Option Pricing in Finance

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1 Motivation

Suppose that you are a risk manager of a pension fund invested in the financial market and you know that in T years the fund needs to pay out, at least, €K million in retirement money to the investors. Moreover, your fund is invested in risky assets like stocks. In T years, the value of the fund could be more than €K million, but it also could be less.

- **Problem**: You face the risk that in T years your investment is worth less than the amount promised to your pensioners.
- **Possible solution**: Enter into a financial contract that guarantees a minimal value of €K million for your assets in T years. Such contract gives you the right to sell the pension fund at a guaranteed price of €K million in T years time, but if the fund is worth more you don have the obligation to do so. Your counterpart, however, is comitted to buy your portfolio (for €K million) if its market value in T years is less than €K million. This financial contract is called a *European put option.*

Note the optionality embbeded in the contract. If at time T your fund has a market value of less than €K million, you exercise the rights of the financial contract. On the other hand, if the market value is higher than €K million you don't use this right, because you can sell your fund at a better value in the market. The put option contract has the following payoff at time T

$$\begin{aligned} P(T,S,K) &= \begin{cases} K - S(T) & \text{if} \quad S(T) < K \\ 0 & \text{if} \quad S(T) \geq K \end{cases} \\ &= \max(0, K - S(T)) =: (K - S(T))^+, \end{aligned}$$

the agreed price K is known as the *strike price*, while T is the *exercise time* of the option.

There are some issues though. First, note the asymmetry in the contract. If you enter into this contract, you have a right while your conterparty has an obligation. Moreover, at time T, this contract will give you a nonnegative payoff while your conterparty will have a negative payoff. Hence, your conterparty will not enter such a contract without receiving a premium from you. This rise the following question:

What should this premium be in order for both of you to accept this deal as fair?

More generally, a *financial derivative* is a financial asset (contract) where the price is dependent on (is derived from) another financial asset (say: bonds, commodities, stocks, exchange rates, etc...) which is usually called the underlying. We can roughly divide the derivatives in two classes: European and American. The key difference between American and European derivatives relates to when the derivatives can be exercised: A European derivative may be exercised only at the expiration date of the derivative, i.e. at a single pre-defined point in time. An American derivative on the other hand may be exercised at any time before the expiration date. In what follows we will denote by S the price of the underlying.

Examples of financial derivatives (European)

- *Call option*: A call option with strike K and exercise time T is a financial derivative with the following payoff

$$
\begin{aligned}
P(T,S,K) &= \begin{cases} S(T)-K & \text{if} \quad S(T) > K \\ 0 & \text{if} \quad S(T) \le K \end{cases} \\
&= \max(0, S(T)-K) = (S(T)-K)^+.
\end{aligned}
$$

  It gives you the right to buy the underlying at a fixed price K at time T.

- Asset or nothing call option: An asset or nothing call option with strike K and exercise time T is a financial derivative with the following payoff

$$
P(T,S,K) = \begin{cases} S(T) & \text{if} \quad S(T) > K \\ 0 & \text{if} \quad S(T) \le K \end{cases}
$$

- Asset or nothing put option: An asset or nothing put option with strike K and exercise time T is a financial derivative with the following payoff

$$
P(T,S,K) = \begin{cases} 0 & \text{if} \quad S(T) \ge K \\ S(T) & \text{if} \quad S(T) < K \end{cases}
$$

- Arithmetic Asian call option: An arithmetic asian option with strike K and exercise time T is a financial derivative with the following payoff

$$
P(T,S,K) = \max(0, \frac{1}{T}\int_0^T S(t)dt - K)
$$

  Similarly we can define the Arithmetic Asian put option.

- A Lookback call option: A Lookback call option with strike K and exercise time T is a financial derivative with the following payoff

$$
P(T,S,K) = \max(0, \max_{0 \le t \le T} S(t) - K).
$$

  Similarly we can define the Lookback put option.

One of the main goals of Mathematical Finance is to solve the following problem:

- What is the price of a derivatives contract?

In other words, what should the buyer pay the seller for such contract, or what premium is the seller willing to accept in order to commit to a derivative contract?

In this course, we will develop a mathematical theory that will provide us with a price for the derivatives which both parties in the transaction will find acceptable. Moreover, we will also show how the seller of the derivative can eliminate the risk in his/her position. This is known as hedging the derivative.

2 Empirical Finance

In order to derive the price of a derivative, we need a stochastic model for the dynamical behaviour of the underlying stock. We have to invent a stochastic model that reproduces the observations of the stock's past history as well as possible from a statistical point of view. At the same time, the stochastic model has to fit into a mathematical framework that is appropriate for analysing derivative prices. We are interested in models that capture the statistical properties of the stock price dynamics and also fit into the theory of stochastic analysis. A good compromise between these two requirements is the Black-Scholes model of stock prices, also known as *geometric Brownian* *motion* or *the lognormal process.* A rule of thumb is that the better the statistical properties of a model the more difficult is the stochastic analysis associated.

The task of empirical finance is to investigate financial data from a statistical point of view. We will introduce some basic tools in empirical finance and we will study some financial datasets with a view towards the Black-Scholes model.The statistical possibilities and limitations of the model will be described, and alternatives will be suggested which fit the data better. In particular, we will introduce the geometric normal inverse Gaussian Lévy process as a very flexible class in a financial context.

3 Stochastic Finance

To find the fair price of an option one constructs a portfolio which replicates perfectly the derivative's payoff. Such a portfolio is called the *hedging portfolio,* and it is an investment in a risk-free asset (bank account) and the underlying stock. The price of the replicating portfolio must be the same as the price of the derivative, otherwise arbitrage can be constructed by trading into the portfolio and the derivative. An arbitrage is a portfolio that costs nothing to set up at the beginning and it has nonnegative payoff at some future time.

This hedging portfolio is mathematically modeled as an stochastic integral. We will need a theory of stochastic integration and a change of variable rule, known as Itô formula. With this change of variable formula we will prove that the price of a derivative can be obtained as the solution of a partial differential equation. Moreover, we will show that the solution to this equation can be stated in terms of an expectation under the so call risk-neutral probability measure Q, which is different from the subjective probability measure P. To prove this results we will rely on basic results on Stochastic Analysis such as Itô integration and martingales.

4 Computational Finance

In general, it is not possible to derive an explicit pricing formula, in terms of elementary functions, for a derivative. Then, we must resort to numerical techniques. We will introduce to different numerical approaches to solve the problem:

- *Finite difference method*: This is a classical method to solve partial differential equations (pdes). The first step is to partition the domain into a grid.Then, on the nodes of the grid, we discretize the partial derivatives using finite differences. Hence, the pde is reduced to recursive system of linear equations that can be solved by linear algebra methods.

- *Monte Carlo method*: The MonteCarlo method is based on the law of large numbers, which roughly speaking states that if $\{X_n\}_{n\geq 1} is$ a sequence or fandom variables then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} g(X_i) = \mathbb{E}\left[g(X)\right].$$

  As the price of a derivative can also be written as the expected value of the payoff under the risk neutral probability Q, the previous result suggests that if we can simulate the derivative's payoff under the risk neutral measure then a simple arithmetic average will yield an approximation of the derivative's price.

References

[1] F. Benth. Option Theory with Stochastic Analysis. An Introduction to Mathematical Finance. Universitext. Springer Verlag, Berlin. (2014)