Introduction and Techniques
in Financial Mathematics
UiO-STK4510
Autumn 2015
Teacher: *S. Ortiz-Latorre*

Exercises
List 1

Probability and Measure Theory

---

1. Is the family $\mathcal{F}$ consisting of all finite subsets of Ω and their complements always a σ-algebra?

2. Is the family $\mathcal{F}$ consisting of all countable subsets of Ω and their complements always a σ-algebra?

3. Let $\mathcal{F}$ be a σ-algebra on $\Omega = [0,1]$ such that $[\frac{1}{n+1}, \frac{1}{n}] \in \mathcal{F}$ for $n \in \mathbb{N}$. Show that:

   (a) $\{0\} \in \mathcal{F}$.
   (b) $\{\frac{1}{n} : n = 2, 3, 4...\} \in \mathcal{F}$.
   (c) $(\frac{1}{n}, 1] \in \mathcal{F}$ for all $n \in \mathbb{N}$.
   (d) $(0, \frac{1}{n}] \in \mathcal{F}$ for all $n \in \mathbb{N}$.

4. Let $\mathcal{F}$ be a σ-algebra. Demonstrate that if $\{A_n\}_{n\geq 1} \subset \mathcal{F}$, then
$$\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}.$$

5. Let Ω and $\tilde{\Omega}$ be arbitrary sets and $X : \tilde{\Omega} \to \Omega$ be any mapping. Show that if $\mathcal{F}$ is a σ-algebra on Ω, then $\tilde{\mathcal{F}} = \{X^{-1}(A) : A \in \mathcal{F}\}$ is a σ-algebra on $\tilde{\Omega}$.

6. Let $\mathcal{F}$ be a σ-algebra on Ω and let $A \subset \mathcal{F}$. Show that $\mathcal{F}_A := \{A \cap B : B \in \mathcal{F}\}$ is a σ-algebra on Ω.

7. Show that if $\{\mathcal{F}_i\}_{i\in I}$ is any collection of σ-algebra defined on the same set Ω, then their intersection $\cap_{i\in I}\mathcal{F}_i$ is also a σ-algebra on Ω.

8. Let $(\Omega, \mathcal{F})$ be a measurable space. We define the *upper limit* or *limit superior* of a sequence of sets $\{A_n\}_{n\geq 1} \subset \Omega$ by
$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k,$$
and the *lower limit* or *limit inferior* by
$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k.$$
Show that if $\{A_n\}_{n\geq 1} \subset \mathcal{F}$, then
$$\limsup_{n\to\infty} A_n \in \mathcal{F}, \quad \liminf_{n\to\infty} A_n \in \mathcal{F}.$$
Show that
$$\limsup_{n\to\infty} A_n = \{A_n \text{ occurs for infinitely many } n\},$$
$$\liminf_{n\to\infty} A_n = \{A_n \text{ occurs for all but finitely many } n\}.$$
and that $\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$.

9. Let $\Omega = \mathbb{N}$, and let $\mathcal{F}$ be the family of all subsets of Ω. Put $P(\{i\}) = \alpha_i, i = 1, 2, ...$ Extend P to a probability measure defined on $\mathcal{F}$. What conditions have to be imposed on the numbers α_i? Can they all be chosen the same?

10. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Show that:

    (a) P is mononotone, that is, if $A, B \in F$ satisfies $B \subset A$ then $P(B) \leq P(A)$.

    (b) P is finitely subadditive, that is, for any $\{A_n\}_{n=1,...,N} \subset \mathcal{F}$ where $N \in \mathbb{N}$, one has that

$$P\left(\bigcup_{n=1}^{N} A_n\right) \leq \sum_{n=1}^{N} P(A_n).$$

11. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{A_n\}_{n\geq 1} \subset \mathcal{F}$ be a sequence of events. Show that:

    (a) If $A_1 \subset A_2 \subset A_3 \subset \cdots$ (that is, the sequence $\{A_n\}_{n\geq 1}$ is increasing) then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

    (b) If $A_1 \supset A_2 \supset A_3 \supset \cdots$ (that is, the sequence $\{A_n\}_{n\geq 1}$ is decreasing) then

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

    (c) We have that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n).$$

    (d) If $P(A_n) = 0, n \geq 1$, then $P\left(\bigcup_{n=1}^{\infty} A_n\right) = 0$.

    (e) If $P(A_n) = 1, n \geq 1$, then $P\left(\bigcap_{n=1}^{\infty} A_n\right) = 1$.

    (f) *First Borel-Cantelli lemma.* If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\limsup_{n\to\infty} A_n) = 0$.

12. Show that if X is a constant function, then it is a random variable with respect to any σ-algebra.

13. Le X be a discrete random variable, that is, X only can take a countable number of values. Describe how is the σ-algebra $\sigma(X)$. Describe how are the $\sigma(X)$ measurable functions.

14. Let $\Omega = \{1, 2, 3, 4\}$ and $\mathcal{F} = \{\varnothing, \Omega, \{1\}, \{2, 3, 4\}\}$. Is $X(\omega) = 1 + \omega$ a random variable with respect to the σ-algebra $\mathcal{F}$? If not give an example of a non-constant function which is.

15. Let X be a random variable and $Y = X^2$. Show that Y is also a random variable. Is X measurable with respect to $\sigma(Y)$?

16. Let X be a random variable such that $X \geq 0, P$-a.s.. Prove that if $\mathbb{E}[X] = 0$ then $X = 0, P$-a.s..

17. Let $X \in L^1(\Omega, \mathcal{F}, P)$. Prove that if $\mathbb{E}[X\mathbf{1}_A] = 0, \forall A \in \mathcal{F}$ then $X = 0, P$-a.s..

18. Let P and Q be two probability measures defined on the same measurable space $(\Omega, \mathcal{F})$. Show that $P \sim Q$ iff $Q \ll P$ and $P(\frac{dQ}{dP} = 0) = 0$. What is the relationship between $\frac{dQ}{dP}$ and $\frac{dP}{dQ}$?

19. Let Q be the distribution on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by

$$Q((a,b]) = \int_a^b \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz, \quad a < b \in \mathbb{R}.$$

Construct a probability space and a random variable Z defined on it such that the law of Z is given by Q. We say that the random variable Z is a standard Normal (or Gaussian) random variable. Compute its mean and variance. Let $X = \sigma Z + \mu$, where $\mu \in \mathbb{R}$ and $\sigma > 0$. Check that X is a random variable, find its distribution function and compute $\mathbb{E}[X]$ and $\mathrm{Var}[X]$. We say that X is normal random variable with mean μ and variance σ^2 and we denote it by $X \sim \mathcal{N}(\mu, \sigma^2)$.

20. Let $X \sim \mathcal{N}(\mu, \sigma^2)$ defined on some probability space $(\Omega, \mathcal{F}, P)$. Compute $\psi(\theta) = \mathbb{E}[e^{\theta X}]$ for $\theta \in \mathbb{R}$. Define $L(X;\theta) := e^{\theta X}/\psi(\theta)$ and show that $Q_\theta(A) = \mathbb{E}[L(X;\theta)\mathbf{1}_A], A \in \mathcal{F}$ defines a probability measure on $(\Omega, \mathcal{F})$. Show that $Q_\theta \ll P$. Find the law of X under Q_θ, that is, the law of X as a random variable defined on $(\Omega, \mathcal{F}, Q_\theta)$.

21. Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Define $Y = \exp(X)$. Show that Y is a random variable. The law of Y is called lognormal with mean μ and variance σ^2, show that $P_Y \ll \lambda$ and find its density function $\frac{dP_Y}{d\lambda}$. Compute $\mathbb{E}[Y^n], n \geq 1$ and $\mathrm{Var}[Y]$.

22. (**Markov inequality**). Let X be a random variable and $f : \mathbb{R} \to \mathbb{R}_+$ an increasing, Borel measurable function. Let $a \in \mathbb{R}$ such that $f(a) > 0$. Prove that

$$P(X \geq a) \leq \frac{\mathbb{E}[f(X)]}{f(a)}.$$

23. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $I \subset \mathbb{R}$ be an open interval and $t_0 \in I$. Let $\{X_t\}_{t \in I}$ be a family of random variables satisfying:

    (a) $X_\cdot(\omega) : I \to \mathbb{R}$ is differentiable on I, P-a.s.

    (b) There exists a random variable Y such that $|X_t| + \left|\frac{d}{dt} X_t\right| \leq Y, P$-a.s.,$\forall t \in I$.

    Show that the function

$$\begin{array}{cccc} F: & I & \to & \mathbb{R} \\ & t & \mapsto & \mathbb{E}[X_t] \end{array},$$

is well defined, differentiable at t_0 with $F'(t_0) = \mathbb{E}[\frac{d}{dt} X_t\big|_{t=t_0}]$.

24. Let X and Y be two independent random variables that are absolutely continuous with respect to λ. Show that $X + Y$ is absolutely continuous with respect to λ and find its density.

25. Let X and Y be two independent and identically distributed (i.i.d.) random variables with law $\mathcal{N}(\mu, \sigma^2)$. Find the density of $(U, V) = (X + Y, X - Y)$. Under which conditions on μ and σ^2 are U and V independent?

26. Let X and Y be two i.i.d. random variables with law $\mathcal{N}(0, \sigma^2)$ independent random variables. Find the density of $(U, V) = (\sqrt{X^2 + Y^2}, X/Y)$, where V is defined as zero if $Y = 0$. Are U and V independent?

27. Let (X, Y) be a Gaussian random vector with mean (μ_X, μ_Y) and covariance matrix

$$Q = \begin{pmatrix} \mathrm{Var}[X] & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(X,Y) & \mathrm{Var}[Y] \end{pmatrix} = \begin{pmatrix} \sigma_X^2 & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(X,Y) & \sigma_Y^2 \end{pmatrix},$$

with $\sigma_X \sigma_Y > 0$. Let ρ be the correlation coeficient

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]}}.$$

Show that if $|\rho| < 1$ the density of (X, Y) exists and it is equal to

$$f_{X,Y}(x,y) = \frac{\exp\left\{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right\}\right\}}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}.$$

Show that if $|\rho| = 1$, then the density of (X, Y) does not exist. In the case $|\rho| < 1$ show that the law of Y conditioned to X has density an it is equal to the density of a univariate Gaussian r.v. with mean $\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X)$ and variance $\sigma_Y^2(1 - \rho^2)$.

28. Let $Y \sim \mathcal{N}(0, 1)$ and for $a > 0$ define $Z = Y\mathbf{1}_{\{|Y|\leq a\}} - Y\mathbf{1}_{\{|Y|>a\}}$. Show that Z is Gaussian r.v. but (Y, Z) is not multivariate Gaussian.

29. Let $Y \in L^2(\Omega, \mathcal{F}, P)$. Show that if $\mathbb{E}[Y|X] = X$ and $\mathbb{E}[Y^2|X] = X^2$ then $X = Y$, P-a.s..

30. Prove that if $X, Y \in L^1(\Omega, \mathcal{F}, P)$ and $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$ then:

    (a) $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]] = \mathbb{E}[X]$.

    (b) If $X \geq 0 \Rightarrow \mathbb{E}[X|\mathcal{G}] \geq 0$, P-a.s.

    (c) If $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{G}$. Then,

    $$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}] = \mathbb{E}[X|\mathcal{H}], P\text{-a.s.}$$

    (d) Assume that $XY \in L^1(\Omega, \mathcal{F}, P)$ and that Y is $\mathcal{G}$-measurable. Then, $\mathbb{E}[XY|\mathcal{G}] = Y\mathbb{E}[X|\mathcal{G}]$, P-a.s.. In particular, if X is $\mathcal{G}$-measurable then $\mathbb{E}[X|\mathcal{G}] = X$, P-a.s..

31. Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $\{A_n\}_{n\geq 1} \subset \mathcal{F}$ be a partition of Ω, (i.e., $\{A_n\}_{n\geq 1}$ are pairwise disjoint and its union is Ω) and let $\mathcal{G}$ be the σ-algebra generated by $\{A_n\}_{n\geq 1}$. Assume that $X \in L^1(\Omega, \mathcal{F}, P)$ and $P(A_n) > 0$. Show that

    $$\mathbb{E}[X|\mathcal{G}] = \sum_{n\geq 1} \frac{\mathbb{E}[X\mathbf{1}_{A_n}]}{P(A_n)}\mathbf{1}_{A_n}, \quad P\text{-a.s.}$$

32. In the setup of Exercise 20, let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Show that for any $Y \in L^1(\Omega, \mathcal{F}, Q_\theta)$ one has that

    $$\mathbb{E}_{Q_\theta}[Y|\mathcal{G}] = \frac{\mathbb{E}[YL(X;\theta)|\mathcal{G}]}{\mathbb{E}[L(X;\theta)|\mathcal{G}]}, \quad Q_\theta\text{-a.s.}$$

33. Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $K > 0$. Compute

    $$\mathbb{E}[\max(0, e^X - K)].$$

    You can express the solution in terms of the cumulative distribution function of a standard normal random variable

    $$\Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz.$$