Introduction and Techniques
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Lecture 2

Measure Theory

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1 Measurable Spaces

Let E denote a set and $\mathcal{P}(E)$ denote the power set of E, that is, the set of all subsets of E. In what follows we will use calligraphic letters to denote a class of subsets of E, that is, a subset of $\mathcal{P}(E)$. Moreover, the reference set E will be called a space.

Definition 1 *A σ-algebra on E is a nonempty class $\mathcal{A}$ of subsets of E satisfying:*

1. $E \in \mathcal{A}$.
2. *If $B \in \mathcal{A}$ then $B^c := E \backslash B \in \mathcal{A}$.(closed under the formation of complements)*
3. *If $B_i \in \mathcal{A}, i = 1, 2, 3, ...,$ then $\bigcup_{i=1}^{\infty} B_i \in \mathcal{A}$.(closed under the formation of countable unions)*

Definition 2 *If condition 3. in the previous definition is only satisfied when considering a finite number of sets we say that $\mathcal{A}$ is an algebra.*

It is easy to show that an arbitrary intersection of σ-algebras is a σ-algebra again. This fact motivates the following definition.

Definition 3 *Given $\mathcal{G}$ any class of subsets we define $\sigma(\mathcal{G})$ the σ-algebra generated by $\mathcal{G}$ as the smallest σ-algebra containing $\mathcal{G}$, which coincides with the intersection of all σ-algebras containing $\mathcal{G}$.*

Definition 4 *Let E be an space and $\mathcal{E}$ a σ-algebra of E. The pair $(E, \mathcal{E})$ is called a measurable space. The elements of $\mathcal{E}$ are the measurable sets of $(E, \mathcal{E})$.*

To give trivial examples of the previous mathematical objects is easy.

Example 5 *For any set E, both the power set $\mathcal{P}(E)$ and the class $\{\varnothing, E\}$ are σ-algebras, the largest and the smallest that can be defined on E, respectively.*

Example 6 *If E is a set such that $\#E := \{\text{number of elements of } E\} < \infty$, we usually take the σ-algebra $\mathcal{P}(E)$ which has $2^{\#E}$ elements.*

The following is a more interesting example that will be crucial when we introduce the concept of random variable. We start by recalling the concept of metric space.

Example 7 *Let (E, d) be a separable metric space. We define the Borel σ-algebra on E to be the σ-algebra generated by all open (or closed) sets and we denoted it by $\mathcal{B}(E)$. Consider the family $\mathcal{T}$ of all open balls*

$$B_r(x) = \{y \in E : d(x, y) < r\}, \quad x \in E, r \in \mathbb{R}_+.$$

One has that $\mathcal{B}(E) = \sigma(\mathcal{T})$.

The previous example covers many interesting cases, in particular $\mathbb{R}$ and $\mathbb{R}^d$ with the usual euclidean distance $d(x, y) = \sqrt{\sum_{i=1}^d (x_i - y_i)^2}$ are separable metric spaces.

2 Measurable Transformations and Functions

Let $T : E \to F$ be a transformation/mapping from a space E to another space F. The inverse image $T^{-1}B$ of a subset $B \subset F$ is defined to be

$$T^{-1}B = \{x \in E : T(x) \in B\}.$$

Definition 8 *Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be two measurable spaces. An mapping $T : E \to F$ is called $(\mathcal{E}, \mathcal{F})$-measurable if $T^{-1}B \in \mathcal{E}$ whenever $B \in \mathcal{F}$. If the σ-algebras involved are clear from the context it is called measurable.*

Note that if $\mathcal{E} = \mathcal{P}(E)$ then any transformation is measurable. The following theorem simplifies checking if a transformation is measurable.

Theorem 9 *Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be two measurable spaces and $T : E \to F$ a transformation. Let $\mathcal{G}$ be a class of subsets of F such that $\sigma(\mathcal{G}) = \mathcal{F}$. Then, T is $(\mathcal{E}, \mathcal{F})$-measurable if and only if $T^{-1}B \in \mathcal{E}$ for all $B \in \mathcal{G}$.*

Definition 10 *Let $(E, \mathcal{E})$ be a measurable space and E a space. A function $f : E \to \mathbb{R}$ is called Borel measurable if it is $(\mathcal{E}, \mathcal{B}(\mathbb{R}))$-measurable.*

Definition 11 *Let $(F, \mathcal{F})$ be a measurable space, E a space and $T : E \to F$ a transformation. We define $\sigma(T)$, the σ-algebra generated by T, as the σ-algebra generated by the family of sets $\{T^{-1}(B) : B \in \mathcal{F}\}$, which is a σ-algebra on E.*

$\sigma(T)$ is the smallest σ-algebra on E that makes the transformation T measurable.

Theorem 12 *Let $(E, \mathcal{E}), (F, \mathcal{F})$ and $(G, \mathcal{G})$ be measurable spaces and $T : E \to F$ and $S : F \to G$, two measurable transformations. Then $S \circ T : E \to G$ is $(\mathcal{E}, \mathcal{G})$-measurable.*

The following lemma simplifies to check if a particular function is measurable.

Theorem 13 *Let $(E, \mathcal{E})$ be a measurable space and $f : E \to \mathbb{R}$ a function on E. Then, f is measurable if and only if $\{x \in E : f(x) \leq a\} \in \mathcal{E}$ for all $a \in \mathbb{R}$.*

Theorem 14 *Let f, g measurable functions on $(E, \mathcal{E})$. Then,*

1. *$af + gf, a, b \in \mathbb{R}$ is a measurable function.*
2. *fg and f/g are measurable functions.*
3. *The functions $f^+ := \max(0, f(x))$ and $f^- := -\min(0, f(x))$ are positive measurable functions. Note that $f = f^+ - f^-$ and $|f| = f^+ + f^-$.*
4. *The sets $\{x \in E : f(x) = g(x)\}, \{x \in E : f(x) < g(x)\}$ and $\{x \in E : f(x) \leq g(x)\}$ are measurable.*

Theorem 15 *Let $\{f_n\}_{n \geq 1}$ be a sequence of measurable functions. Then, the following functions are measurable*

1. $\sup_{n \geq 1} f_n(x)$.
2. $\inf_{n \geq 1} f_n(x)$.
3. $\limsup_{n \to \infty} f_n(x) := \inf_{n \geq 1} \sup_{k \geq n} f_k(x)$.
4. $\liminf_{n \to \infty} f_n(x) := \sup_{n \geq 1} \inf_{k \geq n} f_k(x)$.

3 Measure Spaces

Definition 16 *Let $(E,\mathcal{E})$ a measurable space. A measure μ on $\mathcal{E}$ is a mapping $\mu : \mathcal{E} \to [0,+\infty]$ satisfying $\mu(\varnothing) = 0$ and*

$$\mu(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \mu(B_i), \qquad (\sigma\text{-additivity}),$$

whenever $\{B_i\}_{i\geq 1}$ is a sequence of elements of $\mathcal{E}$ that are disjoint ($B_i \cap B_j = \varnothing$, if $i \neq j$). If $\mu(E) < +\infty$, we say that the measure μ is finite. If there exists $\{B_i\}_{i\geq 1}$ disjoint such that $E = \bigcup_{i=1}^{\infty} B_i$ and $\mu(B_i) < +\infty, i \geq 1$, we say that the measure μ is σ-finite.

Definition 17 *Let E be an space, $\mathcal{E}$ be a σ-algebra of E and μ be a measure on $\mathcal{E}$. The triple $(E,\mathcal{E},\mu)$ is called a measure space.*

Example 18 *Any measurable space with the measure $\mu \equiv +\infty$ or $\mu \equiv 0$ are measure spaces.*

Example 19 *If $\#E < \infty$, then we can define a measure μ by assigning a positive number $\mu(x)$ to each element x of the set E, that is,*

$$\mu(B) = \sum_{x\in B} \mu(x), \quad B \in \mathcal{P}(E),$$

and $(E,\mathcal{P}(E),\mu)$ is a measure space.

The following result shows how to induce a measure on a measurable space from a measure space through a measurable transformation.

Definition 20 *Let $(E,\mathcal{E},\mu)$ be a measure space, $(F,\mathcal{F})$ be a measurable space and $T : E \to F$ be a measurable transformation. Then, the set function*

$$\mu_T(B) := \mu(T^{-1}B), \quad B \in \mathcal{F},$$

is a measure in $\mathcal{F}$. The measure μ_T is called the image measure induced by T on $\mathcal{F}$ or the pushforward measure.

3.1 Carathéodory Theorems*

When E is an uncountable set, to find a measure is much more difficult. To construct measures one makes use of the so called Carathéodory Theorems. These results allow to extend σ-additive positive set functions, defined on some families of sets, to measures on the σ-algebras generated by these families.

Definition 21 *Let E be a space. An exterior (or outer) measure on $\mathcal{P}(E)$ is a mapping $\mu^* : \mathcal{P}(E) \to [0,+\infty]$ such that*

1. *$\mu^*(\varnothing) = 0$,*
2. *μ^* is increasing: $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$,*
3. *μ^* is σ-subadditive: $\{A_n\}_{n\geq 1} \subset \mathcal{P}(E) \Rightarrow \mu^*(\bigcup_{n\geq 1} A_n) \leq \sum_{n\geq 1} \mu^*(A_n)$.*

Definition 22 *Let E be a space and μ^* an exterior measure on $\mathcal{P}(E)$. A set $A \subset E$ is μ^*-measurable iff*

$$\mu^*(B) \geq \mu^*(A\cap B) + \mu^*(A^c \cap B), \quad \forall B \in \mathcal{P}(E).$$

Theorem 23 (First Carathéodory Theorem) *Let E be a space and μ^* an exterior measure on $\mathcal{P}(E)$. We denote by $\mathcal{E}_{\mu^*}$ the collection of μ^*-measurable sets. Then, $\mathcal{E}_{\mu^*}$ is a σ-algebra and $\mu^*_{|\mathcal{E}_{\mu^*}}$ (the restriction of μ^* to $\mathcal{E}_{\mu^*}$) is a measure.*

Definition 24 *An algebra on E is a nonempty class $\mathcal{U}$ of subsets of E satisfying:*

1. $E \in \mathcal{U}$.
2. *If $B \in \mathcal{U}$ then $B^c := E \backslash B \in \mathcal{U}$.(closed under the formation of complements)*
3. *If $B_1, B_2 \in \mathcal{U}$, then $B_1 \cup B_2 \in \mathcal{U}$.(closed under the formation of finite unions)*

Theorem 25 *Let E be a space and $\mathcal{U}$ an algebra on E. If μ_1 and μ_2 are two measures defined on $\sigma(\mathcal{U})$ which coincide on $\mathcal{U}$ and they are σ-finite on $\mathcal{U}$, then $\mu_1 \equiv \mu_2$.*

Theorem 26 (Second Carathéodory Theorem or Carathéodory Extension Theorem) *Let E be a space and $\mathcal{U}$ an algebra on E. Let $\mu : \mathcal{U} \to [0, +\infty]$ be a σ-additive set function, $\mu \not\equiv +\infty$. We define the mapping*

$$\begin{array}{cccc} \mu^* : & \mathcal{P}(E) & \longrightarrow & [0, +\infty] \\ & B & \longmapsto & \inf\limits_{\substack{B \subset \cup_{n=1}^{\infty} A_n \\ \{A_n\}_{n \geq 1} \subset \mathcal{U}}} \sum\limits_{n=1}^{\infty} \mu(A_n) \end{array}$$

Then,

1. *μ^* is an exterior measure.*
2. *$\mathcal{U} \subset \mathcal{E}_{\mu^*}$.*
3. *$\mu^*_{|\mathcal{U}} = \mu$.*

The previous theorem states that if the set function μ is σ-additive on the algebra $\mathcal{U}$, then μ can be extended to a certain σ-algebra $\mathcal{E}_{\mu^*}$ that contains the σ-algebra generated by $\mathcal{U}$. The question is: $\mathcal{E}_{\mu^*} = \sigma(\mathcal{U})$? The answer is no, in general. However, if we want a measure on $\sigma(\mathcal{U})$ is enough to consider $\mu^*_{|\sigma(\mathcal{U})}$. Also, Theorem 25 ensure that if the set function μ is σ-finite on $\mathcal{U}$, then the extension on $\sigma(\mathcal{U})$ is unique.

The relationship between $\sigma(\mathcal{U})$ and $\mathcal{E}_{\mu^*}$ is quite simple and it is explained in the next section

3.2 Complete Measure Spaces

Definition 27 *Let $(E, \mathcal{E}, \mu)$ be a measure space and $N \subset \mathcal{P}(E)$. We say that N is negligible if there exists $A \in \mathcal{F}$ such that $N \subset A$ and $\mu(A) = 0$.*

Definition 28 *A measure space $(E, \mathcal{E}, \mu)$ is complete if all negligible sets are measurable.*

The next proposition shows that one always can construct a complete measure from one that is not complete in a consistent way.

Proposition 29 *Let $(E, \mathcal{E}, \mu)$ be a measure space. We define $\overline{\mathcal{E}} := \{A \cup N : A \in \mathcal{E}, N \text{ negligible}\}$ and the set function*

$$\begin{array}{cccc} \bar{\mu} : & \overline{\mathcal{E}} & \longrightarrow & [0, +\infty] \\ & A \cup N & \longmapsto & \mu(A) \end{array}.$$

Then, $(E, \overline{\mathcal{E}}, \bar{\mu})$ is a complete measure space and its called the completion of $(E, \mathcal{E}, \mu)$.

The next theorem explains the relationship between $\mathcal{E}_{\mu^*}$ and $\sigma(\mathcal{U})$ in the Carathéodory extension theorem

Theorem 30 *Let E be a space and $\mathcal{U}$ an algebra on E. Let $\mu : \mathcal{U} \to [0, +\infty]$ be a σ-additive and σ-finite set function. Then, the measure space $(E, \mathcal{E}_{\mu^*}, \mu^*)$ is the completion of $(E, \sigma(\mathcal{U}), \mu^*)$.*

3.3 Measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

A relevant measurable space where we would like to to construct measures is $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. A non straightforward application of Carathéodory theorems yields the following result.

Theorem 31 *Let $\mathcal{J}$ be the family of half-closed and bounded intervals on $\mathbb{R}$, that is, the sets of the following form*

$$\{x \in \mathbb{R} : x \in (a, b]\}, \quad a \le b.$$

and let $\mu : \mathcal{J} \to [0, +\infty]$ be a σ-additive and σ-finite set function on $\mathcal{R}$. Then μ extends to an unique measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

The previous theorem is interesting because we have reduced the determination of a measure on $\mathcal{B}(\mathbb{R})$ to the determination of a set function (satisfying certain requirements) on a smaller family of sets $\mathcal{J}$. It does not follow directly from Carathéodory extension theorem because $\mathcal{J}$ is not an algebra.

We can proceed even further with the so called Lebesgue-Stieltjes measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Definition 32 *A Lebesgue-Stieltjes measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a measure such that*

$$\forall B \in \mathcal{B}(\mathbb{R}) \text{ bounded}, \quad \mu(B) < \infty.$$

Definition 33 *A distribution function F is a function $F : \mathbb{R} \to \mathbb{R}$ nondecreasing and right continuous,i.e.,*

$$x \le y \Rightarrow F(x) \le F(y) \quad \text{and} \quad \lim_{x \to a^+} F(x) = F(a).$$

Theorem 34 *Let F be a distribution function. Then, there exists a unique (σ-finite) measure μ_F on $\mathcal{B}(\mathbb{R})$ such that*

$$\mu_F((a, b]) := F(b) - F(a), \quad -\infty < a < b < \infty.$$

Conversely, if μ is a Lebesgue-Stieltjes measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ then there exists a distribution function F such that $\mu = \mu_F$. F is unique up to an additive constant.

Example 35 *If we take $F(x) = x$ we get the Lebesgue measure λ.*

The previous theorem just states that to determine a Lebesgue-Stieltjes measure on $\mathbb{R}$ we only need to provide a distribution function, which is a considerably simpler object than a set function.

A similar theory also holds for $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ but the conditions on F are more involved.

3.4 Properties Almost Everywhere "a.e." and Completeness

Definition 36 *Let $(E, \mathcal{E}, \mu)$ be a measure space and suppose that some property hold at all point of $B \in \mathcal{E}$ where $\mu(B^c) = 0$. Then, this property is said to hold almost everywhere (abbreviated a.e. or μ-a.e.). The set B^c is called the exceptional set for that property. If $\mu(E) = 1$, we say that the property holds almost surely (abbreviated a.s. or μ-a.s.).*

For example, if f is a function on E the statement $f \ge 0$, a.e. means that there is a set $B \in \mathcal{E}, \mu(B^c) = 0$, such that $f(x) \ge 0$ for all $x \in B$. Note that the set where $f(x) < 0$ is to be a subset of B^c, but this set does not need to be the whole B^c. The precise set where the property does not hold is not necessarily measurable unless, of course, μ is a complete measure.

One often has several properties withe each hold a.e. and it is desired to say that they hold a.e. as a group. This is, one seeks one exceptional set, rather than several. This is clearly possible for finite or countable infinite set of properties since countably many zero sets can be combined to get a zero measure set. This of course cannot be done in general if there are uncountably many conditions.

Next, suppose that $f = g$, a.e.. Suppose f is known to be measurable. It is then not necessarily true that g is measurable. However, we have the following result.

Theorem 37 *Let $(E, \mathcal{E}, \mu)$ be a measure space, and f, g functions defined on E. If f is measurable and μ is complete, and $f = g$, a.e., then g is measurable.*

Pursuing this line a little further, suppose that $(E, \mathcal{E}, \mu)$ is a measure space and $\bar{\mu}$ is the completion of μ, on the completed σ-algebra $\overline{\mathcal{E}}$. Suppose that f is $\overline{\mathcal{E}}$-measurable. Then, it can be shown that there is an $\mathcal{E}$-measurable function g such that $f = g$, μ-a.e.

Finally, note the important notion of convergence a.e.. Specifically "$f_n \to f$, a.e." means that $f_n(x) \to f(x)$ for all $x \in A$, where $A \in \mathcal{E}, \mu(A^c) = 0$. This does not necessarily imply that f is measurable, even though the function $\lim_{n\to\infty} f_n(x)$ is a measurable function if all f_n are measurable functions. Of course, f is measurable if μ is complete. From the previous discussion though, we have that we always can choose a measurable version of f, that is, we can choose g measurable such that $f = g$, a.e..

In integration theory, many hypothesis can be relaxed to hold only a.e. because it does not matter how the functions are defined in sets of measure 0. Sometimes we will not add the "a.e." to economize the writing.

4 Integration Theory

Assume that we have a measure space $(E, \mathcal{E}, \mu)$.

Definition 38 *Let* $f(x) = \sum_{i=1}^{n} a_i \mathbf{1}_{B_i}(x)$ *be a nonnegative simple function defined on* E*, where* $\{B_i\}_{i=1,\ldots,n} \subset \mathcal{E}$ *are disjoint and* $E = \bigcup_{i=1}^{n} B_i$. *Then, the integral of* f *with respect to* μ *is*

$$\int_E f d\mu = \sum_{i=1}^{n} a_i \mu(B_i).$$

Definition 39 *Let* f *a nonnegative measurable function. The integral of* f *with respect to* μ *is defined to be*

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu,$$

where $\{f_n\}_{n\geq 1}$ *is the sequence of measurable simple functions defined by*

$$f_n(x) = \left\{ \begin{array}{lll} \frac{i-1}{2^n} & if & \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n}, \quad i = 1, ..., n2^n \\ n & if & f(x) \geq n \end{array} \right. .$$

Note that $f_n(x)$ *is an increasing sequence that converges to* $f(x)$.

Definition 40 *Let* f *be a measurable function. We say that the integral of* f *with respect to* μ *exists if* $\int_E f^+ d\mu < \infty$ *or* $\int_E f^- d\mu < \infty$ *and it is defined to be*

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu.$$

We say that f *is integrable if* $\int_E f d\mu$ *exists and it is finite and we say that* $f \in L^1(E, \mathcal{E}, \mu)$.

Note that f is integrable if and only if $\int_E |f| \, d\mu < \infty$.

Theorem 41 *Let* f, g *be measurable functions such that* $f = g$*, a.e.. Then, if the integral of* f *exists then the integral of* g *also exists and they coincide. In particular, if* f *is integrable then* g *is also integrable.*

From the previous theorem one deduces that f only need to be defined a.s. to consider its integral with respect to μ. That is, it may be a set $N \in \mathcal{E}$ with $\mu(N) = 0$ such that f is not defined on N. Define $g = f\mathbf{1}_{N^c} + 0\mathbf{1}_N$. The function g is measurable and defined for all $x \in E$. We can define, in a consistent manner, the integral of f with respect to μ by

$$\int_E f d\mu := \int_E g d\mu.$$

Theorem 42 *Let* $f, g \in L^1(E, \mathcal{E}, \mu)$. *We have that*

1.

$$\left|\int_E f d\mu\right| \leq \int_E |f|\, d\mu.$$

2. $af+bg \in L^1(E,\mathcal{E},\mu)$ *and*

$$\int_E (af+bg)\, d\mu = a\int_E f d\mu + b\int_E g d\mu,$$

3. *If* $f \geq g$, *a.e. then*

$$\int_E f d\mu \geq \int_E g d\mu.$$

4. *If* $f = g$, *a.e. then*

$$\int_E f d\mu = \int_E g d\mu.$$

5. *If* $B \in \mathcal{E}$, *then* $f\mathbf{1}_B$ *is measurable and we may define*

$$\int_B f d\mu = \int_E f\mathbf{1}_B d\mu.$$

Moreover, if $\int_B f d\mu = 0$ *for all* $B \in \mathcal{E}$ *then* $f = 0$, *a.e.*

Definition 43 *For* $p \geq 1$, *we define the space* $L^p(E,\mathcal{E},\mu) := \{f \quad \text{being } (\mathcal{E}, B(\mathbb{R}))\text{-measurable} : \int_E |f|^p\, d\mu < \infty\}$, *abbreviated by* L^p *whenever is clear the underlying space of measure.*

By the previous theorem, L^p is a vector space. Moreover, in L^p we can define a norm $\|\cdot\|_{L^p}$ by

$$\|f\|_{L^p} := \left(\int_E |f|^p\, d\mu\right)^{1/p}.$$

Theorem 44 *The normed vector space* $(L^p, \|\cdot\|)$ *is complete, i.e., every Cauchy sequence is convergent to an element of* L^p. *That means, if* $\lim_{n,m\to\infty} \|f_n - f_m\|_{L^p} = 0$ *then there exist* $f \in L^p$ *such that* $\lim_{n\to\infty} \|f_n - f\|_{L^p} = 0$.

Theorem 45 $L^2(E,\mathcal{E},\mu)$ *is also a Hilbert space with the inner product* $(f,g) := \int_E fg d\mu$.

The following results relate the operation of taking limits and integration.

Theorem 46 (Monotone convergence) *Let* $\{f_n\}_{n\geq 1}$ *be an a.e. increasing sequence of measurable functions such that*

$$\int_E f_1 d\mu \quad \text{exists and} \quad \int_E f_1 d\mu \neq -\infty.$$

Then, the integrals of $f_n, n \geq 1$ *and* $\lim_{n\to\infty} f_n(x)$ *exist and*

$$\int_E \lim_{n\to\infty} f_n(x) d\mu = \lim_{n\to\infty} \int_E f_n d\mu.$$

In particular, if $\lim_{n\to\infty} \int_E f_n d\mu < +\infty$ *then* $f_n, n \geq 1$ *and* $\lim_{n\to\infty} f_n(x)$ *are integrable.*

The Monotone convergence theorem is usually applied when the sequence $\{f_n\}_{n\geq 1}$, in addition of being increasing, is also positive, i.e., $f_1 \geq 0$, a.e.

Theorem 47 (Fatou's lemma) *Let* $\{f_n\}_{n\geq 1}$ *be a sequence of measurable functions such that*

$$\int_E \inf_{n\geq 1} f_n d\mu \quad \text{exists and} \quad \int_E \inf_{n\geq 1} f_n d\mu \neq -\infty.$$

Then, the integrals of $f_n, n \geq 1$ *and* $\liminf_{n\to\infty} f_n(x)$ *exist and*

$$\int_E \lim \inf_{n\to\infty} f(x) d\mu \leq \lim \inf_{n\to\infty} \int_E f_n d\mu.$$

In particular, if $\liminf_{n\to\infty} \int_E f_n d\mu < +\infty$ *then* $\liminf_{n\to\infty} f(x)$ *is integrable.*

Theorem 48 (Dominated convergence) *Let $\{f_n\}_{n\geq 1}$ be a sequence of measurable functions such that $|f_n(x)| \leq g(x)$, a.e., where g is an integrable function. Then, f is integrable and*

$$\int_E \lim_{n\to\infty} f_n(x)d\mu = \lim_{n\to\infty} \int_E f_n d\mu.$$

The following is a general change of variable formula.

Theorem 49 (Image measure Theorem) *Let $(E, \mathcal{E}, \mu)$ be a measure space, $(F, \mathcal{F})$ a measurable space, $T : E \to F$ a measurable transformation and $f : F \to \mathbb{R}$ a measurable function. Then, the integral of f as a function on $(F, \mathcal{F}, \mu_T)$ exists if and only if the integral of $f \circ T$ as a function on $(E, \mathcal{E}, \mu)$, and if this is the case*

$$\int_E f \circ T d\mu = \int_F f d\mu_T.$$

In particular, f is integrable if and only $f \circ T$ is integrable in their respective spaces.

The previous theorem is a powerful tool to compute integrals of functions defined on an arbitrary measure space $(E, \mathcal{E}, \mu)$. The idea is to find an equivalent integral on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$. Assume that we have a measurable function g defined on a measure space $(E, \mathcal{E}, \mu)$ and we want to compute its integral with respect to μ. Applying the image measure theorem with $T = g$ and $f = Id$ (the identity function) we get that

$$\int_E g d\mu = \int_E Id \circ g d\mu = \int_{\mathbb{R}} Id\, d\mu_g = \int_{\mathbb{R}} x \mu_g(dx).$$

We end this section by stating some results comparing the Riemann and Lebesgue integral on $\mathbb{R}$. This is important because it will allow us to use the machinery of classical integral calculus to compute Lebesgue integrals. The Lebesgue integral is an extension of the Riemann integral when where are integrating over a bounded interval $[a, b]$. However, there are functions that are Riemann integrable in the improper sense that are not Lebesgue integrable.

Theorem 50 (Lebesgue characterization of Riemann integrable functions) *Let f be a bounded real-valued function defined on a bounded interval $[a, b]$. Then, f is Riemann integrable iff the set*

$$D = \{x \in [a, b] : f \text{ is not continuous at } x\},$$

has Lebesgue measure zero. If this is the case, one can find a Borel measurable function $g = f$, λ-a.s. such that the Riemann integral of f coincides with the Lebesgue integral of g.

Theorem 51 *Let $f : [a, +\infty) \to \mathbb{R}$ be a Borel measurable function, where $a \in \mathbb{R}$. Suppose that*

1. *f is Riemman integrable on $[a, b]$ for every $b \geq a$.*
2. *There exists a constant M such that $\int_a^b |f(x)|\, dx \leq M$, for all $b \geq a$.*

Then f and $|f|$ are improper Riemann integrable on $[a, b)$. Furthermore, $f \in L^1([a, \infty), \mathcal{B}([a, \infty)), \lambda)$ and

$$\int_{[a,+\infty)} f d\lambda = \int_a^\infty f(x)dx.$$

5 The Radon-Nykodim Theorem

Definition 52 *Let $(E, \mathcal{E}, \mu)$ be a space of measure. Let $f : \mathbb{E} \to \mathbb{R}_+$ be a measurable function. We define on $\mathcal{E}$ the measure*

$$\nu(B) = \int_B f d\mu := \int_E \mathbf{1}_E f d\mu, \quad B \in \mathcal{E}.$$

We say that f is the density of ν with respect to μ and it is written $f = \frac{d\nu}{d\mu}$. We also say that f is the Radon-Nikodym derivative of ν with respect to μ.

Definition 53 *Let μ and ν be two measures on a measurable space $(E, \mathcal{E})$. We say that ν is absolutely continuous with respect to μ if and only if*

$$\forall B \in \mathcal{E}, \quad \mu(A) = 0 \Rightarrow \nu(A) = 0,$$

and we write $\nu \ll \mu$. If $\mu \ll \nu$ and $\nu \ll \mu$ we say that the two measures are equivalent and we write $\mu \sim \nu$.

Theorem 54 (Radon-Nikodym) *Let μ and ν be two measures on a measurable space $(E, \mathcal{E})$ and μ being σ-finite. Then, the following statements are equivalent:*

1. *$\nu \ll \mu$.*
2. *There exists a measurable function $f : E \to \mathbb{R}_+$ such that $f = \frac{d\nu}{d\mu}$.*

Proposition 55 *Let $(E, \mathcal{E}, \mu)$ be a space of measure. Let $\nu \ll \mu$ with density $f = \frac{d\nu}{d\mu}$. Let $g : E \to \mathbb{R}$ measurable. Then the integral of g with respect to ν exists if and only if the integral of gf exists and, if this is the case,*

$$\int_E g d\nu = \int_E g f d\mu.$$

In particular, f is integrable with respect to ν if and only if gf is integrable with respect to μ.

6 Measures on Product Spaces

Definition 56 *Let $\{(E_i, \mathcal{E}_i)\}_{i\in I}$ a family of measurable spaces. Let $E := \underset{i\in I}{\times} E_i$ the Cartesian product of the family $\{E_i\}_{i\in I}$. Let $\pi_i : E \to E_i, i \in E_i$, the projection of E into its factors. We call the product σ-algebra of $\{\mathcal{E}_i\}_{i\in I}$ the σ-algebra generated by $\{\pi_i\}_{i\in I}$ and we will denote it by $\mathcal{E} = \underset{i\in I}{\otimes} \mathcal{E}_i$. Equivalently, $\underset{i\in I}{\otimes} \mathcal{E}_i = \sigma(\{\pi_i^{-1}(A_i) : A_i \in \mathcal{E}_i, \quad i \in I\})$. A set of the form $\pi_i^{-1}(A_i)$ is called a cylinder of base A_i. $(E, \mathcal{E})$ is called the measurable product space of $\{(E_i, \mathcal{E}_i)\}_{i\in I}$.*

Observe that if $(\Omega, \mathcal{F})$ is another measurable space, then $g : \Omega \to E$ is measurable iff $\forall i \in I$, $\pi_i \circ f : \Omega \to E_i$ is measurable.

When we have finite product of measurable spaces the product σ-algebra can be defined, equivalently, in terms of rectangles.

Definition 57 *Let $(E, \mathcal{E})$ the measurable product space of $\{(E_i, \mathcal{E}_i)\}_{i=1,\dots,n}$. A measurable rectangle is a subset of E of the form $A_i \times \cdots \times A_n$ with $A_i \in \mathcal{E}_i$.*

We have that the σ-algebra generated by the measurable rectangles coincides with $\mathcal{E}$.

We are interested in constructing measures on finite product spaces from measures on its factors. The basic tool are the transition measures.

Definition 58 *A transition measure from a measurable space $(E_1, \mathcal{E}_1)$ to a measurable space $(E_2, \mathcal{E}_2)$ is a mapping*

$$\tau : E_1 \times \mathcal{E}_2 \to \mathbb{R}_+,$$

such that

1. *$\forall e_1 \in E_1, \tau(e_1, \cdot) : \mathcal{E}_2 \to \mathbb{R}_+$ is a measure on $(E_2, \mathcal{E}_2)$.*
2. *$\forall A_2 \in \mathcal{E}_2, \tau(\cdot, A_2) : E_1 \to \mathbb{R}_+$ is $(\mathcal{E}_1, \mathcal{B}(\mathbb{R}_+))$-measurable.*

Definition 59 *A family of measures $\{\mu_i\}_{i\in I}$ on a measurable space $(E, \mathcal{E})$ is uniformly σ-finite iff there exists a decomposition $E = \bigcup_{n=1}^{\infty} A_n$, with $A_n \in \mathcal{E}$, and constants $K_n \in \mathbb{R}_+$ such that $\forall n \in \mathbb{N}, \forall i \in I, \mu_i(A_n) \leq K_n$.*

Theorem 60 (Product measure) *Let $(E_1, \mathcal{E}_1, \mu)$ be a measure space with μ being σ-finite. Let $(E_2, \mathcal{E}_2)$ be a measurable space. Let $\tau : E_1 \times \mathcal{E}_2 \to \mathbb{R}_+$ be a transition measure such that the family of measures $\{\tau(e_1, \cdot)\}_{e_1 \in E_1}$ is uniformly σ-finite. Then:*

1. *There exists an unique measure ν on $(E_1 \times E_2, \mathcal{E}_1 \otimes \mathcal{E}_2)$ such that*

$$\nu(A_1 \times A_2) = \int_{A_1} \tau(e_1, A_2) d\mu = \int_{A_1} \tau(e_1, A_2)\mu(de_1).$$

2. *If $g : E_1 \times E_2 \to \mathbb{R}$ is a function $(E_1 \otimes E_2, \mathcal{B}(\mathbb{R}))$-measurable and the integral of g with respect to ν exists, then the function*

$$e_1 \to \int_{E_2} g(e_1, e_2)\tau(e_1, de_2)$$

is well defined μ-a.e., it is measurable and its integral with respect to μ exists and is equal to

$$\int_{E_1 \times E_2} g d\nu = \int_{E_1} \left(\int_{E_2} g(e_1, e_2)\tau(e_1, de_2) \right) \mu(de_1). \tag{1}$$

Note that in the previous theorem we assume that the function g is integrable with respect to ν. A very useful result is the so called Tonelli-Hobson integrability criterion.

Theorem 61 (Tonelli-Hobson) *Assume the same notation as in the product measure theorem. Then, if g is $(E_1 \otimes E_2, \mathcal{B}(\mathbb{R}))$-measurable and*

$$\int_{E_1} \left(\int_{E_2} |g(e_1, e_2)| \, \tau(e_1, de_2) \right) \mu(de_1) < +\infty,$$

then g is integrable with respect to ν and the formula (1) holds.

A particular important case is when $\tau(\cdot, A_2) = \tau(A_2)$ is a constant for every $A_2 \in \mathcal{E}_2$. Then, the product measure ν is denoted by $\mu \otimes \tau$ and

$$\begin{aligned} \int_{E_1 \times E_2} g d\nu &= \int_{E_1} \left(\int_{E_2} g(e_1, e_2)\tau(de_2) \right) \mu(de_1) \\ &= \int_{E_2} \left(\int_{E_1} g(e_1, e_2)\mu(de_1) \right) \tau(de_2), \end{aligned}$$

whenever the function g is ν-integrable (which can be checked using Tonelli-Hobson). This result is known as Fubini's Theorem.

The Product Measure and Tonelli-Hobson's Theorems are easily extended to any finite product of measurable spaces.

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