Introduction and techniqes
in financial mathematics
STK4510
UiO
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Exercises
List 3

Black-Scholes Model and Risk Neutral Pricing

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1. Let C_t be the price of a call option at time t, where K is the strike price and $T > t$ the time of exercise. Show that
$$C_t \geq S_t - Ke^{-r(T-t)},$$
otherwise there exists an arbitrage opportunity. Argue by arbitrage that $C_t \leq S_t$.

2. Transform the Black-Scholes PDE
$$\frac{\partial f}{\partial t}(t,x) + rx\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(t,x) = rf(t,x), \qquad (t,x) \in [0,T) \times \mathbb{R}_+$$
$$f(T,x) = h(x), \qquad x \in \mathbb{R}_+,$$
to the heat equation
$$\frac{\partial}{\partial \tau} v(\tau, z) = \frac{\partial^2}{\partial z^2} v(\tau, z).$$
What is the initial condition for the heat equation?

3. The Black-Scholes formula for for a call option with strike K and exercise time T is
$$C(t, S_t; T, K, r, \sigma) = S_t \Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$
where
$$d_1 = \frac{\log(S_t/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = d_1 - \sigma\sqrt{T-t},$$
and
$$\Phi(x) = \int_{-\infty}^{x} \phi(z)dz = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz.$$
Show that $C(t, S_t; T, K, r, \sigma)$ is an increasing function in T and σ, and decreasing in K. Find
$$\lim_{\sigma \downarrow 0} C(t, S_t; T, K, r, \sigma), \qquad \lim_{\sigma \uparrow \infty} C(t, S_t; T, K, r, \sigma).$$

4. A digital option with strike $K = S_0$ is a T-contingent claim with payoff $H = \mathbf{1}_{\{S_T > S_0\}}$. Find the price of a digital option and the hedging strategy using the density approach.

5. We consider the Black-Scholes model with time dependent parameters. That is, the asset prices follow the following dynamics
$$dB_t = r(t)B_t dt,$$
$$dS_t = S_t\mu(t)dt + S_t\sigma(t)dW_t,$$
where $r(t), \mu(t)$ and $\sigma(t)$ are deterministic, continuous functions on $[0,T]$. Furthermore, we assume that $\min_{t\in[0,T]} \sigma(t) > \sigma^*$, where $\sigma^* > 0$ is a constant.

(a) Prove that

$$S_t = S_0 \exp\left(\int_0^t \mu(s)ds + \int_0^t \sigma(s)dW_s - \frac{1}{2}\int_0^t \sigma^2(s)ds\right).$$

(b) Prove that there exists a probability Q equivalent to P, under which the discounted stock price $\tilde{S}_t = B_t^{-1}S_t$ is a martingale. Give its density with respect to P.

(c) Let $\phi = (\phi^0, \phi^1)$ be a self-financing strategy. Show that if $\tilde{V}_t(\phi) = B_t^{-1}V_t(\phi)$ is a martingale under Q and if $V_T = \max(0, S_T - K)$, then $V_t(\phi) = F(t, S_t)$, where F is the function defined by

$$F(t,x) = \mathbb{E}_Q\left[\max\left(0, x\exp\left(\int_t^T \sigma(s)d\tilde{W}_s - \frac{1}{2}\int_t^T \sigma^2(s)ds\right) - Ke^{-\int_t^T r(s)ds}\right)\right]$$

(d) Give an expression for the function F and compare it with the Black-Scholes formula.

6. Assume we are in the basic Black-Scholes model. Let $0 < t_1 < t_2 < \cdots < t_n < T$ and consider a discrete geometric Asian option with payoff

$$H = \max\left(\left(\prod_{i=1}^n S_{t_i}\right)^{1/n} - K, 0\right),$$

for a constant strike K. Find a formula for the price $\pi_0(H)$ at time 0.

7. Assume we are in the basic Black-Scholes model. Find the price the arbitrage free price of the following options

(a) **Collars**: Let $K_2 > K_1 > 0$ be fixed constants. The payoff at expiry date T is

$$H_1(S_T) = \min\left(\max(S_T, K_1), K_2\right).$$

(b) **Break forwards**: Let $K > 0$ be a fixed constant. The payoff at expiry date T is

$$H_2(S_T) = \max\left(S_T, S_0e^{rT}\right) - K.$$

8. A chooser option is an agreement in which one party has the right to choose at some future time T_0 wether the option is to be a call or a put option with a common strike price K and remaining time to expiry $T - T_0$. The terminal payoff of this option is

$$H = \max(0, S_T - K)\mathbf{1}_{\{C_{T_0} > P_{T_0}\}} + \max(0, K - S_T)\mathbf{1}_{\{C_{T_0} \le P_{T_0}\}},$$

where C_{T_0} and $P_{T_0}$are the arbitrage free prices at time T_0 of a call and a put options, respectively, with strike K and exercise time T. Find the arbitrage free price $\pi_t(H)$ for $0 \le t \le T_0$.