

Brownian Motion and Related Processes

1 Brownian Motion

Definition 1 A stochastic process $W = \{W_t\}_{t \in \mathbb{R}_+}$ is a Brownian motion (or Wiener process) with variance σ^2 starting at $x_0 \in \mathbb{R}$ if its family of finite dimensional distributions is given by

$$P(W_0 \in B_0, W_{t_1} \in B_1, \dots, W_{t_n} \in B_n) \\ = \delta_{x_0}(B_0) \int_{B_1 \times \dots \times B_n} \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2(t_i - t_{i-1})}} \exp\left(-\frac{(x_i - x_{i-1})^2}{2\sigma^2(t_i - t_{i-1})}\right) dx_1 \cdots dx_n,$$

where $0 = t_0 < t_1 < \dots < t_n, n \in \mathbb{N}$ and $B_i \in \mathcal{B}(\mathbb{R})$.

The Dirac's delta in the previous formula tell us that $W_0 = x_0$, P -a.s.. One can check the previous family of finite dimensional distributions is consistent and, hence, we can use the Daniell-Kolmogorov theorem to prove that Brownian motion always exists. That is, we can construct a probability space and a process defined on it such that its finite dimensional distributions coincide with the ones given in the definition. Moreover, Brownian motion is a Gaussian process.

Proposition 2 If W is a Brownian motion there exists a modification \tilde{W} that has continuous paths.

Proof. By the properties of the normal law and as $W_t - W_s \sim \mathcal{N}(0, \sigma^2(t - s))$, we have that

$$\mathbb{E}[|W_t - W_s|^4] = 3\sigma^4(t - s)^2,$$

and applying Kolmogorov's continuity criterion $r = 4, K = \sigma^2$ and $\alpha = 1$ the result follows. ■

Remark 3 Our first definition of Brownian motion only uses the minimal properties that provide certain consistent family of finite dimensional distributions. Then, using the Daniell-Kolmogorov's extension theorem we have proved that there exists a process having this family of finite dimensional distributions. Finally, using Kolmogorov's continuity criterion we have proved that there is a version of the process having continuous sample paths with probability one. From now on we will always assume that we are working with a continuous modification of a Brownian motion and, hence, we will include this property in the definition.

Remark 4 Using the results in the previous lecture regarding the existence of processes satisfying some sample path properties (Lecture 4, section 3.1), one can consider the canonical construction of a Brownian motion on the the space of continuous functions starting at zero (or some $x \in \mathbb{R}$). In this way we have a Brownian motion with all the paths being continuous not only P -a.s.

Proposition 5 Brownian motion can be defined as a stochastic process $W = \{W_t\}_{t \in \mathbb{R}_+}$ satisfying

1. W has continuous paths P -a.s.,
2. $W_0 = x, P$ -a.s.,
3. W has independent increments,

4. For all $0 \leq s < t$, the law of $W_t - W_s$ is a $\mathcal{N}(0, \sigma^2(t-s))$.

Proof. The law of the vector $(W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \dots, W_{t_n} - W_{t_{n-1}})$ is determined by the conditions of the proposition (independent increments plus law of a generic increment) and by a change of variable one can compute the law of $(W_{t_0}, W_{t_1}, \dots, W_{t_n})$, which coincides with the law in the previous definition. ■

Note that property 4. in the previous proposition not only says that Brownian motion has Gaussian increments but also that has stationary increments.

Definition 6 When $\sigma^2 = 1$, we say that W is a standard Brownian motion. If the starting point is not especificed we understand the the process starts at 0. Hence, a standard Brownian motion is a process satisfying

1. W has continuous paths P -a.s.,
2. $W_0 = 0$, P -a.s.,
3. W has independent increments,
4. For all $0 \leq s < t$, the law of $W_t - W_s$ is a $\mathcal{N}(0, (t-s))$.

Remark 7 From now on, when we say Brownian motion we are considering a standard Brownian motion.

Let $\mathbb{F} = \{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$ be a filtration. In stochastic analysis is useful the concept of Brownian motion with respect to a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$.

Definition 8 A \mathbb{F} -Brownian motion W is a real stochastic process adapted to \mathbb{F} satisfying

1. W has continuous paths P -a.s.,
2. $W_0 = 0$, P -a.s.,
3. For all $0 \leq s < t$, the random variable $W_t - W_s$ is independent of \mathcal{F}_s .
4. For all $0 \leq s < t$, the law of $W_t - W_s$ is a $\mathcal{N}(0, (t-s))$.

Remark 9 To have a \mathbb{F} -Brownian motion is a stronger statement that to only have a Brownian motion. If $W_t - W_s$ is independent of \mathcal{F}_s then it is independent from $W_u - W_v$, $0 \leq v \leq u \leq s$ and W is a process with independent increments. If we only have a Brownian motion then we also have a $\mathbb{F}^W = \{\mathcal{F}_t^W\}_{t \in \mathbb{R}_+}$ -Brownian motion but W does not need to be a \mathbb{F} -Brownian motion where \mathbb{F} is an enlargement of \mathbb{F} , i.e., $\mathcal{F}_t^W \subseteq \mathcal{F}_t$, $t \in \mathbb{R}_+$.

Definition 10 A d -dimensional Brownian motion $W = \{(W_t^1, \dots, W_t^d)\}_{t \in \mathbb{R}_+}$ is a stochastic process with values in \mathbb{R}^d such that each component W^i , $i = 1, \dots, d$ is a real Brownian motion independent of each other.

For instance the first component W^1 of a d -dimensional Brownian motion is a \mathbb{F}^{W^1} -Brownian motion but it is also a \mathbb{F}^W -Brownian motion. In general, if we have a \mathbb{F} -Brownian motion and we enlarge the filtration \mathbb{F} with some events independent from \mathbb{F} we still have a Brownian motion with respect to the enlarged filtration.

We end this section by introducing some processes that can be obtained from Brownian motion using simple transformations.

1.1 Brownian Motion with Drift

Definition 11 A process Y is a Brownian motion with drift μ and volatility σ if it can be written as

$$Y_t = \mu t + \sigma W_t, \quad t \in \mathbb{R}_+,$$

where W is a standard Brownian motion.

This process has almost all the properties of Brownian motion. It starts at zero, has independent increments and the increments have Gaussian laws. However, the Gaussian law of $Y_t - Y_s, s < t$ is not centered but has mean $\mu(t - s)$. Brownian motion with drift is also a Gaussian process.

This process was used in the first attempt to model stock prices behaviour by Bachelier. Many people consider Bachelier's PhD dissertation entitled *Théorie de la Espéculation* (1900) the beginning of Mathematical Finance. Brownian motion with drift, as a model for stock prices, allows for negative prices with positive probability, which is major pitfall.

1.2 Geometric Brownian motion

Definition 12 *A process S is a geometric Brownian motion (or exponential Brownian motion) with drift μ and volatility σ if it can be written as*

$$S_t = \exp(\mu t + \sigma W_t), \quad t \in \mathbb{R}_+,$$

where W is a standard Brownian motion.

The law of a geometric Brownian motion is not Gaussian. Actually, the random variable S_t has lognormal distribution with mean μt and variance $\sigma^2 t$, see exercise 21 in List 1. It does not have independent and stationary increments like Brownian motion or Brownian motion with drift. On the other hand, its relative increments

$$\frac{S_{t_n} - S_{t_{n-1}}}{S_{t_{n-1}}}, \frac{S_{t_{n-1}} - S_{t_{n-2}}}{S_{t_{n-2}}}, \dots, \frac{S_{t_1} - S_{t_0}}{S_{t_0}}, \quad 0 \leq t_0 < t_1 < \dots < t_n,$$

are independent and stationary. Equivalently,

$$\frac{S_{t_n}}{S_{t_{n-1}}}, \frac{S_{t_{n-1}}}{S_{t_{n-2}}}, \dots, \frac{S_{t_1}}{S_{t_0}}, \quad 0 \leq t_0 < t_1 < \dots < t_n,$$

and

$$\log\left(\frac{S_{t_n}}{S_{t_{n-1}}}\right), \log\left(\frac{S_{t_{n-1}}}{S_{t_{n-2}}}\right), \dots, \log\left(\frac{S_{t_1}}{S_{t_0}}\right), \quad 0 \leq t_0 < t_1 < \dots < t_n,$$

are also independent and stationary. Moreover, the law of $\frac{S_t}{S_s}, s < t$ is a lognormal distribution with parameters $\mu(t - s)$ and $\sigma^2(t - s)$ or, equivalently, the law of $\log\left(\frac{S_t}{S_s}\right), s < t$ is $\mathcal{N}(\mu(t - s), \sigma^2(t - s))$.

This process is the standard model for stock prices. Note that in this case the prices are always positive. The model was first used by Samuelson (1964) and later on by Black-Scholes (1973) in their theory of option pricing.

1.3 Brownian Bridge

Definition 13 *Let W be a standard Brownian motion. A process $X = \{X_t\}_{t \in [0,1]}$ is a standard Brownian bridge if it can be written as*

$$X_t = W_t - tW_1, \quad t \in [0, 1],$$

where W is a standard Brownian motion.

This process starts at zero and ends at zero, that is, $P(X_0 = 0) = P(X_1 = 0) = 1$. The law of this process can be deduced from its relation with respect to Brownian motion and is given by

$$\begin{aligned} & P(X_0 \in B_0, X_{t_1} \in B_1, \dots, X_{t_n} \in B_n, X_{t_{n+1}} \in B_{n+1}) \\ &= \delta_0(B_0) \delta_0(B_{n+1}) \int_{B_1 \times \dots \times B_n} \prod_{i=1}^{n+1} \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right) dx_1 \dots dx_n, \end{aligned}$$

where $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = 1, n \in \mathbb{N}, x_0 = x_{n+1} = 0$ and $B_i \in \mathcal{B}(\mathbb{R})$.

Note that the family of finite dimensional distributions is Gaussian, like Brownian motion. However the increments cannot be independent because the process must end at zero at time $t = 1$ and, hence, we must have for any $t \in (0, 1)$ that

$$0 = W_1 = W_1 - W_t + W_t - W_0,$$

which yields that

$$W_1 - W_t = -(W_t - W_0).$$

Moreover, one can check that the law of a Brownian bridge coincides with the law of a Brownian motion on $[0, 1]$ conditioned to end at $W_1 = 0$, i.e.,

$$P(X_{t_1} \in B_1, \dots, X_{t_n} \in B_n) = P(W_{t_1} \in B_1, \dots, W_{t_n} \in B_n | W_1 = 0).$$

Besides the importance of Brownian bridge in statistics (the Kolmogorov-Smirnov distribution is the law of the maximum of Bridge) it is also used in simple models of financial markets with inside or privileged information.

2 Poisson process

Definition 14 A stochastic process $N = \{N_t\}_{t \in \mathbb{R}_+}$ is a Poisson process with variance λ and starting at $x \in \mathbb{Z}$ if its law is given by

$$\begin{aligned} P(N_0 = x_0, N_{t_1} = x_1, \dots, N_{t_n} = x_n) \\ = \delta_x(\{x_0\}) \prod_{i=1}^n \frac{e^{-\lambda(t_i - t_{i-1})} (\lambda(t_i - t_{i-1}))^{x_i - x_{i-1}}}{(x_i - x_{i-1})!}, \end{aligned}$$

if $x_i \in \mathbb{Z}$, $x_0 \leq x_1 \leq \dots \leq x_n$ (and zero otherwise), and on $0 = t_0 < t_1 < \dots < t_n, n \in \mathbb{N}$.

We are imposing that $N_0 = x_0, P$ -a.s.. On the other hand, if we compute the law of the random vector $(N_{t_1} - N_{t_0}, N_{t_2} - N_{t_1}, \dots, N_{t_n} - N_{t_{n-1}})$ we get that

$$\prod_{i=1}^n \frac{e^{-\lambda(t_i - t_{i-1})} (\lambda(t_i - t_{i-1}))^{x_i}}{x_i!}, \quad x_i \in \mathbb{Z},$$

and we deduce that is composed of independent random variables with Poisson law with parameter $\lambda(t_i - t_{i-1})$. Hence, the Poisson process has independent increments.

Proposition 15 The Poisson process can be defined as a stochastic process N satisfying:

1. $N_0 = x, P$ -a.s.,
2. N has independent increments,
3. For all $0 \leq s < t$, the law of $N_t - N_s$ has law $\text{Pois}(t - s)$.

Definition 16 When $\lambda = 1$, we say that N is a standard Poisson process. If the starting point is not specified we understand the process starts at 0. Hence, a standard Poisson motion is a process satisfying

1. $N_0 = 0, P$ -a.s.,
2. N has independent increments,
3. For all $0 \leq s < t$, the law of $N_t - N_s$ is a $\text{Pois}((t - s))$.

One can show that the Poisson process exists using the Daniell-Kolmogorov theorem. However, it is not possible to use the Kolmogorov's continuity criterion to show that there exists a continuous modification. Actually, the paths of the Poisson process are discontinuous almost surely. Given a filtration \mathbb{F} one can also define a \mathbb{F} -Poisson process analogously to \mathbb{F} -Brownian motion, see remark 9.

3 Lévy processes

The Brownian motion and the Poisson process are particular cases of a more general type of process.

Definition 17 A stochastic process $X = \{X_t\}_{t \in \mathbb{R}_+}$ is a Lévy process starting at zero if it satisfies:

1. $X_0 = 0$, P -a.s.,
2. X has independent increments,
3. X has stationary increments, i.e., for all $0 \leq s < t$, the law of $X_t - X_s$ coincides with the law of X_{t-s} .
4. X is stochastically continuous, i.e., $\lim_{s \rightarrow t} P(|X_t - X_s| > \varepsilon) = 0, \forall \varepsilon > 0, t \in \mathbb{R}_+$.

Condition 4. in Definition 17 serves to exclude processes with jumps at fixed (nonrandom) times. Given a

Lévy process we can choose a unique modification whose paths are right continuous and with left limits. This property is satisfied by the Brownian motion and the Poisson process. Obviously, conditions 3. and 4. strongly restrict the possible laws of the process X and its family of finite dimensional distributions. Actually, a Lévy process X is determined by the law of X_1 but this law cannot be arbitrary, it must be infinitely divisible. We recall that a distribution F is infinitely divisible iff for any $n \in \mathbb{N}$ there exists a sequence $\{Z_i^n\}_{i=1, \dots, n}$ of i.i.d. random variables such that the law of $Z_1^n + \dots + Z_n^n$ is given by F .

Given a filtration \mathbb{F} one can also define a \mathbb{F} -Lévy process analogously to the \mathbb{F} -Brownian motion, see remark 9.

4 Martingales

The following type of processes will be essential in the theory of stochastic integration and option pricing.

Definition 18 A stochastic process X is a martingale if $\mathbb{E}[|X_t|] < \infty, t \in \mathbb{R}_+$ and

$$\mathbb{E}[X_t | \mathcal{G}_s] = X_s, \quad 0 \leq s \leq t.$$

We can also define a martingale with respect to a filtration \mathbb{F} and this is definition that we will use the most.

Definition 19 Let X be a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. The process X is a \mathbb{F} -martingale if $\mathbb{E}[|X_t|] < \infty, t \in \mathbb{R}_+$, X is \mathbb{F} -adapted and

$$\mathbb{E}[X_t | \mathcal{F}_s] = X_s, \quad 0 \leq s \leq t.$$

The martingale property essentially says that the best prediction of the value (or the expected value) of the process at some future time t given the information up to time s is precisely the value of the process at time s .

Example 20 A \mathbb{F} -Brownian motion is a \mathbb{F} -martingale: by definition W is \mathbb{F} -adapted, as $W_t \sim \mathcal{N}(0, t)$ has finite moments of all orders and

$$\mathbb{E}[W_t - W_s | \mathcal{F}_s] = \mathbb{E}[W_t - W_s] = 0,$$

where we have used that $W_t - W_s$ is independent from \mathcal{F}_s , that if Z is independent from \mathcal{G} we have that $\mathbb{E}[Z | \mathcal{G}] = \mathbb{E}[Z]$ and that $\mathbb{E}[W_t] = 0$, for all $t \in \mathbb{R}_+$. From the previous equality and using that W_s is \mathcal{F}_s -measurable and, hence, $\mathbb{E}[W_s | \mathcal{F}_s] = W_s$, we can conclude.

The following result will be needed in the development of the stochastic integral.

Theorem 21 (Doob's maximal inequalities) If $\{M_t\}_{t \geq 0}$ is a martingale with almost surely continuous paths then we have that:

1. For all $T \geq 0, \lambda > 0$ and $p \geq 1$

$$P\left(\sup_{0 \leq t \leq T} |M_t| > \lambda\right) \leq \frac{1}{\lambda^p} \mathbb{E}[|M_T|^p].$$

2. For all $T \geq 0, p > 1$

$$\mathbb{E}[|M_T|^p] \leq \mathbb{E}\left[\sup_{0 \leq t \leq T} |M_t|^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|M_T|^p].$$

5 Markov processes

Definition 22 A real stochastic process $X = \{X_t\}_{t \in \mathbb{R}_+}$ is a Markov process if

$$P(X_t \in B | \sigma(X_{s_1}, X_{s_2}, \dots, X_{s_n})) = P(X_t \in B | \sigma(X_{s_n})),$$

for all $0 \leq s_1 < s_2 < \dots < s_n \leq t$ and $B \in \mathcal{B}(\mathbb{R})$.

A Markov process is the stochastic analogous of what happens in a system governed by a differential equation with unique solution: A value given at some time determines what happens after that time, we do not need more information from the past. Essentially, a Markov process is independent of the past given the present.

We can also consider the Markov property with respect to a filtration \mathbb{F} .

Definition 23 Let X be a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. The process X is a Markov process with respect to \mathbb{F} if X is \mathbb{F} -adapted and

$$P(X_t \in B | \mathcal{F}_s) = P(X_t \in B | \sigma(X_s)),$$

for all $0 \leq s < t$ and $B \in \mathcal{B}(\mathbb{R})$.

The following proposition provides an alternative (and more useful for our goals) characterization of the Markov property.

Proposition 24 Under the assumptions of the previous definition the following two statements are equivalent:

1. For all $0 \leq s < t$ and $B \in \mathcal{B}(\mathbb{R})$,

$$P(X_t \in B | \mathcal{F}_s) = P(X_t \in B | \sigma(X_s)).$$

2. For all $g : \mathbb{R} \rightarrow \mathbb{R}$ Borel measurable such that $\mathbb{E}[|g(X_t)|] < \infty, t \in \mathbb{R}_+$ and for all $0 \leq s \leq t$

$$\mathbb{E}[g(X_t) | \mathcal{F}_s] = \mathbb{E}[g(X_t) | X_s].$$

Note that if a process X is Markov with respect to a filtration \mathbb{F} then it is Markov with respect to its natural filtration.

Let X be a Markov process and for each $0 \leq s \leq t < \infty, A \in \mathcal{B}(\mathbb{R}), x \in \mathbb{R}$, define

$$p_{s,t}(x, A) = P(X_t \in A | X_s = x).$$

By the properties of conditional probability, we can choose $p_{s,t}(x, A)$ to be a transition probability measure from $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. They give the probabilities of ‘transitions’ of the process from the point x at time s to the set A at time t .

Theorem 25 (The Chapman-Kolmogorov equations) If X is a Markov processes then for each $0 \leq r \leq s \leq t, x \in \mathbb{R}, A \in \mathcal{B}(\mathbb{R})$,

$$p_{r,t}(x, A) = \int_{\mathbb{R}} p_{s,t}(y, A) p_{r,s}(x, dy).$$

Definition 26 Let $\{p_{s,t}; 0 \leq s \leq t < \infty\}$ be a family of mappings from $\mathbb{R} \times \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$. We say that they are a family of Markov transition probabilities if, for each $0 \leq s \leq t$:

1. the maps $x \mapsto p_{s,t}(x, A)$ are measurable for each $A \in \mathcal{B}(\mathbb{R})$;
2. $p_{s,t}(x, \cdot)$ is a probability measure on $\mathcal{B}(\mathbb{R})$ for each $x \in \mathbb{R}$.
3. they satisfy the Chapman-Kolmogorov equations.

Theorem 27 If $\{p_{s,t}; 0 \leq s \leq t < \infty\}$ is a family of Markov transition probabilities and μ is a fixed probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, there exists a probability space $(\Omega, \mathcal{F}, P_\mu)$ and a Markov process $X = \{X_t\}_{t \in \mathbb{R}_+}$ defined on that space such that:

1. $P(X_t \in A | X_s = x) = p_{s,t}(x, A)$ for each $0 \leq s \leq t, x \in \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R})$.
2. X_0 has law μ .

Definition 28 A Markov process is said to be (time-) homogeneous if

$$p_{s,t}(x, A) = p_{0,t-s}(x, A),$$

for each $0 \leq s \leq t, x \in \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R})$. For homogeneous Markov processes we will always write the transition probabilities $p_{0,t}$ as p_t .

Example 29 A \mathbb{F} -Brownian motion is a \mathbb{F} -Markov process with homogeneous transition probabilities given by

$$p_s(x, A) = \int_A \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{(y-x)^2}{2s}\right) dy,$$

and initial distribution the Dirac's delta distribution δ_0 .

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