

# 1 Modelling II: Conditional and non-linear

## 1.1 Introduction

Insurance requires modelling tools different from those of the preceding chapter. Pension schemes and life insurance make use of **life cycle** descriptions. Individuals start as ‘active’ (paying contributions), at one point they ‘retire’ (drawing benefits) or become ‘disabled’ (benefits again) and they may die. To keep track on what happens stochastic models are needed, but those can not be constructed by means of linear relationships like in the preceding chapter. There are no numerical variables to connect! Distributions are used instead.

The central concept is **conditional** probabilities and distributions, expressing mathematically that what has occurred is going to influence (but not determine) what comes next. That idea is the principal topic of the chapter. As elsewhere, mathematical aspects (here going rather deep) are downplayed. Our target is the conditional viewpoint as a modelling tool. Sequences of states in life cycles involve time series (but of kind different from those in Chapter 5) and are treated in Section 6.6. Actually time may not be involved at all. Risk heterogeneity in property insurance is a typical (and important) example. Consider a car owner. What he encounters daily in the traffic is thoroughly influenced by randomness, but so is (from a company point of view) his ability as a driver. These are uncertainties of entirely different origin and define a **hierarchy** (driver comes first). Conditional modelling is the natural way to connect random effects of this kind which operate on different levels. The very same viewpoint is used when estimation and Monte Carlo errors are examined in the next chapter, and there are countless other examples.

Conditional arguments will hang over much of this chapter, and we embark on it in the next section. **Copulas** is an additional tool. The idea behind is very different from conditioning and as a popular approach of fairly recent origin. Yet copulas has without doubt to come to stay. Section 6.7 is an introduction.

## 1.2 Conditional modelling

### Introduction

Conditional probabilities are defined in elementary textbooks in statistics. When an event  $A$  has occurred, the probability of another one  $B$  changes from  $\Pr(B)$  to

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} \quad \text{and also} \quad \Pr(B|A) = \frac{\Pr(A|B)\Pr(B)}{\Pr(A)}, \quad (1.1)$$

where the right hand side is known as **Bayes’ formula**. This is of obvious relevance in gambling where new information leads to new odds. In this book conditional probabilities are above all modelling tools, used to express random relationships between random variables. Note the mathematical notation. The condition is always placed to the right of a vertical bar. Similar notation is  $Y|x$  (the random variable  $Y$  given  $x$ ) with conditional density functions and expectation  $f(y|x)$  and  $E(Y|x)$ .

Conditional modelling is **sequential** modelling, first  $X$  and then  $Y$  given  $X$ . The purpose of

this section is to demonstrate the power in this line of thinking. It is the natural way to describe countless stochastic phenomena, and simulation is easy. Simply

generate  $X^*$  and then  $Y^*$  given  $X^*$ ,

the second drawing being dependent on the outcome of the first. The following examples all play major roles in later chapters.

### The conditional Gaussian

Bivariate normal models were in Chapter 2 defined through

$$X_1 = \xi_1 + \sigma_1 \eta_1 \quad \text{and} \quad X_2 = \xi_2 + \sigma_2(\rho \eta_1 + \sqrt{1 - \rho^2} \eta_2),$$

where  $\eta_1$  and  $\eta_2$  are independent and  $N(0, 1)$ ; see (??). Suppose  $X_1 = x_1$  is fixed. Then  $\eta_1 = (x_1 - \xi_1)/\sigma_1$ , which when inserted for  $\eta_1$  in the representation for  $X_2$  leads to

$$X_2 = \xi_2 + \sigma_2 \left( \rho \frac{x_1 - \xi_1}{\sigma_1} + \sqrt{1 - \rho^2} \eta_2 \right),$$

or after some reorganizing

$$X_2 = \underbrace{(\xi_2 + \rho \sigma_2 \frac{x_1 - \xi_1}{\sigma_1})}_{\text{expectation}} + \underbrace{(\sigma_2 \sqrt{1 - \rho^2})}_{\text{standard deviation}} \eta_2. \quad (1.2)$$

Here  $\eta_2$  is the only random term and, by definition,  $X_2$  is normal with mean and standard deviation

$$E(X_2|x_1) = \xi_2 + \rho \sigma_2 \frac{x_1 - \xi_1}{\sigma_1} \quad \text{and} \quad \text{sd}(X_2|x_1) = \sigma_2 \sqrt{1 - \rho^2}. \quad (1.3)$$

We are dealing with a **conditional distribution**. As  $x_1$  is varied, then so does the expectation and (for other models) also standard deviation.

### Survival modelling

Let  $Y$  be the length of life of an individual. A central quantity in life insurance is

$${}_t p_{y_0} = \Pr(Y > y_0 + t | Y > y_0), \quad (1.4)$$

known as the **survival probability** which defines how often a person of age  $y_0$  reaches age  $y_0 + t$ . If  $F(y)$  is the distribution function of  $Y$ , then from (1.1) left

$${}_t p_{y_0} = \frac{\Pr(Y > y_0 + t)}{\Pr(Y > y_0)} = \frac{1 - F(y_0 + t)}{1 - F(y_0)}. \quad (1.5)$$

Survival probabilities are often used on multiples of a given increment  $h$ , for example

$$\underbrace{y_l = lh}_{\text{age}} \quad l = 0, 1, \dots \quad \text{and} \quad \underbrace{t_k = kh}_{\text{time}} \quad k = 0, 1, \dots,$$

and we shall write  ${}_k p_{l_0} = {}_t p_{y_0}$  when  $y_0 = l_0 h$  and  $t = kh$ . A convenient simplification is  $p_l = {}_1 p_l$  for single periods which aggregates to probabilities over  $k$  periods through

$${}_k p_{l_0} = \underbrace{p_{l_0}}_{\text{first interval}} \times \underbrace{p_{l_0+1}}_{\text{second interval}} \times \dots \times \underbrace{p_{l_0+k-1}}_{\text{k'th interval}}; \quad (1.6)$$

much more on such modelling in Section 12.3.

### Over threshold modelling

Conditional probabilities of the same type is needed in property insurance, particularly in connection with large claims and re-insurance. For a given threshold  $b$  we seek the distribution of

$$Z_b = Z - b \quad \text{given that} \quad Z > b. \quad (1.7)$$

We can write it down by replacing  $t$  and  $y_0$  on the right in (1.5) by  $z$  and  $b$ . Thus

$$\Pr(Z_b > z | Z > b) = \frac{1 - F(b + z)}{1 - F(b)},$$

where  $F(z)$  is the distribution function of  $Z$ . When differentiated with respect to  $z$ , this leads to

$$f_b(z) = \frac{f(z + b)}{1 - F(b)}, \quad z > 0. \quad (1.8)$$

as the density function for the amount exceeding a given threshold. Tail distributions of this type possess a remarkable property, see Pickands (1975). For most distributions used in practice, precisely if  $f(z)$  is *not* identically zero above some upper limit, then  $f_b(z)$  becomes either a *Pareto* density or an *exponential* one as  $b \rightarrow \infty$  no matter which model we started with; see Chapter 9.

### Risk heterogeneity

Claim numbers  $N$  in property insurance was in Chapter 3 described by

$$\Pr(N = n | \mu) = \frac{\lambda^n}{n!} \exp(-\lambda) \quad \text{where} \quad \begin{array}{l} \lambda = \mu T \\ \text{Policy} \end{array} \quad \text{or} \quad \begin{array}{l} \lambda = J\mu T, \\ \text{Portfolio} \end{array}$$

for  $n = 0, 1, \dots$ ; see (??) and (??). Suppose  $\mu$  is random as suggested by the nomenclature. In countless situations this is a natural viewpoint. Consider automobile insurance. Variation in risk among drivers is accounted for by one random  $\mu$  for each individual, but  $\mu$  might also reflect general driving conditions (such as the weather) that affect everybody jointly.

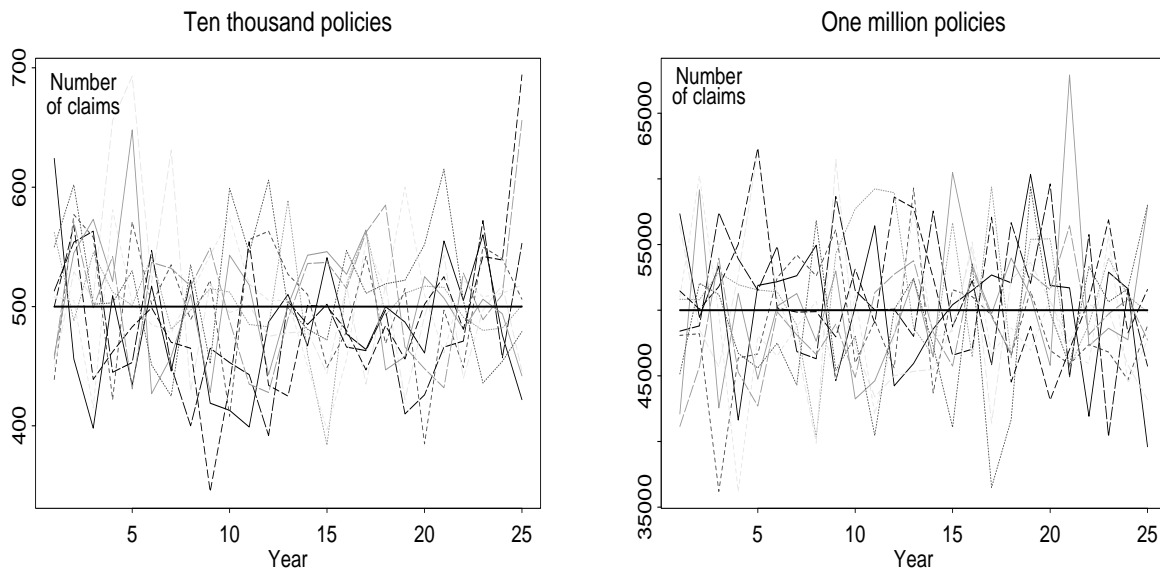
Modelling is the same whether  $\mu$  affects policies *individually* or the entire portfolio *collectively*. The claim frequency observed ( $N$  for an individual or  $\mathcal{N}$  for a portfolio) is the outcome of two experiments in a hierarchy. First  $\mu$  is drawn randomly and *then*  $N$  or  $\mathcal{N}$  given  $\mu$ ; i.e.

$$\mu = \xi Z, \quad \begin{array}{l} N | \mu \sim \text{Poisson}(\mu T) \\ \text{policy level} \end{array} \quad \text{and} \quad \begin{array}{l} \mu = \xi Z, \quad \mathcal{N} | \mu \sim \text{Poisson}(J\mu T). \\ \text{portfolio level,} \end{array} \quad (1.9)$$

where  $E(Z) = 1$  to make  $\xi$  *mean intensity*. The standard model for  $Z$  is Gamma( $\alpha$ ), one of the distributions introduced in Section 2.6. Now

$$E(\mu) = \xi \quad \text{and} \quad \text{sd}(\mu) = \xi / \sqrt{\alpha}, \quad (1.10)$$

and the variability in  $\mu$ , controlled by  $\alpha$ , is removed when  $\alpha \rightarrow \infty$  so that eventually  $\mu = \xi$  becomes fixed; more on this model in Section 8.3.



**Figure 6.1** *Simulated portfolio claim frequency scenarios under annual change of risk*

### Common risk factors

Claim numbers  $N_1, \dots, N_J$  depending on the same random intensity  $\mu$  is a special case of a more general viewpoint. A random variable  $\omega$  is called a **common factor** for  $X_1, \dots, X_J$  if

$$X_1, \dots, X_J \quad \text{are } \mathbf{conditionally} \text{ independent} \quad \text{given } \omega. \quad (1.11)$$

The theory of credibility in Section 10.5 is a classical example in actuarial science, and the market component played the role of  $\omega$  in the CAPM model of Section 5.3. If  $\omega$  isn't directly observable, we are dealing with **hidden** or **latent** factors.

Common factors (whether hidden or not) invariably increase risk and they are impossible to diversify. Figure 6.1 is a simulated example where claim frequencies were generated over 25 years for one 'small' and one 'large' car insurance portfolio. The intensity  $\mu$ , changed every year and was the same for all policies. Suppose  $\mu$  follows a Gamma model. Claim frequencies are then generated through

$$Z^* \sim \text{Gamma}(\alpha), \quad \mu^* \leftarrow \xi Z^* \quad \text{and then} \quad \mathcal{N}^* \sim \text{Poisson}(J\mu^*T).$$

The experiments in Figure 6.1 were run as 25 independent drawings for each of  $m = 20$  scenarios plotted jointly. Underlying parameters were

$$\xi = 5\%, \quad \alpha = 100, \quad T = 1,$$

which means that claim frequency per car is 5% in an average year and the standard deviation 10% of that; see (1.10). Fluctuations in Figure 6.1 match this fairly well<sup>1</sup>, but the main point is

<sup>1</sup>The oscillations in both plots go out to about  $\pm 20\%$  of the position of the straight line, and the 10% relative standard deviation emerges when you divide by two.

the uncertainty which is no smaller (in relative terms) for the large portfolio. That runs contrary to what has seen before (Section 3.2) and is caused by the uncertainty of the the common factors which doesn't go down when the number of policies is raised. The mathematics is given in Section 6.3.

### Monte Carlo distributions

Simulation experiments are often run from parameters that have been estimated from historical data. The distribution of the simulations are then influenced by estimation error *in addition* to ordinary Monte Carlo randomness. To be specific, suppose the claim frequency  $\mathcal{N}$  against a portfolio follows the ordinary Poisson model with fixed intensity  $\mu$  and let  $\hat{\mu}$  be its estimate (method in Chapter 8). The scheme is then

$$\begin{array}{ccccc} \text{historical data} & \longrightarrow & \hat{\mu} & \longrightarrow & \mathcal{N}^*, \\ & \text{estimation} & & \text{Monte Carlo} & \end{array}$$

and the question is how both sources of error are aggregated. A first step is to notice that the model for  $\mathcal{N}^*$  is a *conditional* one; i.e

$$\Pr(\mathcal{N}^* = n | \hat{\mu}) = \frac{(JT\hat{\mu})^n}{n!} \exp(-JT\hat{\mu}), \quad n = 0, 1, \dots,$$

and we must combine with statistical errors in the estimation process. The argument is given in Chapter 7.

## 1.3 Risk from subordinate level

### Introduction

Risk variables  $X$  influenced by a second random factor  $\omega$  on a **subordinate** level were introduced above. Think of  $\omega$  as personal qualities of a policy holder, background conditions affecting an entire insurance portfolio or market risk in finance. The importance of this kind of uncertainty was examined in Section 5.3 through a specific model (CAPM), but it is also possible to proceed more generally though the conditional mean and standard deviation of  $X$  only. Let

$$\xi(\omega) = E(X|\omega) \quad \text{and} \quad \sigma(\omega) = \text{sd}(X|\omega), \tag{1.12}$$

and consider  $\mathcal{X} = X_1 + \dots + X_J$  where  $X_1, \dots, X_J$  have the same distribution as  $X$ . The purpose of this section is to study  $\mathcal{X}$  under different sampling regimes for  $\omega$ . General insurance provides the main example, but the conclusions apply to financial risk too.

### The double rules

Our tool is two operational rules that is best introduced generally. Suppose the distribution of  $Y$  depends on a random vector  $\mathbf{X}$ . The conditional mean  $\xi(\mathbf{x}) = E(Y|\mathbf{x})$  plays a leading role in risk modelling; see Section 6.4. Here the issue is how  $\xi(\mathbf{x})$  and  $\sigma(\mathbf{x}) = \text{sd}(Y|\mathbf{x})$  influence  $Y$ . Much insight is provided by the identities

$$\begin{array}{l} E(Y) = E\{\xi(\mathbf{X})\} \\ \text{double expectation} \end{array} \quad \text{for} \quad \xi(\mathbf{x}) = E(Y|\mathbf{x}) \tag{1.13}$$

and

$$\text{var}(Y) = \underbrace{\text{var}\{\xi(\mathbf{X})\}}_{\text{double variance}} + E\{\sigma^2(\mathbf{X})\} \quad \text{for} \quad \sigma(\mathbf{x}) = \text{sd}(Y|\mathbf{x}), \quad (1.14)$$

which are proved in Appendix A. Both  $\xi(\mathbf{X})$  and  $\sigma^2(\mathbf{X})$  are random variables, and the expectation of the former is the expectation of  $Y$ . The double variance formula decomposes  $\text{var}(Y)$  into two positive contributions and is one of the most useful results in risk theory.

Neither formula requires conditional modelling beyond mean and standard deviation, and a number of useful results can be derived from them. They will in Chapter 7 play a main role when errors of different origin are analysed jointly.

### Impact of subordinate risk

Let  $X_j$  be a random variable depending on  $\omega_j$  for  $j = 1, \dots, J$ . The independence of  $X_1, \dots, X_J$  is still assumed, but this now means that they are **conditionally independent** given  $\omega_1, \dots, \omega_J$ . Consider the two different sampling regimes

$$\underbrace{\omega_1 = \dots = \omega_J = \omega}_{\text{common factor}} \quad \text{and} \quad \underbrace{\omega_1, \dots, \omega_J}_{\text{individual parameters}} \text{ independent.}$$

On the left  $\omega$  is a common background factor affecting the entire portfolio whereas on the right  $\omega_j$  is attached each  $X_j$  individually. Their effect on the portfolio risk  $\mathcal{X} = X_1 + \dots + X_J$  is widely different, as we shall now see. It will be assumed that all  $X_j$  and all  $\omega_j$  follow the same distribution (*not essential*).

Consider first the case where  $\omega$  is a common background factor for the entire portfolio. We are assuming that  $\xi(\omega) = E(X_j|\omega)$  and  $\sigma(\omega) = \text{sd}(X_j|\omega)$ , the same for all  $j$ . When all contributions to  $\mathcal{X}$  are added, then

$$E(\mathcal{X}|\omega) = J\xi(\omega) \quad \text{and} \quad \text{var}(\mathcal{X}|\omega) = J\sigma^2(\omega),$$

where the variance formula requires conditional independence. Invoke the double rules with  $Y = \mathcal{X}$  and  $\mathbf{X} = \omega$ . Then, by (1.14)

$$\text{var}(\mathcal{X}) = \text{var}\{J\xi(\omega)\} + E\{J\sigma^2(\omega)\} = J^2\text{var}\{\xi(\omega)\} + JE\{\sigma^2(\omega)\}$$

which with (1.13) leads to

$$E(\mathcal{X}) = JE\{\xi(\omega)\} \quad \text{and} \quad \text{sd}(\mathcal{X}) = \underbrace{J\sqrt{\text{var}\{\xi(\omega)\} + E\{\sigma^2(\omega)\}}}_{\text{common } \omega} / J, \quad (1.15)$$

and the standard deviation is of the same order of magnitude  $J$  as the expectation itself. *Such risk can not be diversified away* by increasing the portfolio size. Indeed,

$$\frac{\text{sd}(\mathcal{X})}{E(\mathcal{X})} \rightarrow \frac{\text{sd}\{\xi(\omega)\}}{E\{\xi(\omega)\}} \quad \text{as} \quad J \rightarrow \infty$$

which does not vanish if  $\text{sd}\{\xi(\omega)\} > 0$ .

Things change drastically when each  $X_j$  is attached a separate and independently drawn  $\omega_j$ . The mean and variance of each  $X_j$  are now calculated by inserting  $J = 1$  in (1.15). When all of those are added over all policies  $j$ , we obtain mean and variance on portfolio level; i.e

$$E(\mathcal{X}) = JE\{\xi(\omega)\} \quad \text{and} \quad \text{sd}(\mathcal{X}) = \sqrt{J[E\{\sigma^2(\omega)\} + \text{var}\{\xi(\omega)\}]} \quad (1.16)$$

*$\omega$  individual*

The mean is the same as before, but the standard deviation has now the familiar form proportional to  $\sqrt{J}$ .

### Random claim intensity in general insurance.

The preceding argument enables us to understand how random intensities  $\mu_1, \dots, \mu_J$  influence the claim frequency  $\mathcal{N} = N_1 + \dots + N_J$  of a general insurance portfolio under the two sampling regimes above:

$$\mu_1 = \dots = \mu_J = \mu \quad \text{and} \quad \mu_1, \dots, \mu_J \quad \text{independent.}$$

*common factor* *individual parameters*

On the left a common (random) factor  $\mu$  is allocated all policy holders jointly whereas on the right there is one independent intensity for each individual. Claim frequencies  $N_1, \dots, N_J$  are in either case assumed conditionally independent and Poisson given  $\mu_1, \dots, \mu_J$ . In particular

$$E(N_j|\mu_j) = \mu_j T \quad \text{and} \quad \text{var}(N|\mu_j) = \mu_j T.$$

which are the functions  $\xi(\mu_j)$  and  $\sigma^2(\mu_j)$  in (1.15) and (1.16). Let  $\xi_\mu = E(\mu_j)$  and  $\sigma_\mu = \text{sd}(\mu_j)$  be the mean and standard deviation of each  $\mu_j$ . Then (1.15) yields

$$E(\mathcal{N}) = JT\xi_\mu \quad \text{and} \quad \text{sd}(\mathcal{N}) = JT\sqrt{\sigma_\mu^2 + \xi_\mu/(JT)}, \quad (1.17)$$

*common  $\mu$*

and the form of the standard deviation (almost proportional to  $J$ ) explains the simulated patterns in Figure 6.1 where the relative random uncertainty seemed unaffected by  $J$ .

This changes when  $\mu_1, \dots, \mu_J$  are drawn independently of each other. Now

$$E(\mathcal{N}) = JT\xi_\mu \quad \text{and} \quad \text{sd}(\mathcal{N}) = T\sqrt{J(\sigma_\mu^2 + \xi_\mu/T)}, \quad (1.18)$$

*$\mu$  individual*

and the standard deviation has the familiar form proportional to  $\sqrt{J}$ . The practical significance for portfolio risk will be examined below.

### Insurance risk: A simple formula

Another useful consequences of the double rules are simple formulae for mean and standard deviation of the portfolio loss in general insurance. Consider the model from Section 3.2; i.e

$$\mathcal{X} = Z_1 + \dots + Z_{\mathcal{N}}$$

where  $\mathcal{N}, Z_1, Z_2 \dots$  are stochastically independent. Let  $E(Z_i) = \xi_z$  and  $\text{sd}(Z_i) = \sigma_z$ . Elementary rules for expectation and variance of sums yields

$$E(\mathcal{X}|\mathcal{N}) = \mathcal{N}\xi_z \quad \text{and} \quad \text{var}(\mathcal{X}|\mathcal{N}) = \mathcal{N}\sigma_z^2.$$

To incorporate claim frequency  $\mathcal{N}$  as an additional source of randomness take  $Y = \mathcal{X}$  and  $\mathbf{X} = \mathcal{N}$  in (1.13) and (1.14). Then

$$\text{var}(\mathcal{X}) = \text{var}(\mathcal{N}\xi_z) + E(\mathcal{N}\sigma_z^2) = \text{var}(\mathcal{N})\xi_z^2 + E(\mathcal{N})\sigma_z^2$$

so that

$$E(\mathcal{X}) = E(\mathcal{N})\xi_z \quad \text{and} \quad \text{var}(\mathcal{X}) = E(\mathcal{N})\sigma_z^2 + \text{var}(\mathcal{N})\xi_z^2. \quad (1.19)$$

In particular, suppose  $\mathcal{N}$  follows a pure Poisson distribution. Then  $E(\mathcal{N}) = \text{var}(\mathcal{N}) = J\mu T$  and

$$E(\mathcal{X}) = J\mu T\xi_z \quad \text{and} \quad \text{var}(\mathcal{X}) = J\mu T(\sigma_z^2 + \xi_z^2), \quad (1.20)$$

which will be used repeatedly.

### Random claim intensity: Important at portfolio level?

How the portfolio liabilities  $\mathcal{X}$  in general insurance are affected by random claim intensities may be examined by inserting the expressions for  $E(\mathcal{N})$  and  $\text{sd}(\mathcal{N})$  into (1.19). For the mean this leads to

$$E(\mathcal{X}) = J\xi_\mu\xi_z, \quad (1.21)$$

which applies both when  $\mu$  is a common, random value for the entire portfolio and when drawn individually for each policy holder. This changes when we move to the variance. Inserting (1.17) and (1.18) into (1.19) right yields by some algebra (detailed in Section 6.8)

$$\text{sd}(\mathcal{X}) = \underbrace{\sqrt{J\xi_\mu(\sigma_z^2 + \xi_z^2)}}_{\text{pure Poisson}} \times \underbrace{\sqrt{1 + \delta\gamma}}_{\text{due to random } \mu} \quad (1.22)$$

where

$$\delta = T \frac{\sigma_z^2}{\sigma_z^2 + \xi_z^2} \cdot \frac{\sigma_\mu^2}{\xi_\mu} \quad \text{and} \quad \gamma = \begin{cases} 1 & \text{for individual } \mu \\ J & \text{for common } \mu \end{cases} \quad (1.23)$$

This lengthy expression tells a lot.

On the right in (1.22) there is a main, pure Poisson factor and a correction caused by  $\mu$  being random. How important is the latter? In practice  $\delta$  is quite small (hardly more than a few per cent, see Exercise 6.3.2), and when  $\mu_1, \dots, \mu_J$  are drawn independently of each other (so that  $\gamma = 1$ ), the correction factor becomes  $\sqrt{1 + \delta} \doteq 1 + \delta/2$ , *not* a large increase. The other case is different. Now  $\gamma = J$ , and the correction  $\sqrt{1 + J\delta}$  may be huge.



## 1.4 The role of the conditional mean

### Introduction

The conditional mean is much more than a brick in the double rules of the preceding section. Consider

$$\hat{Y} = \xi(\mathbf{X}) = E(Y|\mathbf{X}), \quad (1.24)$$

where  $\mathbf{X}$  is a quantity observed. In theory  $\hat{Y}$  is the best way of predicting the value of an unknown  $Y$  if you know  $\mathbf{X}$ ; see below. This is a celebrated result in engineering and statistics, yet not *that* prominent in actuarial science. When  $Y$  is a future value, we are often more concerned with summaries such as mean and percentiles than with predicting its actual outcome.

But there is another (and important) side to this. If  $\mathbf{X}$  is information available,  $E(Y|\mathbf{X})$  is what is expected given that knowledge and a natural break-even price for carrying the risk  $Y$ . Shouldn't what we charge reflect what we know? In property insurance  $\mathbf{X}$  might be our experience with a policy holder or other information with bearing on risk; see Section 10.5. Here the main example is the pricing of money market products such as bonds which will eventually lead us to the theoretical interest rate curve.

A quick word on the meaning of  $\mathbf{X}$  in the present context: Think of it as all present and past observations with bearing on  $Y$ . Theoretical literature often refers to  $\mathbf{X}$  as a **sigma-field** (typically denoted  $\mathcal{F}$ ), but it is perfectly possible to understand the ideas involved without such formalism from measure theory.

### Optimal prediction and interest rates

Central mathematical properties of the conditional mean are

$$\begin{array}{ccc} E(\hat{Y} - Y) = 0 & \text{and} & E(\hat{Y} - Y)^2 \leq E(\tilde{Y} - Y)^2 \quad \text{for all } \tilde{Y} = \tilde{Y}(\mathbf{X}). \\ \textit{expected error} & & \textit{expected squared error} \end{array} \quad (1.25)$$

Here the left hand side, which is merely a rephrasing of the rule of double expectation (1.13), signifies that the expected prediction error  $\hat{Y} - Y$  is zero. This means that  $\hat{Y}$  is an **unbiased** prediction; more on unbiasedness in Chapter 7. The inequality on the right in (1.25) shows that the conditional mean is on average the most accurate way of utilizing the information  $\mathbf{X}$ . The proof is given in Section 6.8.

This result will now be used to examine interest rate forecasting. Suppose the rates  $\{r_k\}$  follow the Vasiček model of Section 5.6. Then

$$r_k = \xi + \sigma(\varepsilon_k + a\varepsilon_{k-1} + \dots + a^{k-1}\varepsilon_1) + a^k(r_0 - \xi),$$

where  $r_0$  is known and  $\varepsilon_1, \varepsilon_2, \dots$  are independent with zero mean and unit variance; see (??). It follows that

$$E(r_k|r_0) = \xi + a^k(r_0 - \xi) \quad \text{and} \quad \text{sd}(r_k|r_0) = \sigma \sqrt{\frac{1 - a^{2k}}{1 - a^2}};$$

see Section 5.6 for the standard deviation. If the Vasiček model is true,  $\hat{r}_k = E(r_k|r_0)$  is the best prediction of  $r_k$  under the model assumed.

What about the accuracy? A quick look is provided by the formula for the standard deviation. Possible annual parameters *could be*  $\sigma = 0.016$  and  $a = 0.7$ . If so, the standard deviation becomes 1.4% after one year and 2.2% after five. This signifies huge prediction error, up to 3 – 4% and more. Forecasting interest levels through simple statistical techniques is futile.

### Term structure modelling

The conditional mean is in the money market much more important for pricing than for prediction. As above let  $r_0$  be the rate of interest today (known) and  $r_1, \dots, r_k$  those of the future (unknown), running over the time sequence  $t_k = kh$  for  $k = 0, 1, \dots$ . Consider

$$P(r_0, t_k) = E_Q(D_k|r_0) \quad \text{where} \quad D_k = \frac{1}{1+r_1} \cdots \frac{1}{1+r_k}. \quad (1.26)$$

Here  $D_k$  is a discount and had future rates of interest been known, that is what you would have been charged today for the right to receive one money unit at  $t_k$ ; i.e. for a unit-faced, zero-coupon bond expiring at that time. Though the rates  $r_1, \dots, r_k$  are unknown, there is a view on their likely values which leads to the expectation of  $D_k$  on the left in (1.26). This is a theoretical price of a zero-coupon bond. When the situation is analysed in Chapter 14, it will emerge that a **risk-neutral**  $Q$ -model should be employed, hence the subscript  $Q$ . This  $Q$ -model is *not* the same as the one describing real interest rate fluctuations. It was the same with equity option in Section 3.5.

There are now *two* pricing systems for zero-coupon bonds. The preceding theoretical **term structure**  $P(r_0, t_k)$  is based on a mathematical description of the market view, but there are also the observed prices  $P(0:k)$  introduced in Section 1.4. Why bother with the theoretical ones at all? Answer: We need them to describe *future* bond prices and their uncertainty. For example, suppose  $r_k^*$  is a Monte Carlo rate of interest at  $t_k$ . Then  $P(r_k^*, t)$  is a simulated price at time  $t_k$  of a bond expiring at  $t_k + t$ . Such simulations will in Chapter 15 play a crucial role with modern **fair value** accounting and with the coordination of assets and liabilities in the life insurance industry.

Surely the observed bond prices  $P(0:k)$  and the theoretical ones  $P(r_0, t_k)$  at time  $t_0 = 0$  should be equal? They must if the  $Q$ -model correctly reflects the market view, and it is common to calibrate its parameters by matching the two sets of prices.

### Example: The Vasiček term structure

Countless theoretical bond pricing schemes have appeared in the literature; see Section 6.9. They make use of a mathematical limiting process where  $h \rightarrow 0$  and carefully constructed  $Q$ -models that allow explicit formulae. One of the simplest and most widely used is the Vasiček model

$$r_k - r_{k-1} = a_q h (\xi_q - r_{k-1}) + \sqrt{h} \sigma_q \varepsilon_k,$$

which was introduced through (??). The parameters are now subscripted with  $q$  to emphasize risk-neutrality. Calculations of (1.26) under this model are carried out in Exercises 5.7.12-16 which lead to

$$P(r_0, t) = e^{A(t) - B(t)r_0} \quad (1.27)$$

where

$$B(t) = \frac{1 - e^{-a_q t}}{a_q} \quad \text{and} \quad A(t) = (B(t) - t) \left( \xi_q - \frac{\sigma_q^2}{2a_q^2} \right) - \frac{\sigma_q^2 B(t)^2}{4a_q}. \quad (1.28)$$

We may interpret  $P(r_0, t)$  as the price in a Vasiček world of a zero-coupon bond maturing at time  $t$  when  $r_0$  is the present rate of interest. The model goes back to Vasiček (1977).

### Monte Carlo term structures

Modern computational power must diminish the importance of closed bond price formulae, and it has become perfectly feasible to compute  $P(r_0, t_k)$  by Monte Carlo and store it as a table over a suitable set of pairs  $(r_0, t_k)$ . Simulations such as  $P(r_k^*, t)$  may then be read off approximately from the table. The minor numerical inaccuracy is of little practical importance, and we may now employ any  $Q$ -model we like. Here is an implementation for the Black-Karisinsky model (which does not allow simple bond price schemes):

#### Algorithm 6.1 The Black-Karisinsky term structure

```

0 Input:  $m, \xi_q, a_q, \sigma_q, r_0, h$  and
           $\sigma_x = \sigma_q / \sqrt{1 - a_q^2}, x_0 = \log(r_0 / \xi_q) + \sigma_x^2 / 2$ 
1  $P^*(k) \leftarrow 0$  for  $k = 1, \dots, K$                                 %  $P^*(k)$  the theoretical bond price
2 Repeat  $m$  times
3    $X^* \leftarrow x_0, D^* \leftarrow 1/m$                                 %  $D^*$  will serve as discount
4   For  $k = 1, \dots, K$  do
5     Draw  $\varepsilon^* \sim N(0, 1)$  and  $X^* \leftarrow a_q X^* + \sigma_q \varepsilon^*$ 
6      $r^* \leftarrow \xi_q e^{-\sigma_x^2 / 2 + X^*}$  and  $D^* \leftarrow D^* / (1 + r^*)$ 
7      $P^*(k) \leftarrow P^*(k) + D^*$                                 % The  $k$ -step discount summarized

8 Return  $P^*(k)$  for  $k = 1, \dots, K$ 

```

The algorithm simulates future rates of interest and updates the stochastic discounts as it goes through the *inner* loop over  $k$ . Output from the *outer* loop are Monte Carlo approximations  $P^*(k)$  to  $P(r_0, t_k)$  for  $k = 1, \dots, K$ . Re-runs for many different  $r_0$  are necessary.

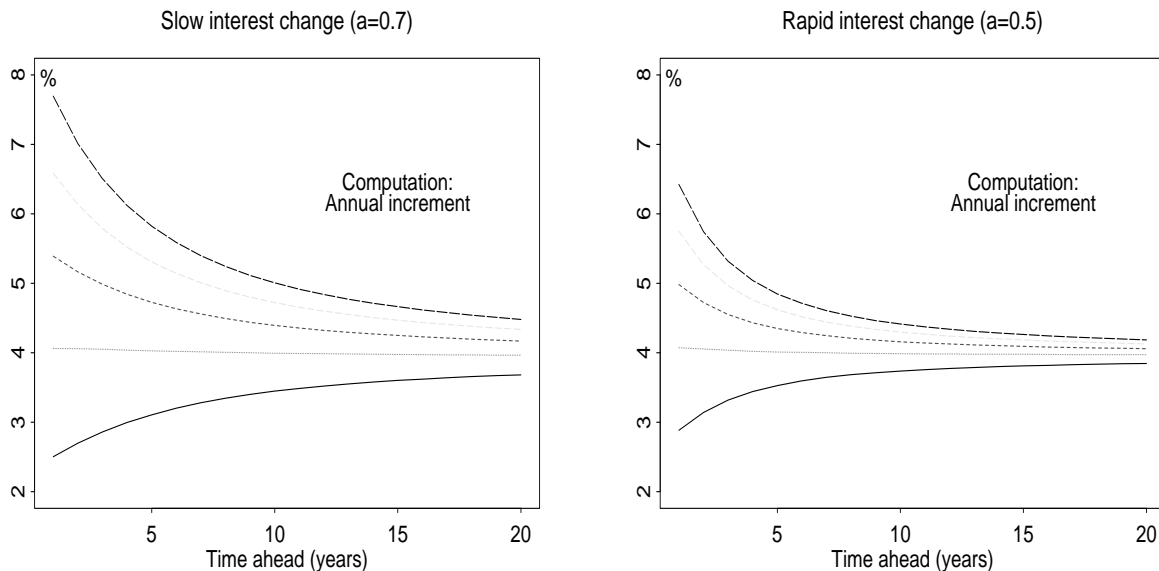
If you want the computations to run on a finely meshed time scale, you must adapt the parameters as explained in Section 5.7. The examples in Figure 6.2 have been run on a crude annual one with parameters

$$\xi_q = 4\%, \quad a_q = 0.7, \quad \sigma_q = 0.25 \quad \text{or} \quad \xi_q = 4\%, \quad a_q = 0.5, \quad \sigma_q = 0.31317.$$

Bond prices have been converted to the yield curve through

$$\bar{r}^*(0:k) = P^*(0:k)^{-1/k} - 1,$$

which is the average rate of interest over the period in question; see Section 1.4. The initial rate  $r_0$  varied between  $r_0 = 2\%, 4\% 6\% 8\%$  and  $10\%$  which produced the different shapes in Figure 6.2. In the long run the average yield tends to  $\xi = 4\%$  with a speed determined by  $a_q$ .



**Figure 6.2** Interest rate curves (from  $m = 10000$  simulations) under the Black-Karinsinsky model when the initial rate of interest is varied.

## 1.5 Stochastic dependence: General

### Introduction

General probabilistic descriptions of dependent random variables  $X_1, \dots, X_n$  are provided by **joint density functions**  $f(x_1, \dots, x_n)$  or **joint distribution functions**  $F(x_1, \dots, x_n)$ . The latter are defined as the probabilities

$$F(x_1, \dots, x_n) = \Pr(X_1 \leq x_1, \dots, X_n \leq x_n).$$

Its  $n$ -fold partial derivative with respect to  $x_1, \dots, x_n$  is (when it exists) the density function  $f(x_1, \dots, x_n)$  which may also be interpreted as the likelihood of the event

$$X_1 = x_1, X_2 = x_2, \dots, X_n = x_n,$$

though formally (in a strict mathematical sense) such probabilities are zero for continuous variables. Textbooks in probability and statistics often *start* with density functions. We need them for parameter estimation in the next chapter, but they are otherwise not much used in this book. Copulas Section 6.7 are examples of modelling joint distribution functions directly.

### Factorization of density functions

Whether  $X_1, \dots, X_n$  is a series in time or not we may always envisage them in a certain order. This observation opens for general simulation technique. Go recursively through the scheme

$$\begin{array}{lllll} \text{Sample} & X_1^* & X_2^* | X_1^* & \cdots & X_n^* | X_1^*, \dots, X_{n-1}^* \\ \text{Probabilities} & f(x_1) & f(x_2 | X_1^*) & \cdots & f(x_n | X_1^*, \dots, X_{n-1}^*), \end{array}$$

where each drawing is conditional on what has come up before. We start by generating  $X_1$  and end with  $X_n$  given all the others. The order selected does not matter in theory, but in practice there is

often a natural sequence to use. If it isn't, look for other ways to do it.

The sampling scheme reflects the general factorization of joint density functions. Multiply the conditional ones together, and you get

$$f(x_1, \dots, x_n) = f(x_1)f(x_2|x_1) \cdots f(x_n|x_1, \dots, x_{n-1}). \quad (1.29)$$

*general factorization*

Note the notation where all density functions are denoted  $f$  though they differ according to their arguments. A special case is **Bayes'** formula. One version is

$$f(x_1|x_2, \dots, x_n) = \frac{f(x_2, \dots, x_n|x_1)f(x_1)}{f(x_2, \dots, x_n)} \quad (1.30)$$

where the conditional density function of  $X_1$  given  $x_2, \dots, x_n$  is referred back to the opposite form of  $X_2, \dots, X_n$  given  $x_1$ . This type of identity is crucial for Bayesian estimation in Section 7.6.

### Types of dependence

Several special cases of (1.29) are of interest. The model with a common random factor in Section 6.2 is of the form

$$f(x_1, \dots, x_n) = f(x_1)f(x_2|x_1) \cdots f(x_n|x_1). \quad (1.31)$$

*Common factor: First variable*

Here the conditional densities only depend on the first variable, and all the variables  $X_2, \dots, X_n$  are conditionally independent given the first. Full independence means

$$f(x_1, \dots, x_n) = f(x_1)f(x_2) \cdots f(x_n). \quad (1.32)$$

*Independence*

Finally, there is the issue of **Markov dependence**, typically associated with time series. If  $X_k$  is attached to time  $t_k$  for  $k = 1, \dots, n$ , the model is now

$$f(x_1, \dots, x_n) = f(x_1)f(x_2|x_1) \cdots f(x_n|x_{n-1}), \quad (1.33)$$

*Markov dependence*

where  $X_k$  depends on the preceding  $x_{k-1}$  only and the earlier  $x_{k-2}, x_{k-3}, \dots$  are irrelevant. Most models in life insurance belongs to this class, and the random walk and first order autoregression models of Section 5.6 do too; see below. How the general sampling scheme above is adapted is obvious, but the Markov situation is so important that the steps have been summarized through the following algorithm:

#### Algorithm 6.2 Markov sampling

0 Input: Conditional models

1 Generate  $X_1^*$

2 For  $k=2, \dots, n$  do

3     Generate  $X_k^*$  given  $X_{k-1}^*$                     %Sampling from  $f(x_k|X_{k-1}^*)$

4 Return  $X_1^*, \dots, X_n^*$

Examples are given in Section 6.6 and in Exercise 6.5.1.

### Linear and normal processes

All the time series models of Chapter 5 could have been introduced as Markov processes through a sequence of conditional distributions. As an example, consider the Vasicěk model (??) which reads

$$r_k = r_{k-1} + (1 - a)(\xi - r_{k-1}) + \sigma\varepsilon_k = \xi + a(r_{k-1} - \xi) + \sigma\varepsilon_k$$

where  $\varepsilon_1, \varepsilon_2, \dots$  are independent with zero mean and unit variance. Suppose they are normal too. Then

$$r_k | r_{k-1} = r \quad \text{has the distribution of} \quad \xi + a(r - \xi) + \sigma\varepsilon_k, \quad \varepsilon_k \sim N(0, 1),$$

and a model for the series  $\{r_k\}$  is constructed by iterating over  $k = 1, 2, \dots$ , first specify  $r_1 | r_0$ , then  $r_2 | r_1$  and so on. This brings no particular benefit over the approach in Chapter 5, and indeed, the dynamic properties of the model were in Section 5.6 derived without introducing the normal.

The sequence  $r_1, \dots, r_k$  inherits Gaussianity from the errors  $\varepsilon_1, \dots, \varepsilon_k$ , and it *is* possible to write down the joint density function. In the general case where  $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)'$  is the vector of expectations and  $\Sigma$  the covariance matrix the Gaussian density function reads

$$f(\mathbf{x}) = (|2\pi\Sigma|)^{-1/2} \exp\left\{-\frac{1}{2}(\mathbf{x} - \boldsymbol{\xi})'\Sigma^{-1}(\mathbf{x} - \boldsymbol{\xi})\right\} \quad (1.34)$$

where  $|2\pi\Sigma|$  is the determinant of the matrix  $2\pi\Sigma$ . This expression, though famous, plays no role in this book.

### The multinomial situation

One joint density function that *will* be used is the multinomial one; for an application consult Section 8.5. This model is based on multinomial sampling (Section 4.2) where one label among  $K$  is selected according to probabilities  $p_1, \dots, p_K$  (here  $p_1 + \dots + p_K = 1$ ). This is repeated  $n$  times with each trial being independent of all others. Let  $N_k$  is the number of times label  $k$  appears. The vector  $(N_1, \dots, N_K)$  is then **multinomially** distributed with density function

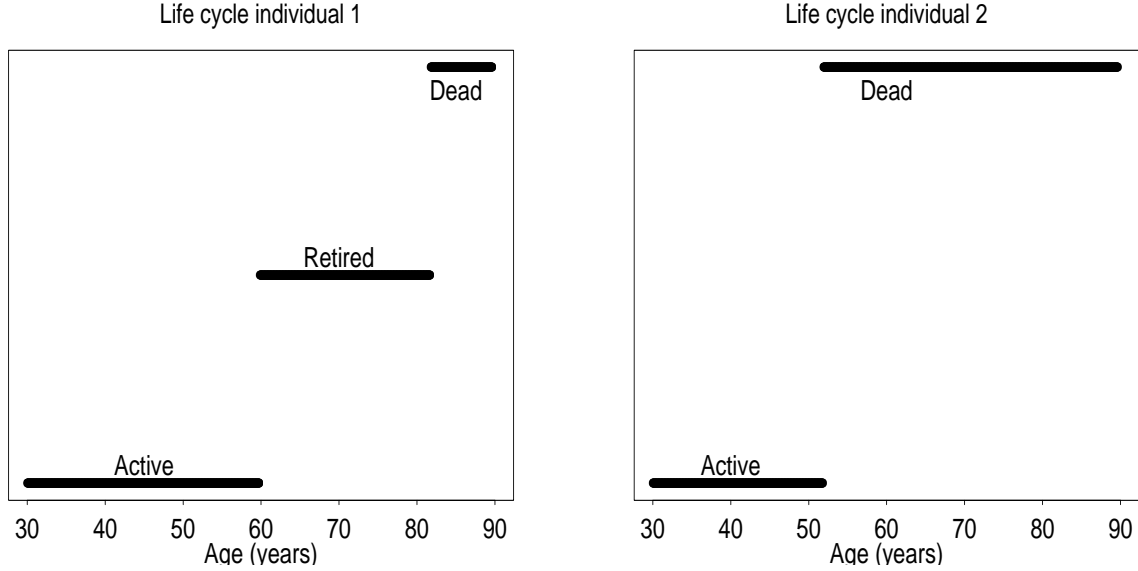
$$\Pr(N_1=n_1, \dots, N_K=n_K) = \frac{n!}{n_1! \dots n_K!} p_1^{n_1} \dots p_K^{n_K} \quad (1.35)$$

where  $n_1 + \dots + n_K = n$ . Take  $K = 2$  and we are back to the ordinary binomial.

The model (1.35) can be justified through a conditional argument. You have to be familiar with the binomial distribution. Let  $\text{bin}(n, p)$  be the binomial density function based on  $n$  trials and success probability  $p$  and suppose (for simplicity) that  $K = 3$ . Then

$$N_1 \sim \text{bin}(n, p_1), \quad N_2 | N_1 = n_1 \sim \text{bin}\left(n - n_1, \frac{p_2}{p_2 + p_3}\right), \quad N_3 = n - n_1 - n_2,$$

where  $N_3$  is fixed by the two others. What lies behind the distribution stated for  $N_2$  in the middle? We know that  $N_1$  has absorbed  $n_1$  trials. There are then  $n - n_1$  of them left for  $N_2$  and  $N_3$  and



**Figure 6.3** *The life cycles of two members of a pension scheme.*

among those label 2 must occur with probability  $p_2/(p_2 + p_3)$ .

The joint density function of  $(N_1, N_2)$  now becomes

$$\frac{n!}{n_1!(n - n_1)!} p_1^{n_1} (1 - p_1)^{n - n_1} \times \frac{(n - n_1)!}{n_2!(n - n_1 - n_2)!} \left(\frac{p_2}{p_2 + p_3}\right)^{n_2} \left(1 - \frac{p_2}{p_2 + p_3}\right)^{n - n_1 - n_2},$$

and if this is multiplied out, you will discover that many of the factors cancel. The expression simplifies to

$$\frac{n!}{n_1!n_2!(n - n_1 - n_2)!} p_1^{n_1} p_2^{n_2} (1 - p_1 - p_2)^{n - n_1 - n_2} = \frac{n!}{n_1!n_2!n_3!} p_1^{n_1} p_2^{n_2} p_3^{n_3}$$

which is (1.35) when  $K = 3$ .

## 1.6 Markov chains and life insurance

### Introduction

Liability risk in life, pension and disability insurance are based on probabilistic descriptions of life cycles like those in Figure 6.3. The individual on the left dies at 82 having retired 22 years earlier at 60, whereas the other is a premature death at 52. A pension scheme consists many (even millions!) of members like those, each with his individual life cycle. Disability is a little more complicated, since there might be transitions back and forth; see below. It is worth noting that a switch from active to retired is determined by a clause in the contract, whereas death and disability must be described in random terms.

Each of the categories of Figure 6.3 is known as a **state**. A life cycle is a sequence  $\{C_l\}$  of such states with  $C_l$  being the category occupied by the individual at age  $y_l = lh$ . We may envisage

$\{C_l\}$  as a step function, jumping occasionally from one state to another. There are three of them in Figure 6.3. This section demonstrates how such schemes are described mathematically. Is it really required? After all, we saw in Section 3.4 that uncertainty due to life cycle movements is rarely very important in insurance. However, that does not make the underlying stochastic model irrelevant. It is needed both to compute the expectations defining the liabilities (Chapter 12) *and* to evaluate portfolio uncertainty due to parameter error (Section 15.2).

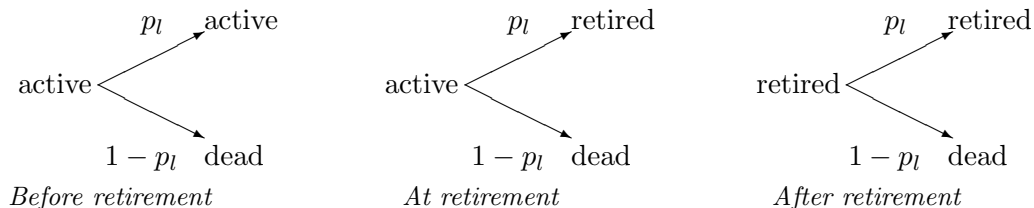
### Markov modelling

Consider random step functions  $\{C_l\}$  jumping between a limited number of states. The most frequently applied model is the **Markov chain**. What makes such time series evolve is the so-called **transition probabilities**

$$p_l(i|j) = \Pr(C_{l+1} = i | C_l = j). \tag{1.36}$$

At each point in time there is a random experiment that takes the state from its current  $j$  to a (possibly) new  $i$ . Note that the probabilities defining the model *do not depend on the track record* of the individual prior to age  $l$ . That is the Markov assumption. Monte Carlo is a good way to find out how such models work; see Exercises 6.6.2 and 6.6.5.

Transition probabilities are usually different for men and women (not reflected in the mathematical notation), and it is (of course) essential that they depend on the age  $l$ . There is always a main role for the survival probabilities  $p_l = {}_1p_l$  introduced in Section 6.2. For a simple pension scheme, such as in Figure 6.3, the three states ‘active’, ‘retired’ and ‘dead’ have the following transition probabilities:



The details differ according to whether we are before, at or after retirement. Note the middle diagram in particular, where the individual from a clause in the contract moves from active to retired (unless he dies).

### A disability scheme

Disability modelling, with movements back and forth between states, is more complicated. Consider the following scheme.



A person may become ‘disabled’ (state  $i$ ), but there is also a chance that he returns to ‘active’ (state  $a$ ). Such rehabilitations are not too frequent as this book is being written (2008), but it



could be different in the future, and we should certainly be able to handle it mathematically. New probabilities are then needed in addition to those describing survival. They have above been denoted  $p_{i|a}$  and  $p_{a|i}$ . Both will in practice depend on age  $l$ , but this aspect is here omitted. The probability of moving from ‘active’ to ‘disabled’ is denoted  $p_{i|a}$  while the opposite is  $p_{a|i}$ .

The transition probabilities for the scheme must combine survival and disability/rehabilitation. The full matrix are as follows:

| From     | To new state              |                           |           | Row sum |
|----------|---------------------------|---------------------------|-----------|---------|
|          | Active                    | Disabled                  | Dead      |         |
| Active   | $p_l \cdot (1 - p_{i a})$ | $p_l \cdot p_{i a}$       | $1 - p_l$ | 1       |
| Disabled | $p_l \cdot p_{a i}$       | $p_l \cdot (1 - p_{a i})$ | $1 - p_l$ | 1       |
| Dead     | 0                         | 0                         | 1         | 1       |

Each entry is the product of input probabilities. For example, to remain active (upper left corner) the individual must survive *and* not become disabled, and similar for the others. Note the row sums. *They are always equal to one* (add them and you see it is true). *Any* set of transition probabilities for Markov chains must satisfy this restriction, which merely reflects that the individual either moves somewhere or remains at the present state.

### Numerical example

Figure 6.4 shows a portfolio development that might occur in practice. The survival model was the same as in Section 3.4, i.e.

$$\log(p_l) = -0.0009 - 0.0000462 \exp(0.090767 \times l)$$

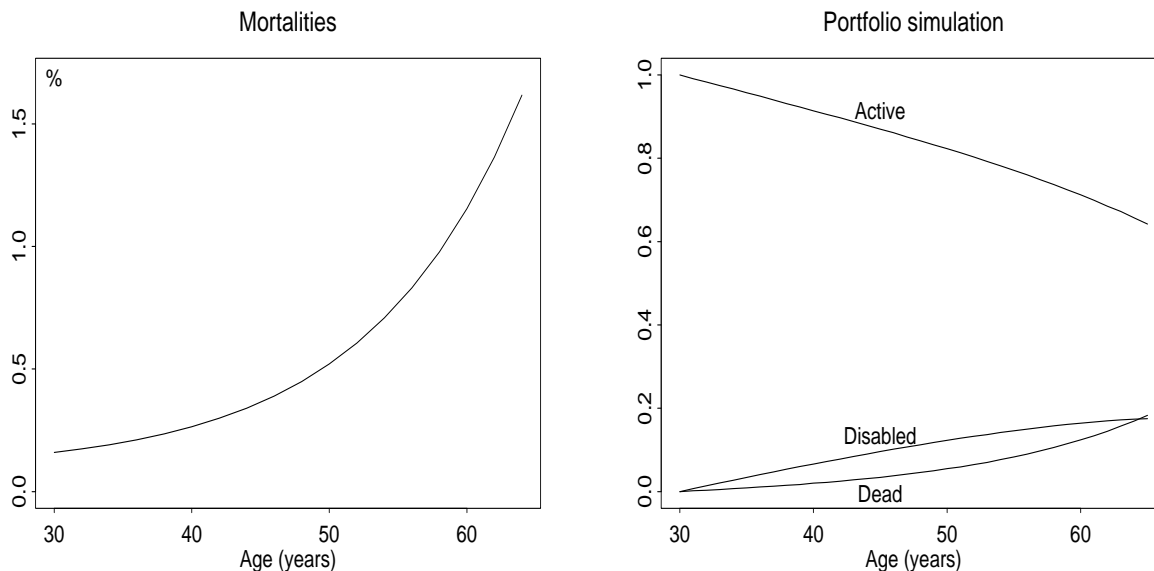
Their corresponding annual **mortalities**  $q_l = 1 - p_l$  are plotted in Figure 6.4 left. Note the steep increase on the right for the higher age groups where the likelihood of dying within the coming year has reached 1.5% and more.

This model corresponds to an average length of life of 75 years and will be further discussed in Chapter 12. It is reasonably realistic for males in a developed country. Disability depends on the current political climate and on economic cycles and is harder to hang numbers on. The computations in Figure 6.4 are based on

$$p_{i|a} = 0.7\%, \quad \text{and} \quad p_{a|i} = 0.1\%,$$

which are values invented. Note the rehabilitation rate, which may be too high. In practice both probabilities might depend on age, as noted above.

How individuals distribute between the three states are shown in Figure 6.4 right for a portfolio originally consisting of one million 30-year males. The scenario has been simulated using Algorithm 6.2 (details in Exercise 6.6.2). There is very little Monte Carlo uncertainty in portfolios this size and one *single* run is enough. At the start all are active, but with age the number of people in the



**Figure 6.4** A disability scheme in life insurance: Mortality model (left) and portfolio simulation (right).

other two classes grow. At 65 years a little over 80% remain alive (distributed between 'active' and 'disabled'), a realistic figure. What may not be true in practice is the downwards curvature in the disability curve which might be turned around if the disability rate is made age-dependent.

## 1.7 Introducing copulas

### Introduction

Let  $U_1$  and  $U_2$  be uniform, *dependent* random variables and consider

$$X_1 = F_1^{-1}(U_1) \quad \text{and} \quad X_2 = F_2^{-1}(U_2), \quad (1.37)$$

where  $F_1^{-1}(u)$  and  $F_2^{-1}(u)$  are the percentiles of distribution functions  $F_1(x)$  and  $F_2(x)$ . This simple set-up defines a modelling strategy that has grown popular. Treat dependence and univariate variation as two separate issues and supply a joint distribution function  $C(u_1, u_2)$ , known as a **copula**, to describe the covariation between  $U_1$  and  $U_2$  which is then passed on to  $X_1$  and  $X_2$ . The inversion algorithm of Chapter 2 tells us that the marginal distributions of  $X_1$  and  $X_2$  are determined by the percentiles used in (1.37).

The idea goes back to the mid twentieth century, originating with the work of Sklar (1959). One of the advantages is that it enables us to tackle situations as those in Figure 6.5 where the variables are more dependent in certain parts of the space than elsewhere. An example in practice is equities that may correlate stronger in falling markets than on average; see Longin and Solnik (2001) for a formal, statistical justification. Such phenomena is of interest in insurance too. Wütrich (2004) is a theoretical contribution.

## Copula modelling

A bivariate copula is the joint distribution function

$$C(u_1, u_2) = \Pr(U_1 \leq u_1, U_2 \leq u_2), \quad 0 < u_1 \leq 1, 0 < u_2 \leq 1, \quad (1.38)$$

where  $0 < u_1 \leq 1$  and  $0 < u_2 \leq 1$ . Any function  $C(u_1, u_2)$  that is to play this role must be increasing in  $u_1$  and  $u_2$  and satisfy

$$C(u_1, 0) = 0, \quad C(u_1, 1) = u_1 \quad \text{and} \quad C(0, u_2) = 0, \quad C(1, u_2) = u_2. \quad (1.39)$$

Simple examples are

$$\begin{array}{ll} C(u_1, u_2) = u_1 u_2 & \text{and} \quad C(u_1, u_2) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta} \\ \text{independent copula} & \text{Clayton copula} \end{array}$$

where  $\theta > 0$  on the right. Both satisfy the side conditions (1.39). Why must we impose the latter? Because  $C(u_1, 0) = \Pr(U_1 \leq u_1, U_2 \leq 0) = 0$  and  $C(u_1, 1) = \Pr(U_1 \leq u_1, U_2 \leq 1) = \Pr(U_1 \leq u_1) = u_1$ , no uniforms without this! The rationale for the other conditions in (1.39) is similar.

The copula approach rests on a representation theorem discovered by Sklar (1959). Any joint distribution function  $F(x_1, x_2)$  with strictly increasing marginal distribution functions  $F_1(x_1)$  and  $F_2(x_2)$  may be written

$$\begin{array}{ll} F(x_1, x_2) = C(u_1, u_2) & \text{where} \quad u_1 = F_1(x_1), \quad u_2 = F_2(x_2). \\ \text{copula modelling} & \text{univariate modelling} \end{array} \quad (1.40)$$

with a modified version even for counts. Working through copulas do not restrict the model at all, and there are additional versions when antitetic twins (Section 4.3) are supplied for either uniform. The copula on the left in (1.40) may be combined with either of

$$\begin{array}{ll} u_1 = F_1(x_1), \quad 1 - u_2 = F_2(x_2) & \text{orientation } (1,2) \\ 1 - u_1 = F_1(x_1), \quad u_2 = F_2(x_2) & \text{orientation } (2,1) \\ 1 - u_1 = F_1(x_1), \quad 1 - u_2 = F_2(x_2) & \text{orientation } (2,2), \end{array} \quad (1.41)$$

and the effect is to rotate the copula patterns  $90^\circ$ ,  $180^\circ$  and  $270^\circ$  compared to the original one (orientation (1,1)); see Figure 6.5.

Extension to many variables is straightforward. A  $J$ -dimensional copula  $C(u_1, \dots, u_J)$  is the distribution function of  $J$  dependent uniform variables  $U_1, \dots, U_J$  and satisfies consistency requirements similar to those in (1.39); see Exercise 6.7.6. Transformations back to the original variables are now through  $X_1 = F_1^{-1}(U_1), \dots, X_J = F_J^{-1}(U_J)$ , and there are  $2^J$  ways of rotating patterns through the use of antitetic twins, not just 4. You calculate copulas of sub-vectors by inserting ones, for example the (joint) distribution function of  $U_1, \dots, U_j$  is  $C(u_1, \dots, u_j, 1, \dots, 1)$ .

### Example: The Clayton copula

The Clayton copula was introduced above above, but its definition can be extended to

$$C(u_1, u_2) = \max\left((u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}, 0\right) \quad (1.42)$$

and it is easy to check that the copula requirements (1.39) are satisfied whenever  $\theta \geq -1$ . Nor is it a difficult exercise to show that  $C(u_1, u_2) \rightarrow u_1 u_2$  as  $\theta \rightarrow 0$  so that  $\theta = 0$  means that  $U_1$  and  $U_2$  are independent. When  $\theta$  is negative,

$$C(u_1, u_2) = 0 \quad \text{if} \quad u_2 > (1 - u_1^{-\theta})^{-1/\theta},$$

and certain pairs  $(u_1, u_2)$  are forbidden territory; see also Figure 6.6 below. Hard restrictions of that kind are often undesirable. Yet when negative  $\theta$  is included, the family in a sense cover the entire range of dependency that is logically possible.; see Exercises 6.7.2. and 6.7.3.

Simulated structures generated by the Clayton copula are shown in Figure 6.5. The marginal distributions of  $X_1$  and  $X_2$  were normal with mean  $\xi = 0.005$  and volatility  $\sigma = 0.05$ , precisely as in Figure 2.5 (and realistic for equity returns). Most striking is the cone-shapes patterns which signify unequal degrees of dependence in unequal parts of the space. Note, for example, the plot in the upper, left corner where downside returns are more highly correlated than upside ones. Such situations have been seen in practice; consult Longin and Solnik (2001). Consequences for downside financial risk could be serious, and ordinary Gaussian models do not capture this. The other plots in Figure 6.5 rotate patterns by varying the orientation of the copula with dependence adjusted by changing the Clayton parameter  $\theta$  (high values go with strong dependence).

### Conditional distributions under copulas

Additional insight is provided by examining conditional distributions, and this is one of the ways copulas are simulated too. We shall need an expression for

$$C(u_J | u_1, \dots, u_{J-1}) = \Pr(U_J \leq u_J | u_1, \dots, u_{J-1}).$$

In the two-dimensional case  $J = 2$

$$c(u_2 | u_1) = c(u_1, u_2) = \frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2},$$

where  $c(u_1, u_2)$  is the joint density function of  $(U_1, U_2)$ . When this is integrated with respect to  $u_2$

$$C(u_2 | u_1) = \int_0^{u_2} c(v | u_1) dv = \int_0^{u_2} \frac{\partial^2 C(u_1, v)}{\partial u_1 \partial v} dv = \frac{\partial}{\partial u_1} \int_0^{u_2} \frac{\partial C(u_1, v)}{\partial v} dv = \frac{\partial C(u_1, u_2)}{\partial u_1},$$

and conditional distribution functions are determined by differentiating the original copula. The general result is

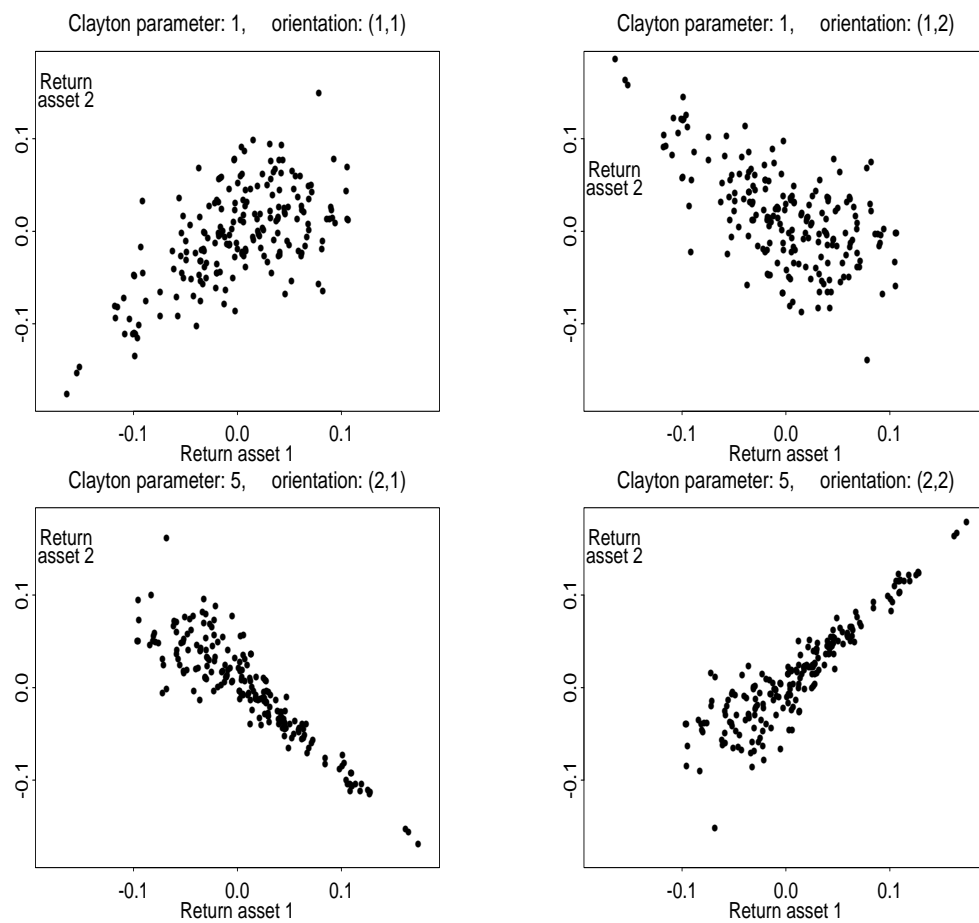
$$C(u_J | u_1, \dots, u_{J-1}) = \frac{\partial^{J-1} C(u_1, \dots, u_{J-1}, u_J) / \partial u_1 \cdots \partial u_{J-1}}{\partial^{J-1} C(u_1, \dots, u_{J-1}, 1) / \partial u_1 \cdots \partial u_{J-1}} \quad (1.43)$$

and you have to differentiate numerator and denominator  $J - 1$  times. This is proved in Section 6.8.

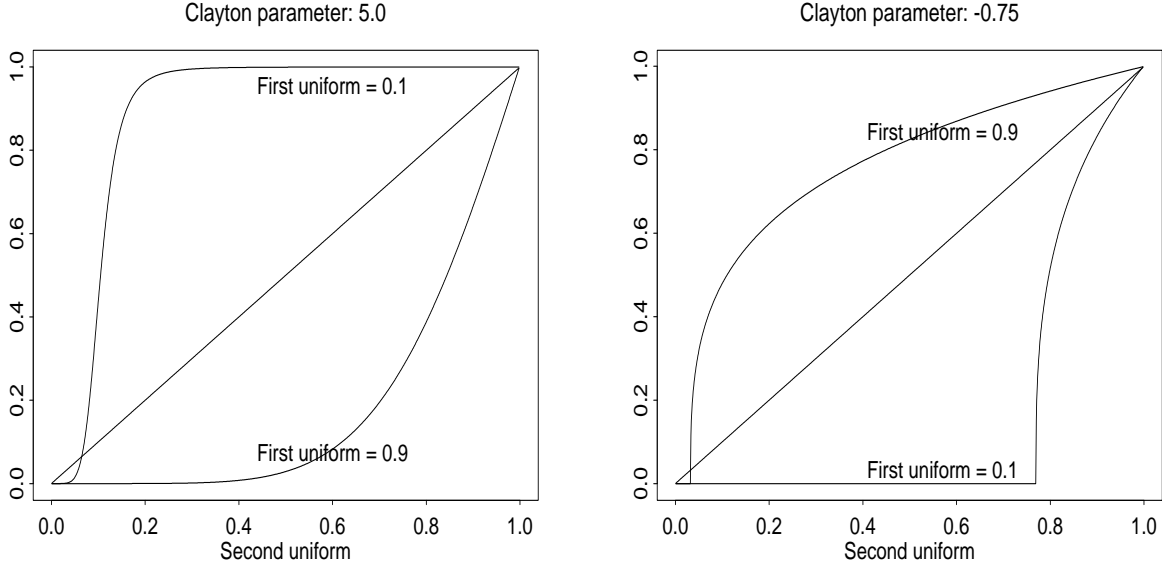
As an example, consider the Clayton copula (1.42) for  $J = 2$ . Straightforward calculations yields

$$C(u_2 | u_1) = u_1^{-(1+\theta)} \max \left( (u_1^{-\theta} + u_2^{-\theta} - 1)^{-(1+1/\theta)}, 0 \right), \quad (1.44)$$

where the expression is zero when  $\theta < 0$  and  $u_2 > (1 - u_1^{-\theta})^{-1/\theta}$ . Conditional distribution functions have been plotted in Figure 6.6 with  $\theta$  is large and positive on the left and large and negative on



**Figure 6.5** *Simulated financial returns from normals and Clayton copula.*



**Figure 6.6** Conditional distribution functions for the **second** variable of a Clayton copula; given first variable marked on each curve.

the right. Shapes under  $u_1 = 0.1$  and  $u_2 = 0.9$  differ markedly, attesting to strong dependence between  $U_1$  and  $U_2$ , but the most notable feature is a lack of symmetry. For the distributions on the left  $U_2$  is located in a narrow strip around  $u_1$  when  $u_1 = 0.1$ , but its variation is much larger when  $u_1 = 0.9$ . It is precisely this feature that creates the cones in Figure 6.5.

### Archimedean copulas

These might be the most widely used of all copulas, and many of the most important ones have a stochastic representation due to Marshall and Olkin (1988). Let  $Z$  be a positive random variable with density function  $g(z)$ . Its moment generating function (or Laplace transform) is

$$M(x) = E(e^{-xZ}) = \int_0^{\infty} e^{-xz} g(z) dz; \quad (1.45)$$

see Section A.1. Only positive  $x$  is of interest, and  $M(x)$  decreases monotonely from one at  $x = 0$  to zero at  $\infty$ . Define

$$U_j = M\left(-\frac{\log(V_j)}{Z}\right), \quad j = 1, \dots, J \quad (1.46)$$

where  $V_1, \dots, V_J$  is a sequence of independent and uniform random variables. Then  $U_1, \dots, U_J$  are uniform too (which isn't obvious!), and their joint distribution function is a copula of the form

$$C(u_1, \dots, u_J) = \phi^{-1}\{\phi(u_1) + \dots + \phi(u_J)\}, \quad (1.47)$$

where  $\phi(u) = M^{-1}(u)$ . Here  $\phi^{-1}(u)$  and  $M^{-1}(u)$  are the inverse functions of  $\phi(u)$  and  $M(u)$ ; for example  $x = \phi^{-1}(u)$  is the solution of the equation  $\phi(x) = u$ . The result is proved in Section 6.8.

The Clayton coula emerges when  $Z$  is Gamma distributed with density function

$$g(z) = \frac{\alpha^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\alpha z} \quad \text{where} \quad \alpha = 1/\theta.$$

Then

$$M(x) = \int_0^\infty e^{-xz} \frac{\alpha^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\alpha z} dz = \left(1 + \frac{x}{\alpha}\right)^{-\alpha} = (1 + \theta x)^{-1/\theta}$$

so that the two functions  $\phi(u)$  and  $\phi^{-1}(x)$  become

$$\phi(u) = \frac{1}{\theta}(u^{-\theta} - 1), \quad \text{and} \quad \phi^{-1}(x) = (1 + \theta x)^{-1/\theta}. \quad (1.48)$$

When these are inserted into (1.47), the earlier expression for the Clayton copula emerges.

The function  $\phi(u)$  is known as the **generator** and the copulas (1.47) as the **Archimedean** class. It was originally obtained by Kimberling (1974). Generators that are inverses of moment generating functions allow easy sampling (see below)<sup>2</sup>, but generators are not confined to this type. Indeed suppose  $\phi(u)$  decreases monotonely from infinity at  $u = 0$  to zero at  $u = 1$ . Then, in the two-dimensional case,

$$C(u_1, 0) = \phi^{-1}\{\phi(u_1) + \phi(0)\} = \phi^{-1}(\infty) = 0$$

and

$$C(u_1, 1) = \phi^{-1}\{\phi(u_1) + \phi(1)\} = \phi^{-1}\{\phi(u_1)\} = u_1$$

which are the copula requirements (1.39). General  $J$  is similar. We may even allow finite  $\phi(0)$ , but the copula is now strictly zero whenever  $\phi(u_1) + \dots + \phi_J(u_J) > \phi(0)$ , and certain combinations of  $u_1, \dots, u_J$  are forbidden. In practice this is rarely what we want. A huge list of generators are compiled in Table 4.1 in Nelsen (2006). The Clayton generator with  $\theta = 0.2$  and the polynomial one  $\phi(u) = (1 - u)^3$  are plotted in Figure 6.7.

### Simulation

A Monte Carlo simulation  $U_1^*, \dots, U_J^*$  of a uniform vector under a copula is passed on to the original variables through

$$X_1^* \leftarrow F_1^{-1}(U_1^*), \dots, X_J^* \leftarrow F_J^{-1}(U_J^*),$$

which is still another application of the inversion algorithm. There are simple exact or approximate ways of calculating percentiles for most distributions used in this book (Gamma and  $t$ -distributions are exceptions). But what about  $U_1^*, \dots, U_J^*$  itself? One class of models that *is* easy to handle are Archimedean copulas satisfying the Marshall-Olkin stochastic representation. Simply copy (1.46) as follows:

#### Algorithm 6.3 Archimedean copulas

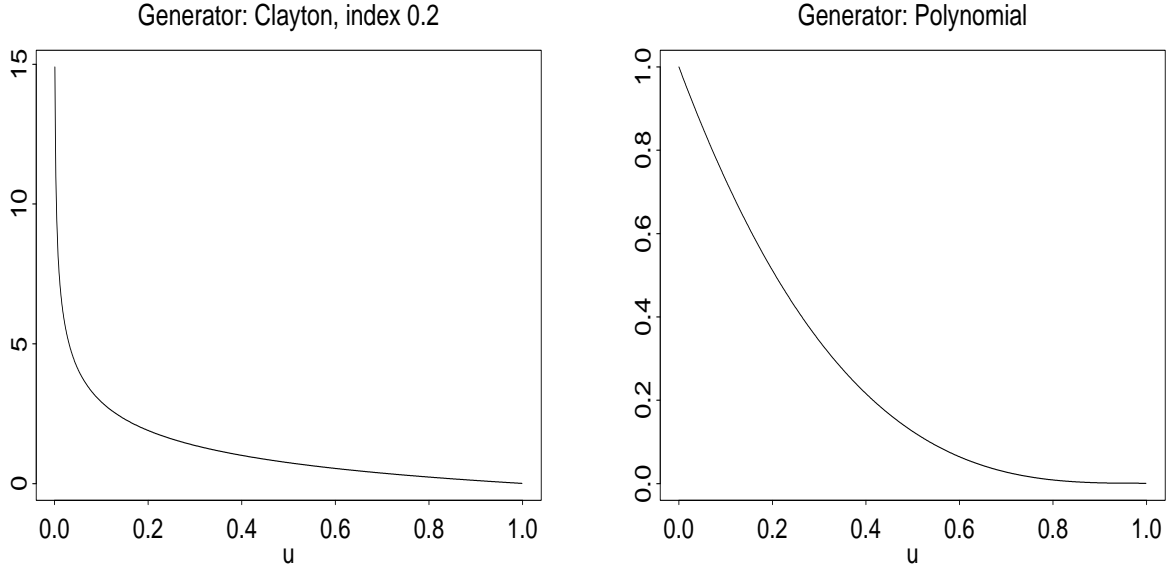
0 Input:  $\phi(u)$

1 Draw  $Z^*$

*%Z with Laplace transform  $M(u) = \phi^{-1}(u)$*

---

<sup>2</sup>For the inverse  $\phi(u)^{-1}(u)$  of a given generator to be a moment generating function of some distribution is must be **totally positive**; i.e its derivatives must satisfy  $(-1)^s \frac{d^s \phi^{-1}(u)}{du^s} \geq 0$  for all  $s$ ; see Feller (1971), p. 439.



**Figure 6.7** Generator functions for Archimedean copulas

- 2 For  $j = 1, \dots, J$  repeat
- 3     Draw  $V^* \sim \text{uniform}$      and      $U_j^* \leftarrow -\log(V^*)/Z^*$
- 4 Return  $U_1^*, \dots, U_J^*$

When  $Z^*$  is drawn from the standard Gamma distribution with shape  $\alpha = 1/\theta$ , a simulation of the Clayton copula is returned.

This doesn't work when  $\theta < 0$ , and other copulas are not in a form convenient for Monte Carlo at all. A general approach is to go recursively through the copula vector, first draw  $U_1^*$  (an ordinary uniform), then  $U_2^*$  given  $U_1^*$  and so on. Let  $C(u_j|u_1, \dots, u_{j-1}) = \Pr(U_j \leq u_j|u_1, \dots, u_{j-1})$  be the conditional distribution of  $U_j$  given the predecessors. Independent uniforms  $V_1^*, \dots, V_J^*$  are passed on to a copula simulation by solving the equations

$$C(U_j^*|U_1^*, \dots, U_{j-1}^*) = V_j^*, \quad j = 1, \dots, J. \quad (1.49)$$

Numerical methods are required in general. There is a neat algorithm for Clayton copulas:

**Algorithm 6.4 The Clayton copula**

- 0 Input:  $\theta$
- 1 Draw  $U_1^* \sim \text{uniform}$      and      $S^* \leftarrow 0$
- 2 For  $j = 2, \dots, J$  do
- 3      $S^* \leftarrow S^* + (U_{j-1}^*)^{-\theta} - 1$      % Updating from preceding uniform
- 3     Draw  $V^* \sim \text{uniform}$
- 4      $U_j^* \leftarrow \{-S^* + (1 + S^*)(V^*)^{-\theta/(\theta+j-1)}\}^{-1/\theta}$      % Next uniform
- 5 Return  $U_1^*, \dots, U_J^*$

This algorithm, which has been used for copula simulations in this book is justified in Section



6.8.

### How important is copula modelling?

Suppose  $R_1$  and  $R_2$  are the returns of two equity investments. Their standard model is

$$R_1 = e^{\xi_1 + \sigma_1 \varepsilon_1} - 1 \quad \text{and} \quad R_2 = e^{\xi_2 + \sigma_2 \varepsilon_2} - 1$$

where  $\varepsilon_1$  and  $\varepsilon_2$  are  $N(0, 1)$  and dependent. If a copula is used, then  $\varepsilon_1 = \Phi^{-1}(U_1)$  and  $\varepsilon_2 = \Phi^{-1}(U_2)$  with a joint distribution function  $C(u_1, u_2)$  supplied for  $U_1$  and  $U_2$  (here  $\Phi^{-1}(u)$  is the Gaussian percentiles). What is the impact of a copula when compared to the traditional Gaussian model? To examine the issue we must compute

$$\rho = \text{cor}(\varepsilon_1, \varepsilon_2) = E\{\Phi^{-1}(U_1), \Phi^{-1}(U_2)\}$$

and use this value as the correlation coefficient of the Gaussian. A practical way is through Monte Carlo. Let  $(U_{1i}^*, U_{2i}^*)$ ,  $i = 1, \dots, m$  be simulations of  $(U_1, U_2)$  which yields as approximation

$$\rho^* = \frac{1}{m} \sum_{i=1}^m \Phi^{-1}(U_{1i}^*) \Phi^{-1}(U_{2i}^*).$$

The experiments in Figure 6.8 used the Clayton copula with  $\theta = 1$ . It is a good idea to consult Figure 6.5 (upper, left corner) to see how the model behaves. The corresponding correlation coefficient of the Gaussian is  $\rho \doteq 0.498$ . Expectations and volatilities were fixed at  $\xi_1 = \xi_2 = 0.005$  and  $\sigma_1 = \sigma_2 = 0.05$ , the same as in Figure 6.5 (monthly values), and the portfolio was equally weighted so that its return is  $\mathcal{R} = R_1/2 + R_2/2$ .

The results in Figure 6.8 are based on one million simulations of  $\mathcal{R}$ . Density functions of  $\mathcal{R}$  below  $-5\%$  monthly return are shown on the left. They were computed by discarding all simulations exceeding this ceiling with ordinary density estimation may applied to on the remaining. Discrepancies under the two models are not negligible, but this changes on the right of Figure 6.8 which shows density functions of ordinary five-year returns. Differences are now hardly visible at all (try to look hard!). Five-year (or sixty-month) returns were simulated through the recursion

$$\mathcal{R}_{0:k}^* = \mathcal{R}_{0:k-1}^* (1 + \mathcal{R}_k^*), \quad k = 1, \dots, 60 \quad \text{starting at} \quad \mathcal{R}_{0:0}^* = 100.$$

Here  $\mathcal{R}_k^*$  is the Monte Carlo return in period  $k$ . The general message is that long-range asset risk may not be too strongly influenced by subtle copula effects. Other values of  $\theta$  gave similar results.

## 1.8 Mathematical arguments

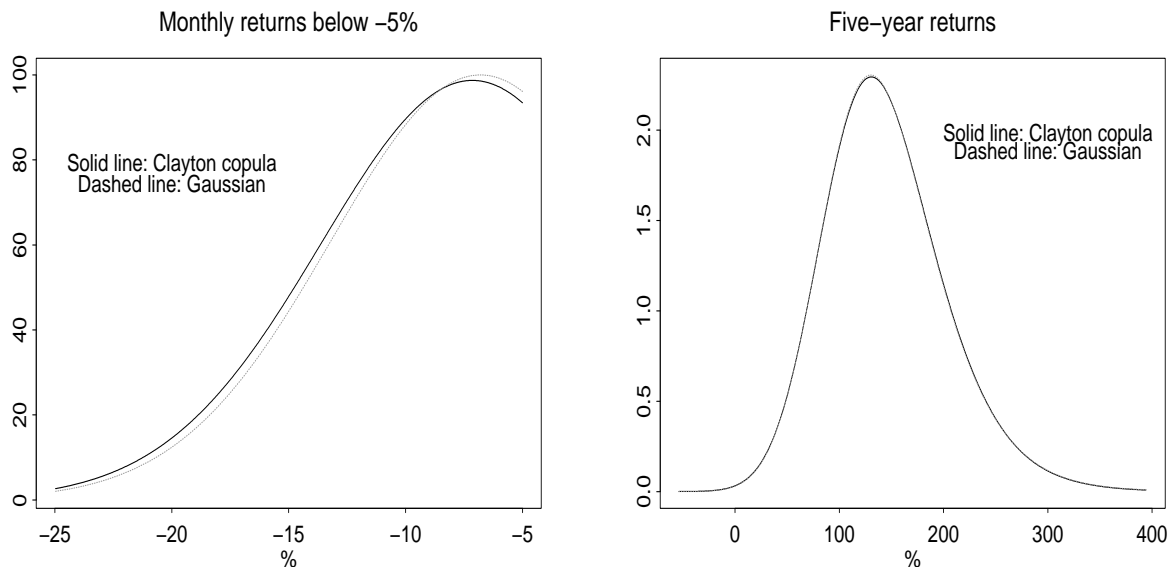
### Section 6.3

**Portfolio risk** We shall verify the formula (1.22) for the standard deviation of the portfolio risk  $\mathcal{X}$ . Start with

$$\text{var}(\mathcal{X}) = \text{var}(\mathcal{N}) \xi_z^2 + E(\mathcal{N}) \sigma_z^2,$$

which is the right hand side of (1.19). Here  $E(\mathcal{N}) = J \xi_\mu T$  whereas the variance varied with the sampling regime. Let  $\gamma = J$  for common and  $\gamma = 1$  for independent sampling of the claim intensity. Then the standard deviation formulae in (1.17) and (1.18) may be summarized as

$$\text{var}(\mathcal{N}) = JT^2(\gamma \sigma_\mu^2 + \xi_\mu/T).$$



**Figure 6.8** Density functions for the monthly returns below 5% (left) and for five-year returns (right) under models described in the text.

Inserting the expressions for  $E(\mathcal{N})$  and  $\text{var}(\mathcal{N})$  into the formula for  $\text{var}(\mathcal{X})$  yields

$$\text{var}(\mathcal{X}) = JT^2(\gamma\sigma_\mu^2 + J\xi_\mu/T)\xi_z^2 + J\xi_\mu T\sigma_z^2 = JT\xi_\mu(\sigma_z^2 + \xi_z^2) + JT^2\gamma\sigma_\mu^2.$$

or

$$\text{var}(\mathcal{X}) = \left(JT\xi_\mu(\sigma_z^2 + \xi_z^2)\right) \times \left(1 + \gamma T \frac{\sigma_\mu^2}{\xi_\mu} \frac{\xi_z^2}{\xi_z^2 + \sigma_z^2}\right),$$

which is (1.22).

#### Section 6.4.

**Optimal prediction** To prove that  $\hat{Y} = E(Y|\mathbf{X})$  is the optimal predictor for  $Y$  start by noting that

$$E(Y - a)^2 = E(Y^2) - 2aE(Y) + a^2, \quad \text{minimized by} \quad a = E(Y),$$

and apply this conditionally given  $\mathbf{X}$ . Then, if  $\tilde{Y} = \tilde{Y}(\mathbf{X})$  is an arbitrary function of  $\mathbf{X}$ ,

$$E\{(Y - \hat{Y})^2|\mathbf{X}\} \leq E\{(Y - \tilde{Y})^2|\mathbf{X}\},$$

and when expectation with respect to  $\mathbf{X}$  is passed over both sides, the rule of double expectation yields

$$E(Y - \hat{Y})^2 \leq E(Y - \tilde{Y})^2$$

which is (1.25).

## Section 6.7.

**Conditional distributions** We shall derive an expression for

$$C(u_J|u_1, \dots, u_{J-1}) = \Pr(U_J \leq u_J | u_1, \dots, u_{J-1}) = \int_0^{u_J} c(v|u_1, \dots, u_{J-1}) dv$$

where  $c(u_J|u_1, \dots, u_{J-1})$  is the conditional distribution function of  $U_J$  given the others. To calculate the integral introduce  $c(u_1, \dots, u_J)$  and  $c(u_1, \dots, u_{J-1})$  as joint density functions of  $U_1, \dots, U_J$  and  $U_1, \dots, U_{J-1}$  and note that

$$c(u_J|u_1, \dots, u_{J-1}) = \frac{c(u_1, \dots, u_J)}{c(u_1, \dots, u_{J-1})} = \frac{\partial^J C(u_1, \dots, u_J) / \partial u_1, \dots, \partial u_J}{\partial^{J-1} C(u_1, \dots, u_{J-1}, 1) / \partial u_1, \dots, \partial u_{J-1}}.$$

If  $D$  is the denominator, then

$$\begin{aligned} \int_0^{u_J} c(v|u_1, \dots, u_{J-1}) dv &= D^{-1} \int_0^{u_J} \frac{\partial^J C(u_1, \dots, u_{J-1}, v)}{\partial u_1, \dots, \partial u_{J-1}, \partial v} dv \\ &= D^{-1} \frac{\partial^{J-1}}{\partial u_1, \dots, \partial u_{J-1}} \int_0^{u_J} \frac{\partial C(u_1, \dots, u_{J-1}, v)}{\partial v} dv \\ &= D^{-1} \frac{\partial^{J-1} C(u_1, \dots, u_J)}{\partial u_1, \dots, \partial u_{J-1}}, \end{aligned}$$

and it follows that

$$C(u_J|u_1, \dots, u_{J-1}) = \frac{\partial^{J-1} C(u_1, \dots, u_{J-1}, u_J) / \partial u_1, \dots, \partial u_{J-1}}{\partial^{J-1} C(u_1, \dots, u_{J-1}, 1) / \partial u_1, \dots, \partial u_{J-1}}$$

as claimed in Section 6.7.

**Stochastic representation of Archimedean copulas** Let  $V_1, \dots, V_J$  be independent and uniform,  $Z$  a positive random variable with moment generating function  $M(x) = \int_0^\infty e^{-xz} g(z) dz$  and define  $U_j = M(-\log(V_j)/Z)$  for  $j = 1, \dots, J$ . We are to prove that  $U_1, \dots, U_J$  follow an Archimedean copula with generator  $\phi(u) = M^{-1}(u)$  provided  $Z$  is independent of  $V_1, \dots, V_J$ . Note that  $V_j = e^{-M^{-1}(U_j)Z}$  so that if  $Z = z$  is fixed, then

$$\Pr(U_1 \leq u_1, \dots, U_J \leq u_J | z) = \Pr(V_1 \leq e^{-M^{-1}(u_1)z} \dots, V_J \leq e^{-M^{-1}(u_J)z} | z)$$

and since  $V_1, \dots, V_J$  are independent,

$$\Pr(U_1 \leq u_1, \dots, U_J \leq u_J | z) = e^{-M^{-1}(u_1)z - \dots - M^{-1}(u_J)z}.$$

But

$$\Pr(U_1 \leq u_1, \dots, U_J \leq u_J) = \int_0^\infty \Pr(U_1 \leq u_1, \dots, U_J \leq u_J | z) g(z) dz$$

and hence

$$\Pr(U_1 \leq u_1, \dots, U_J \leq u_J) = \int_0^\infty e^{-\{M^{-1}(u_1) + \dots + M^{-1}(u_J)\}z} g(z) dz$$

so that

$$\Pr(U_1 \leq u_1, \dots, U_j \leq u_j) = M\{M^{-1}(u_1) + \dots + M^{-1}(u_j)\}$$

which is a Archimedean copula with generator  $\phi(u) = M^{-1}(u)$ .

**Algorithm 6.4** The Clayton copula for  $U_1, \dots, U_j$  is

$$C(u_1, \dots, u_j) = \left( \max\left(\sum_{i=1}^j u_i^{-\theta} - j + 1, 0\right) \right)^{-1/\theta}.$$

which must be differentiated with respect to  $u_1, \dots, u_{j-1}$ . This yields

$$\frac{\partial^{j-1} C(u_1, \dots, u_j)}{\partial u_1 \dots \partial u_{j-1}} = \left( \max\left(\sum_{i=1}^j u_i^{-\theta} - j + 1, 0\right) \right)^{-(1/\theta-j+1)} \times \prod_{i=1}^{j-1} \left( u_i^{-(1+\theta)} (1 + (i-1)\theta) \right),$$

and from (1.43) the conditional distribution function of  $U_j$  given  $u_1, \dots, u_{j-1}$  becomes

$$\begin{aligned} C(u_j | u_1, \dots, u_{j-1}) &= \left( \frac{\max(\sum_{i=1}^j u_i^{-\theta} - j + 1, 0)}{\max(\sum_{i=1}^{j-1} u_i^{-\theta} - j + 2, 0)} \right)^{1/\theta-j+1} \\ &= \left( \frac{\max(u_j^{-\theta} + s_{j-1}, 0)}{\max(s_{j-1} + 1, 0)} \right)^{1/\theta-j+1} \end{aligned}$$

where  $s_{j-1} = \sum_{i=1}^{j-1} u_i^{-\theta} - (j-1)$ . A Monte Carlo drawing of  $U_j^*$  given  $U_1^*, \dots, U_{j-1}^*$  is therefore the solution of the equation

$$\left( \frac{\max((U_j^*)^{-\theta} + S_{j-1}^*, 0)}{\max(S_{j-1}^* + 1, 0)} \right)^{1/\theta-j+1} = V^* \quad \text{where} \quad S_{j-1}^* = \sum_{i=1}^{j-1} (U_i^*)^{-\theta} - (j-1)$$

and  $V^*$  is another uniform. It follows that

$$U_j^* = \{-S_{j-1}^* + (1 + S_{j-1}^*)(V^*)^{-\theta/(\theta+j-1)}\}^{-1/\theta}$$

and the entire scheme can be organized as in Algorithm 6.4.

## 1.9 Bibliographical notes

**General references** Markov models and other basic topics in probability can be found in almost any textbook. The classic treatise Feller (1968) and (1971) is still a good place to start. Specialist monographs on multivariate distributions are Johnson, Kotz and Balakrishnan (1997) (the discrete type) and Kotz, Balakrishnan and Johnson (2000) (continuous ones); see also Balakrishnan (2004a) and (2004b) for the actuarial context on this. Among the applications you will encounter the hierarchical risk model of Section 6.3 in most textbooks on general insurance; see Mikosch (2004) for example. The Vasiček term structure in Section 6.4 is only one among a huge number of possible models, see James and Webber (2000), Briga and Mercurio (2001) and Cairns (2004) for others.

**Copulas** The interest in copulas exploded around the turn of the century, although Mikosch (2006) raises a skeptical eye. General introductions for insurance and finance are Embrechts, P, Lindskog and McNeil (2003) and Cherbini, Lucciano and Vecchiato (2004); see also Nelsen (2006) for a more mathematical treatment or even Joe (1997) for a broader angle on dependence modelling. You will find a good discussion of how Archimedean copulas are simulated in Whelan (2004); see also Frees and Valdez (1998). For applications in insurance consult (among others) Klugman and Parsa (1999), Carrière (2000), Venter (2003), Bäuerle and Grüber (2005) and Escarela and Carrière (2007).

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## 1.10 Exercises

### Section 6.2

**Exercise 6.2.1** The following experiment illustrates the concept of conditional distributions. Let  $a_j = -0.5 + j/10$ , for  $j = 0, 1, \dots, 10$ . **a)** Simulate  $(X_{1i}^*, X_{2i}^*)$  for  $i = 1, \dots, 10000$  from the bivariate normal with  $\xi_1 = \xi_2 = 5\%$ ,  $\sigma_1 = \sigma_2 = 25\%$  and  $\rho = 0.5$ . **b)** For  $j = 1, 2, \dots, 9$ , select those pairs for which  $a_{j-1} < X_{1i}^* \leq a_j$  and compute their mean  $\xi_{|j}$  and standard deviation  $\sigma_{|j}$ . **c)** Plot  $\xi_{|j}$  and  $\sigma_{|j}$  against the mid-points  $(a_{j-1} + a_j)/2$ , and interpret the plots in terms of the conditional density function (1.3). **d)** repeat a), b) and c) with  $\rho = 0.9$  and comment on how the plot changes.

**Exercise 6.2.2** Consider a time series  $\{X_k\}$  of random variables such that the conditional distribution of  $X_k$  given all *preceding* ones are normal with

$$E(X_k | x_{k-1}, x_{k-2}, \dots) = x_{k-1} + \xi \quad \text{and} \quad \text{sd}(X_k | x_{k-1}, x_{k-2}, \dots) = \sigma.$$

Which of the times series models in Chapter 5 is this? see also Exercise 6.5.1.

**Exercise 6.2.3** Let  $Z$  be a positive random variable and suppose  $X$  given  $Z = z$  is normal with

$$E(X|z) = \xi \quad \text{and} \quad \text{sd}(X|z) = \sigma_0 \sqrt{z}.$$

Which model from Chapter 2 is this?

**Exercise 6.2.4** Let the survival probabilities be those used in Section 3.4.; i.e.

$$\log({}_1p_l) = -0.0009 - 0.0000462 \exp(0.090767 \times l).$$

**a)** For  $l = 40$  and  $l = 70$  years, compute  ${}_k p_l$  as given in (1.6) and plot them as a function of  $k$  for  $k = 1, 2, \dots, 30$ .

**Exercise 6.2.5** Let  $N$  be an integer-valued random variable. **a)** Show that

$$\sum_{n=1}^{\infty} \Pr(N \geq n) = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \Pr(N = k) = \sum_{k=1}^{\infty} \sum_{n=1}^k \Pr(N = k) = \sum_{k=1}^{\infty} k \Pr(N = k)$$

so that

$$E(N) = \sum_{n=1}^{\infty} \Pr(N \geq n).$$

Let  $N_l$  be the remaining length of life for somebody having reached  $l$  years. **b)** Use a) to establish that

$$E(N_l) = \sum_{k=1}^{\infty} k p_l.$$

**Exercise 6.2.6** Let  $X$  be an exponentially distributed random variable with density function  $f(x) = \beta^{-1} \exp(-x/\beta)$  for  $x > 0$ . Show that in (1.8)  $f_a(y) = f(y)$ .

**Exercise 6.2.7** Suppose that  $f(x) = \beta^{-1} \alpha / (1 + x/\beta)^{1+\alpha}$  for  $x > 0$  (this is the Pareto density). **a)** Show that

$$f_a(y) = \frac{\alpha/(a + \beta)}{\{1 + y/(a + \beta)\}^{1+\alpha}} \quad \text{if} \quad f_a(y) = \frac{\alpha/\beta}{(1 + y/\beta)^{1+\alpha}}$$

**b)** Interpret this result; i.e. what is the over-threshold distribution if the parent model is Pareto?

**Exercise 6.2.8** A simple (but much less used) alternative to the gamma model to describe variation in the claim intensity  $\mu$  is the log-normal. The model for portfolio claims then reads

$$\mathcal{N}|\mu \sim \text{Poisson}(J\mu T) \quad \mu = \xi \exp(-\frac{1}{2}\sigma^2 + \sigma\varepsilon), \quad \varepsilon \sim N(0, 1).$$

**a)** Show that

$$E(\mu) = \xi \quad \text{and} \quad \text{sd}(\mu) = \xi \sqrt{\exp(\sigma^2) - 1}$$

**b)** Determine  $\sigma$  so that  $\text{sd}(\mu) = 0.1 \times \xi$ . **c)** Run and plot simulations of  $\mathcal{N}$  similar to those in Figure 6.2, using  $\xi = 5\%$  and  $\sigma$  as you determined it in b). Take both  $J = 10^4$  and  $J = 10^6$  as portfolio size. **d)** Any conclusions that differ from those connected to Figure 6.2 in the text?

### Section 6.3

**Exercise 6.3.1** Suppose claim frequency  $\mathcal{N} \sim \text{Poisson}(J\mu T)$ . Show that the formulas (1.19) for mean and variance of the total claim  $\mathcal{X}$  now become

$$E(\mathcal{X}) = J\mu T \xi_z \quad \text{and} \quad \text{sd}(\mathcal{X}) = \sqrt{J\mu T (\xi_z^2 + \sigma_z^2)}.$$

**Exercise 6.3.2** Suppose claim intensities  $\mu$  vary independently from one policy holder to another so that

$$\text{sd}(\mathcal{X}) = \sqrt{J\xi_\mu (\sigma_z^2 + \xi_z^2)} \times \sqrt{1 + \delta} \quad \text{where} \quad \delta = \frac{\sigma_z^2}{\sigma_z^2 + \xi_z^2} \times \frac{\sigma_\mu^2}{\xi_\mu^2};$$

see (1.21) and (1.22). **a)** Show that  $\delta \leq \sigma_\mu^2 / \xi_\mu$ . **b)** Argue that the case  $\xi_\mu = 5\%$  and  $\sigma_\mu = 5\%$  would exhibit huge variability in claim intensity. **c)** Use a) to show that  $\sqrt{1 + \delta} \leq 1.023$  under the specification in b) and argue that the *added* portfolio risk due to the heterogeneity in  $\mu$  accounts for no more than 2.3% of the total value of  $\text{sd}(\mathcal{X})$ . This strongly suggests that at portfolio level the impact of risk heterogeneity usually can be ignored. The next exercise treats a related case where the conclusion is very different.

**Exercise 6.3.3** As in Exercise 6.3.2 assume that  $\mu$  varies randomly, but now as a common parameter for all policy holders. **a)** Go back to (1.21) and explain why the factor

$$\sqrt{1 + J\delta} = \sqrt{1 + J \frac{\sigma_z^2}{\sigma_z^2 + \xi_z^2} \times \frac{\sigma_\mu^2}{\xi_\mu}};$$

accounts for the effect of the  $\mu$ -variability on  $\text{sd}(\mathcal{S})$ . **b)** Compute it when

$$\xi = 5\%, \quad \sigma_\mu = 1\%, \quad \frac{\sigma_z}{\xi_z} = 0.5 \quad J = 100000.$$

Any comments? **c)** Show that the factor  $\sqrt{1 + J\delta}$  increases with the ratio  $\sigma_z/\xi_z$ . Is the impact of  $\mu$ -variability larger or smaller for heavy-tailed claim size distributions than for lighter ones?

**Exercise 6.3.4** Suppose that  $X_1, \dots, X_J$  are conditionally independent and identically distributed given a common factor  $\omega$ . **a)** Explain that (1.15) now becomes

$$E(\mathcal{X}) = JE\{\xi(\omega)\} \quad \text{and} \quad \text{sd}(\mathcal{X}) = J\sqrt{\text{var}\{\xi(\omega)\} + E\{\sigma^2(\omega)\}/J},$$

where  $\xi(\omega)$  and  $\text{sd}(\omega)$  are the conditional mean and standard deviation. **b)** Show that

$$\frac{\text{sd}(\mathcal{X})}{E(\mathcal{X})} \rightarrow \frac{\text{sd}\{\xi(\omega)\}}{E\{\xi(\omega)\}} \quad \text{as} \quad J \rightarrow \infty.$$

**c)** What does this tell you about risk diversification models with common factors? This result throws light on the conclusion in Exercise 6.3.3

**Exercise 6.3.5** Let  $\mathcal{N}^*$  be a simulation of a Poisson claim frequency  $\mathcal{N}$  where the intensity  $\mu$  has been estimated as  $\hat{\mu}$ . If  $T = 1$ , this means that  $\mathcal{N}^*|\hat{\mu}$  is  $\text{Poisson}(J\hat{\mu})$ . **a)** Use the double rules to prove that

$$E(\mathcal{N}^*) = JE(\hat{\mu}) \quad \text{and} \quad \text{var}(\mathcal{N}^*) = JE(\hat{\mu}) + J^2\text{var}(\hat{\mu}).$$

**b)** Recall that  $E(\mathcal{N}) = \text{var}(\mathcal{N})$  for a Poisson variable  $\mathcal{N}$  whereas  $E(\mathcal{N}^*) < \text{var}(\mathcal{N}^*)$ . What causes the difference? Integration of random error from different sources is discussed in Chapter 7.

**Exercise 6.3.6** Suppose  $X^* \sim N(\hat{\xi}, \hat{\sigma})$  where  $\hat{\xi}$  and  $\hat{\sigma}$  are estimates of  $\xi$  and  $\sigma$  from historical data. This should be interpreted as  $X^*$  having a conditional distribution given the estimates. **a)** Argue, using the double rules, that

$$E(X^*) = E(\hat{\xi}) \quad \text{and} \quad \text{var}(X^*) = E(\hat{\sigma}^2) + \text{var}(\hat{\xi})$$

**b)** Suppose that  $\text{var}(\hat{\xi}) = \sigma^2/n$  and that  $E(\hat{\sigma}^2) = \sigma^2$  (which you recognize as a standard situation with  $n$  historical observations). Show that

$$\text{var}(X^*) = \sigma^2\left(1 + \frac{1}{n}\right).$$

**Exercise 6.3.7** The double rule for variances can be extended to a version for covariances. Let

$$\xi_1(\mathbf{x}) = E(Y_1|\mathbf{x}), \quad \xi_2(\mathbf{x}) = E(Y_2|\mathbf{x}) \quad \text{and} \quad \sigma_{12}(\mathbf{x}) = \text{cov}(Y_1, Y_2|\mathbf{x})$$

for random variables  $Y_1, Y_2$  conditioned on  $\mathbf{X} = \mathbf{x}$ . Then

$$\text{cov}(Y_1, Y_2) = \text{cov}\{\xi_2(\mathbf{X}), \xi_1(\mathbf{X})\} + E\{\sigma_{12}(\mathbf{X})\};$$



see Appendix A. Use this to find the covariances between returns  $R_1$  and  $R_2$  satisfying the stochastic volatility model in Section 2.4; i.e

$$R_1 = \xi_1 + \sigma_{01}\sqrt{Z}\varepsilon_1 \quad \text{and} \quad R_2 = \xi_2 + \sigma_{02}\sqrt{Z}\varepsilon_2$$

where  $\varepsilon_1$ ,  $\varepsilon_2$  and  $Z$  are independent and the two former are  $N(0, 1)$  with correlation  $\rho$ .

### Section 6.4

**Exercise 6.4.1** Let  $X_1$  and  $X_2$  be dependent normal variables with expectations  $\xi_1$  and  $\xi_2$ , standard deviations  $\sigma_1$  and  $\sigma_2$  and correlation  $\rho$ . **a)** Use (1.3) to justify that

$$\hat{X}_2 = \xi_2 + \rho\sigma_2 \frac{x_1 - \xi_1}{\sigma_1} \quad \text{for} \quad X_1 = x_1$$

is the most accurate prediction of  $X_2$  if  $X_1$  is observed. **b)** Show that

$$\frac{\text{sd}(\hat{X}_2|x_1)}{\text{sd}(X_2)} = \sqrt{1 - \rho^2}.$$

**c)** By how much is the uncertainty in  $X_2$  reduced by knowing  $X_1$  if  $\rho = 0.3, 0.7$  and  $0.9$ ? Argue that  $\rho$  should from this viewpoint be interpreted through  $\rho^2$ , as claimed in Section 5.2.

**Exercise 6.4.2** Claim intensities  $\mu$  in automobile insurance depends on factors such as age and sex. Consider a female driver of age  $x$ . A standard way to formulate the link between  $x$  and  $\mu$  goes through the conditional mean  $E(N|x)$ , where  $N$  is claim frequency. One possibility is

$$\mu = \mu_0 e^{-\beta(x-x_0)},$$

where  $x_0$  is the starting age for drivers and  $\mu_0$  and  $\beta_0$  are parameters. **a)** What is the meaning of the parameters  $\mu_0$  and  $\beta$ ? **b)** Determine them so that  $\mu = 10\%$  at age 18 and  $5\%$  at age 60 and plot the relationship between  $x$  and  $\mu$ . In practice a more complex relationship is often used; see Chapter 8.

**Exercise 6.4.3** Let

$$\xi = 5\%, \quad a = 0.5 \quad \sigma = 0.016, \quad r_0 = 2\%$$

in the Vasicek model for interest rates. **a)** Write down predictions for the rate of interest  $r_k$  at  $k = 1, 2, 5$  and  $k = 10$  years, using (??). **b)** What is standard deviation of the prediction error? Use (??) and compare the assessment for  $k = 1$  and  $k = 5$  with those in Section 6.4 coming from a related (but different) set of parameters.

**Exercise 6.4.4** Consider the Black-Karinsinski model defined in Section 5.7 under which

$$r_k = \xi \exp\left(-\frac{1}{2}\sigma_x^2 + X_k\right) \quad \text{where} \quad \sigma_x = \frac{\sigma}{\sqrt{1-a^2}}, \quad X_k = aX_{k-1} + \sigma\varepsilon_k.$$

Here  $\varepsilon_1, \varepsilon_2, \dots$  are all independent and  $N(0, 1)$ . **a)** If  $r_0$  is the current rate of interest observed in the market, argue that

$$\hat{r}_k = \xi \exp\left(-\frac{1}{2}\sigma_x^2 + a^k x_0\right) \quad \text{where} \quad x_0 = \log\left(\frac{r_0}{\xi} + \frac{1}{2}\sigma_x^2\right)$$

is a prediction of the future rate  $r_k$ . **b)** Make the prediction for  $k = 1, 2, 5$  and  $k = 10$  years as in the preceding exercise and use the same parameters as there. Compare forecasts under the two models. This example will be examined further in Exercise 7.?

**Exercise 6.4.5** Algorithm 6.1 dealt with the forward rate of interest under the Vasicek model. **a)** Modify it so that it applies to the Black-Karistinski model [Hint: You replace Line 3 with parts of Algorithm 5.4]. **b)** ???

### Section 6.5

**Exercise 6.5.1** Suppose the time series  $\{X_k\}$  is a Gaussian Markov process for which

$$X_k | X_{k-1} = x \sim N(ax, \sigma).$$

Which model from Chapter 5 is this?

**Exercise 6.5.2** Suppose  $X_1, \dots, X_J$  are conditionally normal given  $Z = z$  with expectations  $\xi_i$  and variance/covariances  $\sigma_{ij}z$ . **a)** Which model from earlier chapters is this? **b)** Do the *correlations* depend on  $z$ ? Which model from Chapter 5 is this?

**Exercise 6.5.3** Consider Algorithm 6.2, the skeleton for Markov sampling. **a)** Modify it to deal with *common factors*; i.e explain that  $X_k^*$  on Line 3 now is drawn from  $f(x_k | X_1^*)$ .

**Exercise 6.5.4** This exercise shows how a stochastic volatility model for *log*-returns are sampled by means of the preceding exercise. Suppose

$$Z = \exp(-\frac{1}{2}\tau^2 + \tau\varepsilon), \quad \varepsilon \sim N(0, 1)$$

is log-normal and that

$$X_1 = \log(1 + R_1), \quad X_2 = \log(1 + R_2), \quad X_3 = \log(1 + R_3)$$

are conditionally normal with expectations  $\xi_1, \xi_2, \xi_3$ , volatilities  $\sigma_{01}\sqrt{z}, \sigma_{02}\sqrt{z}, \sigma_{03}\sqrt{z}$  and correlations  $\rho_{ij}$ .

**a)** Explain how the log-returns are sampled. **b)** Carry out the sampling 1000 times when

$$\xi_1 = \xi_2 = \xi_3 = 5\%, \quad \sigma_{01} = \sigma_{02} = \sigma_{03} = 0.2, \quad \text{all } \rho_{ij} = 0.5 \quad \text{and} \quad \tau = 0.5.$$

**c)** Use b) to compute the 5% lower percentile of the portfolio with equal weights on the three risky assets.

**Exercise 6.5.5** Stochastic volatility in finance is in reality a *dynamic* phenomenon where the random variable  $Z = Z_k$  being responsible are correlated in time. The first model proposed to deal with this is known as **ARCH**<sup>3</sup> and can be formulated as follows:

$$R_k = \xi + \sigma_0 \sqrt{Z_k} \varepsilon_k \quad \text{where} \quad Z_k = \sqrt{1 + \theta(R_{k-1} - \xi)^2}$$

where  $\varepsilon_1, \varepsilon_2, \dots$  are independent and  $N(0, 1)$ . **a)** Argue that returns deviating strongly from the mean  $\xi$  makes volatility go up next time. **b)** Why is this a Markov model for the series  $\{R_k\}$ ? **c)** Simulate the model and plot the against time  $k$  for  $k = 1, \dots, 30$  when

$$\xi = 5\%, \quad \sigma_0 = 0.2 \quad \text{and} \quad \theta = 0.2 \quad \text{starting at} \quad R_0 = 5\%.$$

These are annual parameters. Plot ten different scenarios.

---

<sup>3</sup>ARCH stands for **autoregressive, conditional, heteroscedastic**.

**Exercise 6.5.6** An alternative to ARCH of the preceding is to use the Black-Karaisinski model from Section 5.7 for  $\{Z_k\}$ , i.e to take

$$Z_k = \exp\left(-\frac{1}{2}\tau_y^2 + \tau_y Y_k\right) \quad \text{where} \quad \tau_y = \frac{\tau}{\sqrt{1-a^2}}, \quad Y_k = aY_{k-1} + \tau\eta_k.$$

Here both sequences  $\eta_1, \eta_2, \dots$  and  $\varepsilon_1, \varepsilon_2, \dots$  are independent  $N(0, 1)$  and independent from each other. **a)** Simulate and plot ten realisations of this model under the same conditions as in the previous exercise using  $a = 0.6$  and  $\tau = 0.1$ . **b)** Is there in behaviour a principal difference from the ARCH model. This model type, though less used than the former (and, especially its extensions) is drawing much interest as this book is being written (2004).

**Exercise 6.5.7** The multinomial model illustrates the factorization (1.29). Start by noting that  $N_0 \sim \text{Binomial}(n, q_0)$ . **a)** Then argue that

$$N_1 | n_0 \sim \text{Binomial}(n - n_0, \tilde{q}_1) \quad \text{where} \quad \tilde{q}_1 = \frac{q_1}{1 - q_0}.$$

[Hint: From  $n$  trials originally, subtract those ( $= n_0$ ) with no delay. Among the *remaining*  $n - n_0$  trials the likelihood is  $\tilde{q}_1$  for delay exactly one year.]. Suppose a binomial sampling procedure is available. **b)** Justify that  $(N_0, N_1)$  can be sampled through

$$N_0^* \sim \text{Binomial}(n, q_0) \quad \text{and} \quad N_1^* \sim \text{Binomial}(n - N_0^*, \tilde{q}_1)$$

The next step is

$$N_2^* | n_0, n_1 \sim \text{Binomial}(n - n_0 - n_1, \tilde{q}_2) \quad \text{where} \quad \tilde{q}_2 = \frac{q_2}{1 - q_0 - q_1}.$$

**c)** Explain why the general case can be run as follows:

**Algorithm 6.6 Multinomial sampling**

```

0 Input  $n$  and  $q_0, \dots, q_K$ 
1  $S^* \leftarrow 0, \quad d \leftarrow 1$ 
2 For  $k = 1, \dots, K - 1$  do
3     Draw  $N_k^* \sim \text{Binomial}(n - S^*, p_k^*/d)$ 
4      $S^* \leftarrow S^* + N_k^*, \quad d \leftarrow d - p_k$ 
5 Return  $N_1^*, \dots, N_{K-1}^*$  and  $N_K^* \leftarrow n - S^*$ .
```

This is inefficient for large  $K$ , but tolerable for delay. **d)** Run the algorithm 10000 times when

$$K = 4, \quad q_0 = 0.1, \quad q_1 = 0.3, \quad q_2 = 0.25, \quad q_3 = 0.2, \quad q_4 = 0.15.$$

and compare relative frequencies with the underlying probabilities.

**Exercise 6.5.8** We know from the preceding exercise that

$$\Pr(N_0 = n_0) = \frac{n!}{n_0!(n - n_0)!} q_0^{n_0} (1 - q_0)^{n - n_0}$$

and that

$$\Pr(N_1 = n_1 | n_0) = \frac{(n - n_0)!}{n_1!(n - n_0 - n_1)!} \tilde{q}_1^{n_1} (1 - \tilde{q}_1)^{n - n_0 - n_1}.$$

Multiply the two probabilities together and verify that

$$\Pr(N_0 = n_0, N_1 = n_1) = \frac{n!}{n_0!n_1!(n - n_0 - n_1)!} q_0^{n_0} q_1^{n_1} (1 - q_0 - q_1)^{n - n_0 - n_1}.$$

This is the multinomial density function (1.35) for  $K = 2$  (note that  $N_2 = n - n_0 - n_1$  is fixed by the two first). The general case is established by continuing in this way.

### Section 6.6

**Exercise 6.6.1** Consider a Markov chain  $\{C_k\}$  running over the three states “active”, “disabled” and “dead” with  $p^{a|d}$  and  $p^{d|a}$  as probabilities of going from “disabled” to “active” and “active” to “disabled” and with probability of survival  ${}_1p_l$  at age  $l$ . **a)** Argue, using conditioning, that the probability at age  $l$  of remaining active must be  ${}_1p_l(1 - p^{a|d})$ . **b)** Fill out the rest of the table of transition probabilities at page ?, using the same reasoning. **c)** Verify that the row sums are equal to one. **d)** What does the matrix become when

$$\log({}_1p_l) = -0.0009 - 0.0000462 \exp(0.090767 \times l), \quad p^{d|a} = 0.7\%, \quad p^{a|d} = 0.35\% \quad ?$$

**Exercise 6.6.2** Let the three states of the preceding exercise be labeled 0 (for “active”), 1 (“disabled”) and 2 (“dead”) and let  $p_l(i|j)$  be their transition probabilities at age  $l$ . **a)** Implement Algorithm 6.2 for the model of the preceding exercise. For example, argue that the following recursive step can be used on Line 3:

Draw  $U^* \sim \text{uniform}$  and  $l \leftarrow l + 1$   
 If  $U^* < p_l(0|C_{k-1}^*)$  then  $C_k^* \leftarrow 0$   
 else if  $U^* < p_l(0|C_{k-1}^*) + p_l(2|C_{k-1}^*)$  then  $C_k^* \leftarrow 1$   
 else  $C_k^* \leftarrow 2$  and stop.

**b)** Run the algorithm ten times with the model of Exercise 6.1, each time starting at age  $l = 30$  years and plotting the the simulated scenarios 50 years ahead. **c)** Change the model unrealistically!) to  $p^{d|a} = 0.4$  and  $p^{a|d} = 0.20$ , re-compute the transition matrix and re-run the simulations to see different patterns.

**Exercise 6.6.3** The expected remaining life-time at age  $l$  was derived in Exercise 6.2.5 as

$$E(N_l) = \sum_{k=1}^{\infty} {}_k p_l \quad \text{where} \quad {}_k p_l = {}_1 p_{l+k-1} \times \cdots \times {}_1 p_l$$

Consider the recursion

$$P \leftarrow {}_1 p_l \times P, \quad E \leftarrow E + P, \quad l \leftarrow l + 1$$

starting at  $P = 1$ ,  $E = 0$ . **a)** Argue that it yields  $E = E(N_l)$  at the end. **b)** Implement the recursion, compute  $E(N_l)$  for  $l = 20, 25, 30, \dots$  up to  $l = 70$  for the survival model in Exercise 6.6.1. **c)** Plot the computed sequence against  $l$  and explain why it is decreasing.

**Exercise 6.6.4** One of the issues with potentially huge impact on the business of life and pension insurance is the fact that in most countries length of life is steadily prolonged. Suppose we want to change our current survival model into a related one in order to get a rough picture of the economic consequences. A simple way is to introduce

$${}_1 \tilde{p}_l = \frac{\theta {}_1 p_l}{\theta {}_1 p_l + (1 - {}_1 p_l)},$$

where  $\theta$  is a parameter. **a)** Show that the new survival probability  ${}_1 \tilde{p}_l$  decreases with age  $l$  if the original model had that property. **b)** Also show that it increases with  $\theta$  and coincides with the old one if  $\theta = 1$ . **c)**

Let  ${}_1p_l$  be the model of Exercise 6.6.1. Use the program of Exercise 6.6.3 to compute the average, remaining length of life for a twenty-year for  $\theta = 1.0, 1.1, 1.2, \dots$  up to  $\theta = 2$  and plot the relationship. **d)** Use the plot to find out roughly how large  $\theta$  must be for the average age to be five years more than it was.

**Exercise 6.6.5** Consider a policy holder entering a pension scheme at time  $k = 0$  at age  $l_0$  and making a contribution (premium) at the start of each period. From age  $l_r$  he draws benefit  $\zeta$  (also at the start of each period) which lasts until the end of his life. There is a fixed rate of interest  $r$ . Let  $V_k$  be the value of his account after time  $k$ . **a)** Argue that *as long as the member stays alive*, his account develops according to the recursion

$$\begin{aligned} V_k &= (1+r)V_{k-1} + \pi, & k < l_r - l_0 \\ &= (1+r)V_{k-1} - \zeta, & k \geq l_r - l_0 \end{aligned} \quad \text{starting at} \quad V_0 = \pi.$$

**a)** Write a program that allows the account to build up and then decline, the scheme terminating upon death. **b)** Simulate and plot the movements of the account against time when

$$l_0 = 30, \quad l_r = 65, \quad \pi = ? \quad \zeta = ? \quad r = 3\%$$

and the survival model is the one in Exercise 6.6.1. **c)** Repeat b) nine times to judge variability. **d)** If you apply the program ?? on ?? under the Cambridge website you can see how much the status of the account varies when the scheme stops at the death of the policy holder. The plot is based on 10000 simulations under the conditions above.

## Section 6.7

**Exercise 6.7.1 a)** Show that when  $U_1$  is uniform and  $U_2 = U_1$ , then

$$H^{\text{ma}}(u_1, u_2) = \min(u_1, u_2), \quad 0 \leq u_1, u_2 \leq 1.$$

is the copula for the pair  $(U_1, U_2)$ . **b)** Prove the first half of the **Frechet-Hoeffding** inequality; i.e.

$$H(u_1, u_2) \leq \min(u_1, u_2), \quad 0 \leq u_1, u_2 \leq 1.$$

for an *arbitrary* copula  $H(u_1, u_2)$ . This shows that  $H^{\text{ma}}(u_1, u_2)$  is a *maximum* copula.

**Exercise 6.7.2** The second half of the Frechet-Hoeffding inequality apply to antitetic variables, introduced in Chapter 4 to produce negatively correlated random variables. Let  $U_1$  be uniform and  $U_2 = 1 - U_1$ . **a)** Show that the copula is

$$H^{\text{mi}}(u_1, u_2) = \max(u_1 + u_2 - 1, 0),$$

For an *arbitrary* copula  $H(u_1, u_2)$  fix  $u_2$  and define the function

$$G(u_1) = H(u_1, u_2) - (u_1 + u_2 - 1).$$

**b)** Show that  $G(1) = 0$  and that  $G'(u_1) < 0$  [Hint: Recall (??)]. **c)** Explain that this means that  $G(u_1) > 0$  so that

$$H(u_1, u_2) \geq \max(u_1 + u_2 - 1, 0),$$

and the antitetic pair defines a *minimum* copula.

**Exercise 6.7.3** We might use the the preceding two exercises used to check whether a family of copulas capture the entire range of dependency. **a)** Show that the Clayton copula (1.42) coincides with the

minimum (antitetic) copula when  $\theta = -1$  and **b)** that it converges to the maximum copula as  $\theta \rightarrow \infty$  [Hint: Utilize that the Clayton copula for  $\theta > 0$  may be written

$$\exp\{L(\theta)\} \quad \text{where} \quad L(\theta) = \log(u_1^{-\theta} + u_2^{-\theta} - 1)/\theta$$

and apply l'Hôpital's rule to  $L(\theta)$ .]

**Exercise 6.7.4** Show that the Clayton copula (1.40) approaches the independent copula as  $\theta \rightarrow 0$  [Hint: Use the argument of the preceding exercise.].

**Exercise 6.7.5** One of the most popular copula models is the **Gumbel** family for which

$$H(u_1, u_2) = \exp\{-Q(u_1, u_2)\} \quad \text{where} \quad Q(u_1, u_2) = \{(-\log u_1)^\theta + (-\log u_2)^\theta\}^{1/\theta}.$$

**a)** Verify that this is a valid copula when  $\theta \geq 1$  by checking (1.39). **b)** Which model corresponds to the special case  $\theta = 1$ ? **c)** Which model appears as  $\theta \rightarrow \infty$ ? [Hint: One way is to utilize that

$$Q(u_1, u_2) = \exp\{L(\theta)\} \quad \text{where} \quad L(\theta) = \log\{(-\log u_1)^\theta + (-\log u_2)^\theta\}/\theta.$$

Apply l'Hôpital's rule to  $L(\theta)$ .]

**Exercise 6.7.6** Show that the Gumbel family of the preceding exercise belongs to the Archimedean class with generator  $\phi(u) = (-\log u)^\theta$ .

**Exercise 6.7.7** Let  $H(u_1, u_2) = \phi^{-1}\{\phi(u_1) + \phi(u_2)\}$  be a general Archimedean copula where it is assumed that the generator  $\phi(u)$  decreases continuously from infinity at  $u = 0$  to zero at  $u = 1$ . **a)** Calculate  $H(u_1, 0)$  and  $H(0, u_2)$  and verify that the first line in (1.39) is satisfied. **b)** Same question for the second line and  $H(u_1, 1)$  and  $H(1, u_2)$ .

**Exercise 6.7.8** Consider the Archimedean copula based on the generator  $\phi(u) = (1 - u)^3$ . Derive an expression  $H(u_1, u_2)$  and b) show that it is zero whenever  $u_2 \leq \{1 - (1 - u_1)^3\}^{1/3}$ .

**Exercise 6.7.9** Suppose an Archimedean copula is based on a generator for which  $\phi(0)$  is *finite*. Use the fact that the generator is strictly decreasing to explain that the copula  $H(u_1, u_2)$  is positive if and only if

$$\phi(u_1) + \phi(u_2) < \phi(0) \quad \text{true if and only if} \quad u_2 > \phi^{-1}\{\phi(0) - \phi(u_1)\},$$

and the lower bound on  $u_2$  is normally positive. We rarely want models with this property.

**Exercise 6.7.10** Consider the Clayton copula (??) with positive  $\theta$  with generator  $\phi(u) = (u^{-\theta} - 1)/\theta$ . Show that the key part of Algorithm 6.4 (lines 2 and 3) is solved by

$$U_2^* = \{1 + (U_1^*)^{-\theta}[(V^*)^{-\theta/(1+\theta)} - 1]\}^{-1/\theta}.$$