Introduction and Techniques
in Financial Mathematics
UiO-STK4510
Autumn 2015
Teacher: *S. Ortiz-Latorre*

Lecture 4

Stochastic Processes

---

1 Stochastic Processes

Definition 1 *Let $(E, \mathcal{E})$ be a measurable space and I a set. An E-valued stochastic process X indexed by I is a family of E-valued random variables $X = \{X_t\}_{t\in I}$ defined on the same probability space $(\Omega, \mathcal{F}, P)$.*

We can think of a stochastic process as a function

$$X: \begin{array}{ccc} I \times \Omega & \longrightarrow & E \\ (t,\omega) & \mapsto & X_t(\omega) \end{array}.$$

For every $\omega \in \Omega$ fixed, the process X defines a function

$$X_\cdot(\omega): \begin{array}{ccc} I & \longrightarrow & E \\ t & \mapsto & X_t(\omega) \end{array},$$

which is called a *trajectory* or a *sample path* of the process. Hence, we can look at X as a mapping

$$X: \begin{array}{ccc} \Omega & \longrightarrow & E^I \\ \omega & \mapsto & X_\cdot(\omega) \end{array},$$

where E^I is the cartesian product of I copies of E, which is the set of all functions from I to E. That is, we can see X as a mapping from Ω to a space of functions.

Remark 2 *In this course we will usually take $E = \mathbb{R}$ or $E = \mathbb{R}^d$. Moreover, we wil usually take $I = \mathbb{R}_+, I = \mathbb{N}$ or $I = [0, T]$, where $T \in \mathbb{R}_+$.*

2 Law of a Stochastic Process

If we think of a stochastic process as a mapping $X : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ and we put on $\mathbb{R}^{\mathbb{R}_+}$ a σ-algebra such that X is measurable, we will have an $\mathbb{R}^{\mathbb{R}_+}$-valued random variable and we can talk about the law of the process X. We expect the law of the process X to be related to the laws of the random variables X_t. For instance, what is the value of $P(X_t \in B)$ should make sense as a question on the r.v. X_t as well as a question on the law of X. Moreover, the answer must coincide.

We will consider in $\mathbb{R}^{\mathbb{R}_+}$ the product σ-algebra of $\mathbb{R}_+$ copies of $\mathcal{B}(\mathbb{R})$, denoted by $\mathcal{B}(\mathbb{R})^{\mathbb{R}_+}$. Recall that this σ-algebra is the smallest σ-algebra that makes the projections $\{\pi_t\}_{t\in\mathbb{R}}$, that is,

$$\pi_t: \begin{array}{ccc} \mathbb{R}^{\mathbb{R}_+} & \longrightarrow & \mathbb{R} \\ f & \mapsto & f(t) \end{array},$$

is a Borel function for all $t \in \mathbb{R}_+$. It is also characterized as the σ-algebra generated by the cylinders with one dimensional base, i.e, $\sigma(\{\pi_t^{-1}(B) : B \in \mathcal{B}(\mathbb{R}), t \in \mathbb{R}_+\})$. More interestingly, it coincides with the σ-algebra generated by the σ-cylinders, that is,

$$\sigma\left(\left\{\pi_J^{-1}(B) : B \in \mathcal{B}(\mathbb{R})^{\#J}, J \subset \mathbb{R}_+ \text{ countable}\right\}\right),$$

where $\mathcal{B}(\mathbb{R})^{\#J}$ is the product of $\#J$ (cardinal of J) copies of $\mathcal{B}(\mathbb{R})$ and π_J is the natural projection from $\mathbb{R}^{\mathbb{R}_+}$ on $\mathbb{R}^{\#J}$.

Proposition 3 *If $X = \{X_t\}_{t\in\mathbb{R}_+}$ is a real stochastic process and we consider in $\mathbb{R}^{\mathbb{R}_+}$ the product σ-algebra $\mathcal{B}(\mathbb{R})^{\mathbb{R}_+}$, then the application $X : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ is measurable.*

Definition 4 *The law of real stochastic process $X = \{X_t\}_{t\in\mathbb{R}_+}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is the law of the random variable $X : \Omega \to \mathbb{R}^{\mathbb{R}_+}$, where the σ-algebra on $\mathbb{R}^{\mathbb{R}_+}$ is $\mathcal{B}(\mathbb{R})^{\mathbb{R}_+}$.*

Let μ be a probability measure on $(\mathbb{R}^{\mathbb{R}_+}, \mathcal{B}(\mathbb{R})^{\mathbb{R}_+})$. We can consider, for each $J \subset \mathbb{R}_+$ finite, the image measure of μ by the projection π_J, which we will denote by μ_J. That is, μ_J is the probability measure on $(\mathbb{R}^{\#J}, \mathcal{B}(\mathbb{R})^{\#J})$ defined by $\mu_J(B) := \mu(\pi_J^{-1}(B)), B \in \mathcal{B}(\mathbb{R})^{\#J}$.

Definition 5 *The family of probabilities $\{\mu_J : J \subset \mathbb{R}_+, \#J < +\infty\}$ is called the family of finite dimensional distributions of μ.*

This family satisfies that if $K \subset J$ then

$$\mu_K(B) = \mu_J(\pi_{J,K}^{-1}(B)), \quad B \in \mathcal{B}(\mathbb{R})^{\#K}, \tag{1}$$

where $\pi_{J,K}$ is the natural projection of $\mathbb{R}^{\#J}$ on $\mathbb{R}^{\#K}$.

Theorem 6 (Daniell-Kolmogorov) *Let $\{\mu_J : J \subset \mathbb{R}_+, \#J < +\infty\}$ be a family of probabilities satisfying the consistency property* (1)*. Then there exists an unique probability measure μ on $(\mathbb{R}^{\mathbb{R}_+}, \mathcal{B}(\mathbb{R})^{\mathbb{R}_+})$ such that the family of its finite dimensional distributions coincides with the initially given family. The probability measure μ is called the projective limit probability measure of the family.*

Corollary 7 *The law of a stochastic process it is completely determined by the associated family of finite dimensional distributions, that is, by*

$$\mu_{X;t_1,...,t_d}(B) := P((X_{t_1}, ..., X_{t_d}) \in B), \quad B \in \mathcal{B}(\mathbb{R}^d), \quad t_1, ..., t_d \in \mathbb{R}, \quad d \in \mathbb{N}.$$

The previous theorem shows that provided we have a reasonabe (consistent) family of finite dimensional distributions we can construct a real stochastic process (actually a probability space containing it) such that its law has as family of finite dimensional distributions this family. Note that we can always realize the process on $(\Omega, \mathcal{F}, P) = (\mathbb{R}^{\mathbb{R}_+}, \mathcal{B}(\mathbb{R})^{\mathbb{R}_+}, \mu)$ where μ is the projective limit probability measure and the process is $X(f) = f$, where $f \in \mathbb{R}^{\mathbb{R}_+}$.

The following set of definitions only depend on the law of a process X.

Definition 8

1. *A process X is called Gaussian if its family of finite dimensional distributions are multivariate Gaussian distributions.*

2. *A process X has independent increments from the past if*

$$\forall s \le t, \quad X_t - X_s \text{ is independent of } \{X_r\}_{0\le r\le s}.$$

   When we talk about independent increments we understand that $X_t - X_s$ is independent of $X_r - X_u$, for all $u \le r \le s \le t$.

3. *A process X is stationary if for all $h > 0$ and for all $t_1, ..., t_n \in \mathbb{R}_+, n \in \mathbb{N}$, we have that*

$$\mathcal{L}(X_{t_1}, ..., X_{t_n}) = \mathcal{L}(X_{t_1+h}, ..., X_{t_n+h}).$$

4. *A process X is stationary in wide sense if $\mathbb{E}[X_t^2] < \infty$ and for all $h > 0$*

$$\begin{aligned} E[X_t] &= E[X_{t+h}], \\ \operatorname{Cov}(X_t, X_s) &= \operatorname{Cov}(X_{t+h}, X_{s+h}), \quad t, s \in \mathbb{R}_+. \end{aligned}$$

5. *A process X has stationary increments if for all $s \le t \in \mathbb{R}_+$ we have that*

$$\mathcal{L}(X_t - X_s) = \mathcal{L}(X_{t-s}).$$

Note that, as the multivariate Gaussian distributions are characterized by the moments of first and second orde, the law of a Gaussian process X is characterized by $E[X_t]$ and $\operatorname{Cov}(X_t, X_s), s, t \in \mathbb{R}_+$.

3 Properties with Probability 1

Suppose that we have two processes X and Y defined on the same probability space and with the same law. As application from Ω to $\mathbb{R}^{\mathbb{R}_+}$, X and Y may be very different. For instance, Let X_t be Unif$[0,1]$, independent and we define $Y_t = 1 - X_t$. Then, X and Y have the same law, but $P(X_t = Y_t) = 0$ for all $t \in \mathbb{R}_+$. We need stronger concepts of equality for processes than equality in law.

Definition 9 *Let X and Y be two processes defined on the same probability space $(\Omega, \mathcal{F}, P)$. We say that Y is a modification or a version of X if*

$$\forall t \in \mathbb{R}_+, \quad P(X_t = Y_t) = 1.$$

Definition 10 *Let X and Y be two processes defined on the same probability space $(\Omega, \mathcal{F}, P)$. We say that Y is indistinguishable of X if the set*

$$\{\omega : X_t(\omega) \neq Y_t(\omega), \quad \forall t \in \mathbb{R}_+\},$$

is a negligible set.

If X and Y are indistinguishable then one is a version of the other. If X is a version of Y then X and Y have the same law.

3.1 Path Properties

Let $\Gamma \subset \mathbb{R}^{\mathbb{R}_+}$. The following question does not make much sense: Given a process X determine, using only the law μ_X, if $P(X \in \Gamma) = 1$ or not. The law of the process does not give us much information about the properties of its paths. In addition, if $\Gamma \notin \mathcal{B}(\mathbb{R})^{\mathbb{R}_+}$ (for instance Γ is the set of all continuous paths) we can not answer the question by looking at μ_X. On the other hand, it makes sense to ask the following questions:

1. Given a law on $(\mathbb{R}, \mathcal{B}(\mathbb{R})^{\mathbb{R}_+})$, determine if there exists some stochastic process X with this law and such that $P(X \in \Gamma) = 1$.
2. Given a process X, determine, from the law of the process X, if there exists a modification $\tilde{X}$ of X such that $P(\tilde{X} \in \Gamma) = 1$.

To answer the first question we have the following result.

Theorem 11 *Let μ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R})^{\mathbb{R}_+})$. Let $\Gamma \subset \mathbb{R}^{\mathbb{R}_+}$. Then, there exists a process X, defined on some probability space $(\Omega, \mathcal{F}, P)$, with law μ such that $P(X \in \Gamma) = 1$ iff $\mu^*(\Gamma) = 1$, where μ^* is the exterior measure associated to μ, that is a set function defined by*

$$\mu^*(B) = \inf\left\{\mu(A) : B \subset A, \quad A \in \mathcal{B}(\mathbb{R})^{\mathbb{R}_+}\right\}, \quad B \subset \mathbb{R}^{\mathbb{R}_+}.$$

In the proof of the previous theorem one construct a probability space and a process X that will be "canonical" for the given situation. One sets $\Omega = \Gamma, \mathcal{F} = \mathcal{B}(\mathbb{R})^{\mathbb{R}_+}_{|\Gamma} := \{A \subset \Gamma : A = B \cap \Gamma, \quad B \in \mathcal{B}(\mathbb{R})^{\mathbb{R}_+}\}$ and we define P to be $\mu_{|\Gamma}$,i.e., $P(A) = \mu(B)$ if $A = B \cap \Gamma$. Finally, one sets $X_t(\omega) = \omega(t)$. In this way, one constructs a canonical process X on a canonical probability space $(\Gamma, \mathcal{B}(\mathbb{R})^{\mathbb{R}_+}_{|\Gamma}, \mu_{|\Gamma})$ such that *all* the sample paths of the processes are in $\Gamma \subset \mathbb{R}^{\mathbb{R}_+}$.

To answer the second question, when $\Gamma = C(\mathbb{R}_+)$, we have the Kolmogorov continuity criterion.

Theorem 12 (Kolmogorov's continuity criterion) *Let X be a stochastic process defined in some probability space $(\Omega, \mathcal{F}, P)$. Assume that for every $T > 0$, there exist $\alpha > 0, r > 0$ and $K > 0$ such that*

$$\mathbb{E}\left[|X_t - X_s|^r\right] \leq K\,|t-s|^{1+\alpha}, \quad 0 \leq s < t \leq T.$$

Then, there exists a modification $\tilde{X}$ of X with all the paths being continuous.

Regarding path regularity properties the most important classes of processes are the following.

Definition 13 *Let $X = \{X_t\}_{t\in\mathbb{R}_+}$ be a real valued stochastic process.*

1. *We say that X is continuous if its sample paths are continuous.*

2. *We say that X is càdlàg or RCLL if its sample paths are continuous from the right and have limits from the left. That is $X_t = \lim_{s\downarrow t} X_s$ and $\lim_{s\uparrow t} X_s$ exists and is finite.*

3. *We say that X is càglàd or LCRL if its sample paths are continuous from the left and have limits from the right. That is $X_t = \lim_{s\uparrow t} X_s$ and $\lim_{s\downarrow t} X_s$ exists and is finite.*

The properties in the previous definition may hold only P-a.s and then we will say that a proces P-a.s continuous or P-a.s. RCLL, etc... Finally, if we have a process Y that is version of a process X and both are P-a.s. RCLL (or P-a.s. LCRL) then X and Y are indistinguishable. In particular, if X and Y are P-a.s. continuous and Y is a version of X then X and Y are indistinguishable.

4 Measurability Properties of Stochastic Processes

Note that from the measurability of each X_t is not possible to deduce the measurability of X as a function from $\mathbb{R}_+ \times \Omega$ to $\mathbb{R}$.

Definition 14 *We say that a process X is measurable when it is $(\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable, that is, it is measurable as a function $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$.*

Theorem 15 *Let X be a measurable stochastic processes. If*

$$\int_{\mathbb{R}_+\times\Omega} |X(t,\omega)|\,\lambda(dt)\otimes P(d\omega) < \infty,$$

where λ is the Lebesgue measure, then

$$\int_{\mathbb{R}_+\times\Omega} X(t,\omega)\lambda(dt)\otimes P(d\omega) = \int_{\mathbb{R}_+} \mathbb{E}\left[X_t\right] dt = \mathbb{E}\left[\int_{\mathbb{R}_+} X_t dt\right] < \infty.$$

Definition 16 *Let $(\Omega, \mathcal{F}, P)$ be a probability space. A filtration on $(\Omega, \mathcal{F}, P)$ is a family $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s \leq t$. We say that $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is a filtered probability space.*

Definition 17 *A filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ is called right continuous if $\mathcal{F}_t = \mathcal{F}_{t+}$ for every $t \in \mathbb{R}_+$, where $\mathcal{F}_{t+} = \cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}$. The σ-algebra $\mathcal{F}_{t+}$ represents what we know if we look ahead an infinitesimal amount of time.*

Definition 18 *A filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is said to satisfy the "usual conditions" if $(\Omega, \mathcal{F}, P)$ is complete and*

1. *$\mathcal{F}_0$ contains all negligible sets in $\mathcal{F}$.*

2. *$\mathbb{F}$ is right continuous.*

Definition 19 *A stochastic process X is $\mathbb{F}$-adapted if for all $t \in \mathbb{R}_+$ the r.v. X_t is $\mathcal{F}_t$-measurable.*

Note that we can always find a filtration with respect to which a process X is adapted. We can consider the *natural* filtration generated by X, denoted by $\mathbb{F}^X$ and defined by

$$\mathbb{F}^X = \{\mathcal{F}_t^X := \sigma(X_s, s \leq t)\}_{t\in\mathbb{R}_+}.$$

The natural filtration $\mathbb{F}^X$ represents the information revealed by the process X as time evolves. For instance, the process $Y = \{Y_t := X_t^2\}_{t\in\mathbb{R}_+}$ is $\mathbb{F}^X$-adapted while the process $Z = \{Z_t = X_1 + X_t^2\}_{t\in\mathbb{R}_+}$ is not $\mathbb{F}^X$-adapted.

Given a process X, the filtration $\mathbb{F}^X$ does not need to satisfy the usual conditions. However, we can always consider (and we do it) the *minimal augmented* (also know as *the usual augmentation of the*) filtration generated by X, which is the smallest filtration satisfying the usual conditions and with respect to which the process X is adapted. This is done by enlarging $\mathbb{F}^X$ with $\mathcal{N}$, the P-negligible sets in $\mathcal{F}$, and then taking its limit from the right, that is, considering the filtration

$$\left\{ \bigcap_{\varepsilon>0} \sigma(\mathcal{F}^X_{t+\varepsilon} \cup \mathcal{N}) \right\}_{t\in\mathbb{R}_+}.$$

We will use the same notation to denote the natural filtration generated by X and the minimal augmented filtration generated by X, that is, $\mathbb{F}^X$. Moreover, many (but not all) of the processes that we will be dealing with in this course are examples of the so called strong Markov processes: Brownian motion, solutions of stochastic differential equations, Lévy processes... For these processes, it suffices to augment their natural filtration with the P-negligible sets of $\mathcal{F}$ to automatically get a right continuous filtration. Many results that we will see in this course need the filtration under consideration to satisfy the usual conditions to be true, but not all. That the filtration $\mathbb{F}$ contains all the P-negligible sets ensure, for instance, that if a process X is $\mathbb{F}$-adapted and Y is a version of X, then Y is also $\mathbb{F}$-adapted.

Stopping times are one of the most useful concepts in stochastic analysis as they are the basic tool for localisation procedures.

Definition 20 *A random variable $\tau : \Omega \to [0, +\infty]$ is a stopping time with respect to the filtration $\mathbb{F}$ if $\{\tau \le t\} \in \mathcal{F}_t$ for all $t \in \mathbb{R}_+$.*

Definition 21 *Let τ be a $\mathbb{F}$-stopping time. Then, the σ-algebra of events ocurring up to time τ can be defined by*

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : \text{ for each } t > 0, A \cap \{\tau \le t\} \in \mathcal{F}_t\}.$$

One can check that if $\tau = t, P$-a.s. then $\mathcal{F}_\tau = \mathcal{F}_t$ and that if $\tau_1 \le \tau_2, P$-a.s. then $\mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_2}$. Therefore, one can use stopping times to randomize the flow of information, that is, to randomize a filtration.

The right continuity of the filtration ensure that some random times, for instance the first time that a $\mathbb{F}$-adapted process with right continuous paths X hits a open set $A \in \mathcal{B}(\mathbb{R})$, mathematicall defined by $\tau_A(\omega) = \inf\{t \in \mathbb{R}_+ : X_t(\omega) \in A\}$, are actually stopping times.

Definition 22 *A stochastic process X is progressively measurable or progressive with respect to $\mathbb{F}$ if, for all $t \in \mathbb{R}_+$, the restriction*

$$X|_{[0,t]\times\Omega} : \begin{array}{ccc} [0,t]\times\Omega & \longrightarrow & \mathbb{R} \\ (s,\omega) & \mapsto & X_s(\omega) \end{array},$$

is $(\mathcal{B}([0,t]) \otimes \mathcal{F}_t, \mathcal{B}(\mathbb{R}))$-measurable.

The notion of progressive measurability is useful to ensure that some processes are adapted. For instance, if X is progressive then the process $Y_t = \int_0^t X_s ds$ is adapted. Moreover, if X is progressive and τ is a stopping time with respect to the same filtration, then the random variable $X_\tau := X_{\tau(\omega)}(\omega)$ is $\mathcal{F}_\tau$-measurable and the process stopped at X, defined by $X^\tau := X_{\min(t,\tau)}$ is also progressive.

Proposition 23

1. *If X is progressive, then it is measurable and adapted.*
2. *If X is adapted and has right continuous paths (or left continuous paths) then it is progressive.*
3. *There are adapted processes that are not measurable.*
4. *There are measurable processes that are not adapted.*
5. *There are adapted and measurable processes that are not progressive, but one can always find a modification that it is progressive.*

5 Random Analysis

In this brief section we will look at a stochastic processes as a function from $\mathbb{R}_+$ to an space of random variables.

Assume that we have a stochastic process $X = \{X_t\}_{t\in\mathbb{R}_+}$ such that each X_t is a random variable without any additional conditions,i.e.,

$$\begin{array}{rccc} X: & \mathbb{R}_+ & \longrightarrow & L^0(\Omega, \mathcal{F}, P) \\ & t & \mapsto & X_t \end{array},$$

where $L^0(\Omega, \mathcal{F}, P)$ is the space of all random variables defined on probability space $(\Omega, \mathcal{F}, P)$.

This space is a metric space when considering the convergence in probability. Moreover, the metric can be given by the following distance

$$d(X,Y) = E\left[\frac{|X-Y|}{1+|X-Y|}\right],$$

which yields a complete metric space.

Definition 24 *A stochastic process $X = \{X_t\}_{t\in\mathbb{R}_+}$ is continuous in probability (or stochastically continuous) at $t_0 \in \mathbb{R}_+$ if*

$$\lim_{t\to t_0} P\left(|X_t - X_{t_0}| > \varepsilon\right) = 0, \quad \forall \varepsilon > 0.$$

It is continuous in probability if it is continuous in probability at every $t_0 \in \mathbb{R}_+$. Equivalently, if the application $t \longmapsto X_t$ is continuous from $\mathbb{R}_+$ to $(L^0(\Omega, \mathcal{F}, P), d)$.

Note that a process can be stochastically continuous and all its paths be discontinuous functions.

Assume now that all the random variables X_t of a process belong to the space $L^p := L^p(\Omega, \mathcal{F}, P)$ of p-integrable random variables, i.e., $\mathbb{E}\left[|X_t|^p\right] < \infty, \forall t \in \mathbb{R}_+$. The number p can be any number $0 \leq p < \infty$. When $p \geq 1$, L^p is a Banach space with norm

$$\|X\|_{L^p} = \mathbb{E}\left[|X|^p\right]^{1/p}.$$

Definition 25 *A stochastic process $X = \{X_t\}_{t\in\mathbb{R}_+}$ is in L^p if $X_t \in L^p$ for all $t \in \mathbb{R}_+$.*

Definition 26 *Let $p \geq 1$. A stochastic process $X = \{X_t\}_{t\in\mathbb{R}_+}$ in L^p is continuous in L^p (or continuous in p-th mean) at $t_0 \in \mathbb{R}_+$ if the application*

$$\begin{array}{rccc} X: & \mathbb{R}_+ & \longrightarrow & L^p(\Omega, \mathcal{F}, P) \\ & t & \mapsto & X_t \end{array},$$

is continuous at t_0, that is, if

$$\lim_{t\to t_0} \|X_t - X_{t_0}\| = 0.$$

It is continuous in L^p if it is continuous in L^p at every $t_0 \in \mathbb{R}_+$.

The most interesting case is when $p = 2$ because the space L^2 is a Hilbert space with scalar product

$$\langle X, Y\rangle := \mathbb{E}\left[XY\right].$$

Definition 27 *Let $X = \{X_t\}_{t\in\mathbb{R}}$ be a stochastic process in L^2.*

1. *The mean function of X is the function from $\mathbb{R}_+$ to $\mathbb{R}$ given by*

$$m(t) := \mathbb{E}\left[X_t\right].$$

2. *The covariance function of X is the function from $\mathbb{R}_+ \times \mathbb{R}_+$ to $\mathbb{R}$ given by*

$$K(t,s) := \mathrm{Cov}(X_t, X_s) = \mathbb{E}[X_t X_s] - \mathbb{E}[X_t]\mathbb{E}[X_s].$$

Proposition 28 *If the mean function is continuous then the process X is continuous in L^2 at $t_0 \in \mathbb{R}_+$ iff the covariance function $K(t,s)$ is continuous in (t_0, t_0).*

References

[1] A. Alabert. *Difusions I*. Unpublished. You cand find it at *http://gent.uab.cat/aureli_alabert/*. (In Catalan)

[2] A. Alabert and M. Farré. *Moviment Brownià i Càlcul Estocàstic*. Unpublished. You cand find it at *http://gent.uab.cat/aureli_alabert/*. (In Catalan)

[3] R. F. Bass. *Stochastic Processes*. Cambridge University Press, New York. 2011.

[4] I. Karatzas and S. E. Shreve. Brownian Motion and Stochastic Calculus. Second Edition. Springer, New York. 1998.