Introduction and techniqes
in financial mathematics
STK4510
UiO
Autumn 2015

Exercises
List 2

Brownian Motion and Stochastic Calculus

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1. Recall that $X = \{X_t\}_{t\in\mathbb{R}_+}$ is a Gaussian process if for any $\{t_i\}_{i=1,...,n} \subset \mathbb{R}_+, n \in \mathbb{N}$ we have that the vector $(X_{t_1}, ..., X_{t_n})$ is multivariate Gaussian. Prove that the following alternative definition of Brownian motion is equivalent to the one given in class. A real valued process X is a Brownian motion if X has P-a.s. continuous paths and is a centered Gaussian process with covariance function $K(s,t) = \mathbb{E}[X_s X_t] = \min(s,t)$.

2. Let $W = \{W_t\}_{t\in\mathbb{R}_+}$ be a Brownian motion and $a > 0$. Let $\mathbb{F}$ be the minimal augmented filtration generated by W. Which of the following processes are Brownian motions? Which are $\mathbb{F}$-Brownian motions?

   (a) $X_t = -W_t, t \in \mathbb{R}_+$.

   (b) $X_t = W_{a+t} - W_a, t \in \mathbb{R}_+,$.

   (c) $X_t = W_{at^2}, t \in \mathbb{R}_+$.

3. Let $f \in L^2([0,T])$ and W be a Brownian motion. Prove that the process $X_t = \int_0^t f(s)dW_s$ is a Gaussian process. Compute the mean and covariance function.

4. Let $W = \{W_t\}_{t\in[0,T]}$ be a Brownian motion. Prove that the following processes are $\mathbb{F}^W$-martingales:

   (a) $X_t = \exp(\theta W_t - \frac{\theta^2}{2}t), t \in [0,T]$.

   (b) $Y_t = e^{\frac{t}{2}}\cos(W_t), t \in [0,T]$.

   (c) $Z_t = W^2 - t, t \in [0,T]$.

   (d) $G_t = e^{W_t} - 1 - \frac{1}{2}\int_0^t e^{W_s}ds, t \in [0,T]$.

   (e) $H_t = \exp\left(\int_0^t f_s dW_s - \frac{1}{2}\int_0^t f_s^2 ds\right), t \in [0,T]$ where $f \in L^2([0,T])$ is deterministic.

5. Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}, P)$ and let $M = \{M_t\}_{t\in\mathbb{R}_+}$ be a $\mathbb{F}$-martingale such that $\mathbb{E}[|M_t|^2] < \infty$. Prove that
$$\mathbb{E}[(M_t - M_s)^2 \,|\mathcal{F}_s] = \mathbb{E}[M_t^2 - M_s^2|\mathcal{F}_s], \qquad s \le t.$$

6. Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}, P)$ be a filtered probability space. We say that an $\mathbb{F}$-adapted and integrable process $X = \{X_t\}_{t\in\mathbb{R}_+}$ is a submartingale if $\mathbb{E}[X_t|\mathcal{F}_s] \ge X_s, 0 \le s \le t$ (and we say that X is a supermartingale if $-X$ is a submartingale). Let $M = \{M_t\}_{t\in\mathbb{R}_+}$ be a $\mathbb{F}$-martingale. Show that if φ is convex and $Y_t = \varphi(M_t)$ is integrable then Y is a submartingale.

7. Prove that a process X with independent increments and with constant mean is a martingale.

8. Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}, P)$ be a filtered probability space and let $X \in L^p(\Omega, \mathcal{F}, P)$ for some $p \ge 1$. Show that $M_t = \mathbb{E}[X|\mathcal{F}_t]$ is a $\mathbb{F}$-martingale and $M_t \in L^p$, for all $t \in \mathbb{R}_+$.

9. Let $W = \{(W_t^1, W_t^2, W_t^3)\}_{t\in[0,T]}$ be a 3-dimensional Brownian motion. Use Itô's formula to express the following processes as Itô processes:

   (a) $u_1(t, W_t) = 5 + 4t + \exp(3W_t^1)$.

(b) $u_2(t, W_t) = \left(W_t^2\right)^2 + \left(W_t^3\right)^2$.

(c) $u_3(t, W_t) = \log(u_1(t, W_t)u_2(t, W_t))$.

10. Let $X = \{X_n\}_{n\geq 0}$ be a discrete time martingale with respect to some filtration $\{\mathcal{F}_n\}_{n\geq 0}$ and let $H = \{H_n\}_{n\geq 0}$ be *predictable* process with respect to the same filtration,i.e., $H_0 = 0$ and H_n is $\mathcal{F}_{n-1}$-measurable $\forall n \geq 1$. Show that if H is bounded ($|H_n| < C_n$, P-a.s. for some constants $C_n, n \geq 1$), then the process $G = \{G_n\}_{n\geq 0}$ defined by

$$G_0 = 0, \qquad G_n = (H \cdot M)_n = \sum_{i=1}^{n} H_i(M_i - M_{i-1}),$$

is a martingale.

11. Let W be a Brownian motion. Find the Itô representation on the time interval $[0, T]$ of the following random variables:

(a) $F_1 = W_T$,

(b) $F_2 = W_T^2$,

(c) $F_3 = e^{W_T}$,

(d) $F_4 = \int_0^T W_t dt$,

(e) $F_6 = \int_0^T t^2 W_t^2 dt$.

12. Let $\left(\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}_{t\in[0,T]}, P\right)$ be a filtered probability space. Assume that $Q \ll P$ with $Z_T := \frac{dQ}{dP}$. Prove that $Q|_{\mathcal{F}_t} \ll P|_{\mathcal{F}_t}$ for all $t \in [0, T]$ and, if we define $Z_t := \frac{dQ|_{\mathcal{F}_t}}{dP|_{\mathcal{F}_t}}$, then Z_t is a $\mathbb{F}$-martingale under P. Prove that a $\mathcal{F}_t$-adapted process Y is a martingale under Q if and only if ZY is a martingale under P.