$$\mathcal{R}_{0:k} = (1 + \mathcal{R}_k)\mathcal{R}_{0:k-1}, \quad k = 1, 2, \ldots \qquad\qquad \mathcal{R}_{0:0} = 1.$$

a) Simulate the k-step retun over 30 years development when

$$r = 4\%, \quad \xi = 6\%, \quad \sigma = 25\%, \quad w = 0.3$$

and plot 20 replications jointly. **b)** Repeat when $w = 0.1$.

Exercise 3.6.9 a) Add financial income to Algorithm 3.5; i.e implement Algorithm 3.6 for property insurance with the investment strategy introduced above. **b)** Run scenarios with $w = 0$ and $w = 0.3$ under the conditions in Exercise 3.5.2. Plot joint scenarios and discuss the effect of including financial earnings.

Exercise 3.6.10 a) Use the the model scenario in Exercise 3.6.2 to compute the probability of ruin over 10 years when

$$w = 0.3 \text{ and } v_0 = 10, 20, 30 \qquad \text{and} \qquad w = 0 \text{ and } v_0 = 10, 20, 30.$$

b) Comments? What is approximately the required initial capital to keep the ruin probability at 5%?

Exercise 3.6.11 Combine the financial asset model of Exercise 3.6.1 with the liability model of Exercise 3.4.5. This means that the portfolio account evolves according to

$$\mathcal{V}_k = \mathcal{V}_{k-1} + \Pi_k - \mathcal{O}_k - E(\mathcal{X}_k), \quad k = 1, 2, \ldots \qquad\qquad \mathcal{V}_0 = v_0;$$

The difference from property insurance is that only the*expected* portfolio payment is included, since the uncertainty inthe sequence $\{\mathcal{X}_k\}$ is small compared to asset risk. **a)** How do you simulate $\mathcal{V}_k$ now? **b)** Run simulations over 30 years when the conditions in Exercise 3.4.7 b) are combined with those in Exercise 3.6.2. Starting at $v_0 =?$. Repeat 20 times and plot.

Exercise 3.6.12 a) Repeat the simulations in Exercise 3.6.4 for $v_0 =??, ??, ??and??$. **b)** How much initial capital is needed to for the scheme to be solvent with probability 95%? and 20%. Comment?