The Itô Integral

1 Introduction

The goal of the Itô integral is to give mathematical sense to an expression as follows

$$\int_0^t X_s dW_s,$$

where X is a stochastic process and W is a Brownian motion. It is not straightforward to define such and object because, in general, one cannot construct the integral pathwise (that is, considering the Lebesgue-Stieltjes integral $\int_0^t X_s(\omega)dW_s(\omega)$ for each $\omega \in \Omega$ fixed). One needs to use a global probabilistic approach that takes into account all the ω's to construct a stochastic process $\mathcal{I}_t[X](\omega) = (\int_0^t X_s dW)(\omega)$, which has the minimal properties to deserve the name of "integral". Basically, these properties are the linearity with respect to the integrands and the possibility to take limits under the integral sign,i.e., the integral is a bounded linear operator). Moreover, it has some good probabilistic properties, like being a martingale.

The Itô integral is at heart of stochastic analysis and it was initially motivated by the need of having a meaningful concept for differential equation involving stochastic processes. Later on, it became the main tool in mathematical finance as it can be used to describe the gains of a portfolio in a continuous time market.

2 Motivation

Stochastic differential equations.
Consider the ordinary differential equation

$$\frac{dX_t}{dt} = f(X_t), \qquad X_0 = x_0.$$

We want to introduce random perturbations on the system

$$\frac{dX_t}{dt} = f(X_t) + \sigma(X_t)\xi_t, \qquad X_0 = x_0$$

where, ideally, ξ_t should be a continuous, stationary and centered process such if $t_1 \neq t_2$ then ξ_{t_1} and ξ_{t_2} are independent. Unfortunately such a process does not exist. An alternative way is to define

$$X_t = x_0 + \int_0^t f(X_s)ds + \int_0^t \sigma(X_s)dW_s,$$

where W is a Brownian motion.

Pathwise integration is impossible
Consider a sequence of partitions of the interval $[0, T]$

$$\tau_n : 0 = t_0^n < t_1^n < \cdots < t_{k_n-1}^n < t_{k_n}^n = T$$

and intermediate points

$$\sigma_n : t_i^n \leq s_i^n \leq t_i^n, \quad i = 0, ..., k_n - 1,$$

such that $\|\pi_n\| = \max_{i=1,\dots,k_n} \left| t_i^n - t_{i-1}^n \right| \longrightarrow 0$ when n tends to infinity. Let f and g two functions on $[0, T]$, the Riemann-Stieltjes integral is defined as

$$\int_0^T f dg = \lim_{n \to \infty} \sum_{i=0}^{k_n - 1} f(s_{i-1})(g(t_{i+1}) - g(t_i)),$$

provided this limit exists and it is independent of the sequences τ_n and σ_n. If one requires the Riemann-Stieltjes integral $\int_0^T f dg$ to exist for any function f continuous on $[0, T]$, then a necessary and sufficient condition is that the function g has bounded variation, that is

$$\sup_{\tau} \sum_i |g(t_{i+1}) - g(t_i)| < \infty.$$

Unfortunately, with probability 1 the paths of the Brownian motion have infinite variation on any finite interval. As a consequence, if $h = \{h_t\}_{0 \leq t \leq T}$ is a process with continuous paths, the Riemann-Stieltjes integral

$$\int_0^T h_t(\omega) dW_t(\omega),$$

does not exist with probability 1.

Martingale transforms

Let $M = \{M_n\}_{n \geq 0}$ be a discrete time martingale with respect to some filtration $\{\mathcal{F}_n\}_{n \geq 0}$. A process $H = \{H_n\}_{n \geq 0}$ is said to be *predictable* with respect to a filtration $\{\mathcal{F}_n\}_{n \geq 0}$ if $H_0 = 0$ and H_n is $\mathcal{F}_{n-1}$-measurable $\forall n \geq 1$. The process $G = \{G_n\}_{n \geq 0}$ defined as

$$G_0 = 0, \qquad G_n = (H \cdot M)_n = \sum_{i=1}^n H_i(M_i - M_{i-1}),$$

is called the **martingale transform** of M by H. Gambling interpretation of G :

1. - H_i : amount bet in day i ($\mathcal{F}_{i-1}$-measurable).
   - $M_i - M_{i-1}$: outcome of the game in day i.
   - G_n : Gains up to day n.

Questions:

- Is it possible to contruct a continuous time analogue of G?
- If this is the case, is the martingale property preserved?

3 Construction of the Itô Integral

Consider $(\Omega, \mathcal{F}, P)$ a complete probability where is defined a Brownian motion W and let $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ be the minimal augmented filtration generated by W, i.e., $\mathbb{F} = \mathbb{F}^W$.

Definition 1 *Let $L^2_{a,T}$ the class of processes that are measurable, $\mathbb{F}$-adapted and square integrable (with respect to $\lambda \otimes P$). That is, the class of measurable processes h such that h_t is $\mathcal{F}_t$-measurable for all $t \in [0, T]$ and*

$$\|h\|^2_{L^2_{a,T}} := \mathbb{E}\left[\int_0^T |h_t|^2 \, dt\right] < \infty.$$

We want to construct a linear bounded operator

$$\begin{array}{rccl} \mathcal{I}: & L^2_{a,T} & \longrightarrow & L^2(\Omega, \mathcal{F}, P) =: L^2 \\ & h & \longmapsto & \mathcal{I}[h] = \int_0^T h_t dW_t. \end{array}$$

satisfying

$$\mathbb{E}\left[\left(\int_0^T h_t dW_t\right)^2\right] = \mathbb{E}\left[\int_0^T |h_t|^2 \, dt\right]. \qquad \text{(Itô isometry)}$$

Definition 2 *A process h in $L^2_{a,T}$ is **simple** if it is of the form*

$$h_t = \sum_{i=0}^{n-1} h_i \mathbf{1}_{(t_i,t_{i+1}]}(t), \quad 0 \le t \le T, \tag{1}$$

where $0 = t_0 < t_1 < \cdots < t_n = T$ and h_i are $\mathcal{F}_{t_i}$-measurable bounded random variables.

Remark 3 *The representation (1) of a simple process is not unique. However, we can consider some sort of canonical or minimal representation in the following way. If there exists an $i \in \{0, ..., n-1\}$ such that $h_i = h_{i+1}$, P-a.s. then we can replace $h_i \mathbf{1}_{(t_i,t_{i+1}]} + h_{i+1}\mathbf{1}_{(t_{i+1},t_{i+2}]}$ by $h_i \mathbf{1}_{(t_i,t_{i+2}]}$ and repeat this procedure till such i does not exist.*

Definition 4 *We will denote by $\mathcal{S}_T$ the class of all simple processes.*

Definition 5 *The Itô integral of $h \in \mathcal{S}_T$ is defined as*

$$\int_0^T h_t dW_t = \sum_{i=0}^{n-1} h_i \left(W_{t_{i+1}} - W_{t_i}\right).$$

It is easy to check that this definition does not depend on the particular representation of h. The idea is to check that the integral for any representation of h coincides with the integral of the canonical or minimal representation of h.

Proposition 6 *For $h, g \in \mathcal{S}_T$ and $a, b \in \mathbb{R}$:*

1. *(Mean zero)* $\mathbb{E}\left[\int_0^T h_t dW_t\right] = 0;$
2. *(Itô Isometry)* $\mathbb{E}\left[\left(\int_0^T h_t dW_t\right)^2\right] = \mathbb{E}\left[\int_0^T |h_t|^2 dt\right].$
3. *(Linearity)* $\int_0^T (ah_t + bg_t)\, dW_t = a\int_0^T h_t dW_t + b\int_0^T g_t dW_t.$

Proof. Set $\Delta W_i = W_{t_{i+1}} - W_{t_i}, i = 0, ..., n-1$. As W is a Brownian motion, its increments ΔW_i are independent from $\mathcal{F}_{t_i}$ and by the basic properties of the conditional expectations we have that $\mathbb{E}[\Delta W_i|\mathcal{F}_{t_i}] = \mathbb{E}[\Delta W_i] = 0$ and $\mathbb{E}[\,(\Delta W_i)^2\,|\mathcal{F}_{t_i}] = \mathbb{E}[\,(\Delta W_i)^2] = t_{i+1} - t_i$, where we also have used that $\Delta W_i \sim \mathcal{N}(0, t_{i+1} - t_i)$. ∎

1. We can write
$$\mathbb{E}\left[\int_0^T h_t dW_t\right] = \sum_{i=0}^{n-1} \mathbb{E}[h_i \Delta W_i] = \sum_{i=0}^{n-1} \mathbb{E}[h_i \mathbb{E}[\Delta W_i|\mathcal{F}_{t_i}]] = 0.$$

2. One has that
$$\mathbb{E}[h_i h_j \Delta W_i \Delta W_j] = \begin{cases} 0 & \text{if} \quad i \neq j \\ \mathbb{E}[h_i^2]\,(t_{i+1} - t_i) & \text{if} \quad i = j \end{cases},$$
because if $i < j$ the random variables $h_i h_j \Delta W_i$ and ΔW_j are independent and if $i = j$
$$\mathbb{E}[h_i^2\,(\Delta W_i)^2] = \mathbb{E}[h_i^2 \mathbb{E}[\,(\Delta W_i)^2\,|\mathcal{F}_{t_i}]] = \mathbb{E}[h_i^2]\,(t_{i+1} - t_i)$$
Therefore, we obtain
$$\begin{aligned} \mathbb{E}\left[\left(\int_0^T h_t dW_t\right)^2\right] &= \sum_{i,j=0}^{n-1} \mathbb{E}[h_i h_j \Delta W_i \Delta W_j] \\ &= \sum_{i=0}^{n-1} \mathbb{E}[h_i^2]\,(t_{i+1} - t_i) = \mathbb{E}\left[\int_0^T |h_t|^2 dt\right]. \end{aligned}$$

3. Exercise.

The following technical lemma is crucial to extend the Itô integral for processes in $\mathcal{S}_T$ to processes in $L^2_{a,T}$.

Lemma 7 *If h is a process in $L^2_{a,T}$ then there exists a sequence $\{h^n\}_{n\geq 1}$ of simple processes such that*

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T |h_t - h_t^n|^2\, dt\right] = \lim_{n\to\infty} \|h - h^n\|_{L^2_{a,T}} = 0.$$

Definition 8 (The Itô integral) *Let $h \in L^2_{a,T}$. Then the Itô integral of h is defined by*

$$\mathcal{I}_T[h] = \int_0^T h_t dW_t = L^2\text{-}\lim_{n\to\infty} \int_0^T h_t^n dW_t,$$

where $\{h_n\}_{n\geq 1}$ is a sequence of simple processes such that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T |h_t - h_t^n|^2\, dt\right] = \lim_{n\to\infty} \|h - h^n\|_{L^2_{a,T}} = 0. \tag{2}$$

Recall that the notation $L^2\text{-}\lim_{n\to\infty} X_n = X$ means that

$$\lim_{n\to} \mathbb{E}[|X_n - X|^2] = 0.$$

We have to check two things in the definition of Itô integral. First that the limit exists and second that the limit does not depend on the approximating sequence.

1. *The limit exists*: Using the Itô isometry we have that

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T h_t^n dW_t - \int_0^T h_t^m dW_t\right)^2\right] &= \mathbb{E}\left[\left(\int_0^T h_t^n - h_t^m dW\right)^2\right] \\
&= \mathbb{E}\left[\int_0^T |h_t^n - h_t^m|^2\, dt\right] \\
&= \mathbb{E}\left[\int_0^T |h_t^n - h_t + h_t - h_t^m|^2\, dt\right] \\
&\leq 2\mathbb{E}\left[\int_0^T |h_t - h_t^n|^2\, dt\right] + 2\mathbb{E}\left[\int_0^T |h_t - h_t^m|^2\, dt\right],
\end{aligned}$$

   where the right hand side of the previous inequality converge to zero when $n, m \to \infty$ because $\{h^n\}_{n\geq 1} \in \mathcal{S}_T$ is an approximating sequence of h. This show that $\int_0^T h_t^n dW_t$ is a Cauchy sequence in L^2 and, therefore, converges to an element of L^2.

2. *The limit does not depend on the approximating sequence.* Let $\{h^n\}_{n\geq 1}$ and $\{g^n\}_{n\geq 1}$ be two sequence of elements of $\mathcal{S}_T$ such that satisfy (2). Then, by the Itô isometry again,

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T h_t^n dW_t - \int_0^T g_t^n dW_t\right)^2\right] &= \mathbb{E}\left[\int_0^T |h_t^n - g_t^n|^2\, dt\right] \\
&\leq 2\mathbb{E}\left[\int_0^T |h_t - h_t^n|^2\, dt\right] + 2\mathbb{E}\left[\int_0^T |h_t - g_t^n|^2\, dt\right] \\
&\underset{n\to\infty}{\longrightarrow} 0,
\end{aligned}$$

   which shows that the L^2 limits of $\int_0^T h_t^n dW_t$ and $\int_0^T g_t^n dW_t$ coincide.

The properties of the Itô integral for processes in $\mathcal{S}_T$ easily extend to processes in $L^2_{a,T}$.

Proposition 9 *For $h, g \in L^2_{a,T}$ and $a, b \in \mathbb{R}$:*

1. *(Mean zero)* $\mathbb{E}[\int_0^T h_t dW_t] = 0;$
2. *(Itô Isometry)* $\mathbb{E}[\left(\int_0^T h_t dW_t\right)^2] = \mathbb{E}[\int_0^T h_t^2 dt].$
3. *(Linearity)* $\int_0^T (ah_t + bg_t)\, dW_t = a\int_0^T h_t dW_t + b\int_0^T g_t dW_t.$

Proof. It follows by considering approximating sequences in $\mathcal{S}_T$. ■

Remark 10 *The previous reasonings to define the Itô integral for processes $h \in L^2_{a,T}$ and the previous proposition also follows from basic functional analysis results. Indeed, our approach has been to construct a bounded linear operator $\mathcal{I}$ defined on a set $\mathcal{S}_T$ of $L^2_{a,T}$ with values in the Hilbert space $L^2(\Omega, \mathcal{F}, P)$. A basic result in functional analysis states that you can uniquely extend the operator $\mathcal{I}$ to an operator $\overline{\mathcal{I}}$ from $\overline{\mathcal{S}_T}$ (the completion of $\mathcal{S}_T$ in the norm of $L^2_{a,T}$) to $L^2(\Omega, \mathcal{F}, P)$. Moreover, this extension preserves the operator norm , i.e., $\|\mathcal{I}\| = \|\overline{\mathcal{I}}\|$. By Lemma 7 we have that $\overline{\mathcal{S}_T} = L^2_{a,T}$. Hence, we have a linear extension $\overline{\mathcal{I}}$, which we will denote again by $\mathcal{I}$, defined on the whole $L^2_{a,T}$ and satisfying the Itô's isometry due to the preserving operator norm property of the extension.*

Example 11 *We have that $\int_0^T W_s dW_s = \frac{1}{2}(W_T^2 - T)$. Let $\{t_i^n = \frac{iT}{n}, i = 1, ..., n\}_{n\geq 1}$ be a sequence of partitions of the interval $[0, T]$. We choose as an approximating sequence*

$$h_t^n = \sum_{i=0}^{n-1} W_{t_i^n} \mathbf{1}_{(t_i^n, t_{i+1}^n]}.$$

We have that

$$\begin{aligned}
\mathbb{E}[\int_0^T |W_t - h_t^n|^2\, dt] &= \mathbb{E}[\sum_{i=0}^{n-1} \int_{t_i^n}^{t_{i+1}^n} \left|W_t - W_{t_i^n}\right|^2 dt] \\
&= \sum_{i=0}^{n-1} \int_{t_i^n}^{t_{i+1}^n} \mathbb{E}[\left|W_t - W_{t_i^n}\right|^2] dt \\
&= \sum_{i=0}^{n-1} \int_{t_i^n}^{t_{i+1}^n} t - t_i^n dt = \frac{1}{2}\sum_{i=0}^{n-1} (t_{i+1}^n - t_i^n)^2 \\
&\leq \frac{T}{2n} \sum_{i=0}^{n-1} (t_{i+1}^n - t_i^n) \leq \frac{T^2}{2n} \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Moreover, taking into account that

$$W_{t_i^n}(W_{t_{i+1}^n} - W_{t_i^n} = \frac{1}{2}\left\{(W^2_{t_{i+1}^n} - W^2_{t_i^n}) - (W_{t_{i+1}^n} - W_{t_i^n})^2\right\},$$

and the definition of stochastic integral

$$\begin{aligned}
\int_0^T W_s dW_s &= L^2\text{-}\lim_{n\to\infty} \sum_{i=0}^{n-1} W_{t_i^n}(W_{t_{i+1}^n} - W_{t_i^n}) \\
&= \frac{1}{2} L^2\text{-}\lim_{n\to\infty} \sum_{i=0}^{n-1} (W^2_{t_{i+1}^n} - W^2_{t_i^n}) - \frac{1}{2} L^2\text{-}\lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{t_{i+1}^n} - W_{t_i^n})^2 \\
&= \frac{1}{2} W_T^2 - \frac{1}{2} L^2\text{-}\lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{t_{i+1}^n} - W_{t_i^n})^2.
\end{aligned}$$

One can show that $L^2\text{-}\lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{t_{i+1}^n} - W_{t_i^n})^2 = T$ and we get that

$$\int_0^T W_s dW_s = \frac{1}{2}(W_T^2 - T).$$

From the previous example one sees that the definition of the Itô integral is not very practical. More interestingly, if $x(t)$ is a $C^1([0,T])$ then

$$\int_0^T x(t)dx(t) = \frac{x^2(T)}{2},$$

which is different from the Itô integral where a correction term $\frac{T}{2}$ appear. This additional term comes from the fact that the paths of the Brownian motion do not have finite variation with probability 1.

4 The Itô integral as a process

Consider $h \in L^2_{a,T}$. Then, for any $t \in [0,T]$ the process $h\mathbf{1}_{[0,t]} = \{h_s \mathbf{1}_{[0,t]}(s)\}_{0\leq s\leq T}$ also belongs to $L^2_{a,T}$ and, hence, we can define its stochastic integral by

$$\mathcal{I}_t[h] := \int_0^t h_s dW_s := \int_0^T h_s \mathbf{1}_{[0,t]}(s) dW_s.$$

We call the stochastic processes $\{\mathcal{I}_t[h]\}_{0\leq t\leq T}$, the **indefinite integral** of h with respect to W.

Theorem 12 (The Itô integral as a continuos martingale) *Let $h \in L^2_{a,T}$. Then, there exists a martingale M, with almost surely continuous paths, that satisfies*

$$P(M_t = \int_0^t h_s dW_s) = 1,$$

for each $t \in [0,T]$. That is, $\mathcal{I}_t[h]$ has a continuous version.

Proof. Consider a sequence $\{h^n\}_{n\geq 1}$ of simple processes approximating $h \in L^2_{a,T}$. Then, for every n, if $t^n_k \leq t \leq t^n_{k+1}$, we have

$$h^n_s \mathbf{1}_{[0,t]}(s) = \sum_{i=0}^{k-1} h^n_i \mathbf{1}_{(t_i,t_{i+1}]}(s) + h^n_k \mathbf{1}_{(t_k,t]}(s),$$

which is a simple process in $L^2_{a,t}$ (and also in $L^2_{a,T}$, you may add $0\mathbf{1}_{(t,T]}(s)$). Moreover, for n fixed

$$M^n_t := \int_0^t h^n_s dW_s = \sum_{i=0}^{k-1} h^n_i \Delta W_i + h^n_k (W_t - W_{t_k}), \qquad 0 \leq t \leq T,$$

is a process with continuous paths (because W has continuous paths) and it is a martingale (because it is the martingale transform of W by the *predictable* sequence $\{h^n_i\}_{i=0,\ldots,k}$). Hence, for every $n \geq 1$, we have that

$$\mathbb{E}[M^n_t | \mathcal{F}_s] = M^n_s, 0 \leq s < t \leq T$$

and, as for every $t \in [0,T]$, M^n_t converges in L^2 to $M_t := \int_0^t h_s dW_s$, we also have that $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$, P-a.s. Applying Doob's martingale inequality to the continuous martingale $M^n - M^m$ yields

$$\begin{aligned} P\left(\sup_{0\leq t\leq T} |M^n_t - M^m_t| > \lambda\right) &\leq \frac{1}{\lambda^2}\mathbb{E}[|M^n_T - M^m_T|^2] \\ &= \frac{1}{\lambda^2}\mathbb{E}[\int_0^T |h^n_t - h^m_t|^2 \, dt] \overset{n,m\to\infty}{\longrightarrow} 0. \end{aligned}$$

Hence, we can choose an increasing sequence of natural numbers $n_k, k = 1, 2, ...$ such that

$$P\left(\sup_{0\leq t\leq T} \left|M^{n_{k+1}}_t - M^{n_k}_t\right| > 2^{-k}\right) \leq 2^{-k}.$$

The events

$$A_k := \{\sup_{0\leq t\leq T} \left|M^{n_{k+1}}_t - M^{n_k}_t\right| > 2^{-k}\}$$

verify $\sum_{k=1}^{\infty} P(A_k) \leq \sum_{k=1}^{\infty} 2^{-k} = 1 < \infty$. By the Borel-Cantelli lemma

$$P(\omega \in A_k \text{ for infinitely many } k) = 0,$$

or

$$P(\omega \in A_k^c \text{ for all but finitely many } k) = 1$$

Hence, for ω in a set of probability 1, there exists $L(\omega) \in \mathbb{N}$ such that

$$\sup_{0\leq t\leq T} \left| M_t^{n_{k+1}}(\omega) - M_t^{n_k}(\omega) \right| \leq 2^{-k} \qquad \text{for } k \geq L(\omega).$$

This implies that $M_t^{n_k}(\omega)$ converges uniformly to a continuous function $J_t(\omega)$ on $[0,T]$ when $k \to \infty$, for ω in a set of probability 1. Since $M_t^{n_k}$ converges to M_t in L^2 for all $t \in [0,T]$, when $k \to \infty$, we must have that $\int_0^t h_s dW_s = M_t = J_t$,$P$-a.s. ■

Remark 13 *The Itô integral as a process has the following additivity property*

$$\int_a^b h_t dW_t + \int_b^c h_t dW_t = \int_a^c h_t dW_t.$$

5 Extensions

Itô's stochastic integral $\int_0^T h_s dW_s$ can be defined for classes of processes larger than $L^2_{a,T}$. One way is to relax the measurability conditions and the other way is to relax the integrability conditions.

5.1 First Extension

We can replace the minimal augmented filtration generated by W, $\mathbb{F}^W = \{\mathcal{F}_t^W\}_{t\in[0,T]}$, by a largest one $\mathbb{H} = \{\mathcal{H}_t\}_{t\in[0,T]}$ (satisfying the usual conditions) such that the Brownian motion W is a martingale respect to $\mathbb{H}$, i.e., the following property is satisfied

$$\mathbb{E}[W_t - W_s | \mathcal{H}_s] = 0.$$

If h is $\mathbb{H}$-adapted and W is an $\mathbb{H}$-martingale then the Itô integral $\int_0^T h_s dW_s$ exists and satisfies the same properties as in the previous section but now is an $\mathbb{H}$-martingale.

This extension is useful, for example, when $W = (W^1, ..., W^d)$ is a multidimensional Brownian motion. If h is a $p \times d$-dimensional matrix with components in $L^2_{a,T}$ we can construct the p-dimensional integral

$$\mathcal{I}_T[h] = \int_0^T h_t dW_t = \begin{pmatrix} \sum_{j=1}^d \int_0^T h_t^{1,j} dW_t^j \\ \vdots \\ \sum_{j=1}^d \int_0^T h_t^{p,j} dW_t^j \end{pmatrix},$$

where the filtration used in the definition is $\mathbb{F}^W$ the minimal augmented filtration generated by W. However, note that $\mathbb{F}^W$ is strictly larger than $\mathbb{F}^{W^i}$ but we can define integrals such as

$$\int_0^T W_t^2 dW_t^1 \qquad \text{or} \qquad \int_0^T \{\cos(W_t^2) + \sin(W_t^1)\} dW_t^1,$$

because they are thougth as integrals with respect to the larger filtration $\mathbb{F}$.

5.2 Second Extension*

The second extension consists in weakening the integrability assumption $\mathbb{E}\left[\int_0^T h_t^2 dt\right] < \infty$.

Definition 14 *Let $L^0_{a,T}$ the class of measurable and $\mathbb{F}$-adapted processes $h = \{h_t\}_{t\in[0,T]}$ such that*

$$P\left(\int_0^T h_t^2 dt < \infty\right) = 1. \tag{3}$$

The Itô integral can be defined for $h \in L^0_{a,T}$ using a localisation argument. This integral is an extension of the L^2 Itô integral because $L^2_{a,T} \subsetneq L^0_{a,T}$ and if $h \in L^2_{a,T}$ the integral coincides with the L^2 Itô integral. However, it does not need to be a martingale or even have finite moments of any order. Let us sketch this construction.

Assume that $h \in L^0_{a,T}$. Then, for each $n \geq 1$ we define the stopping times (see Lecture 4)

$$\tau_n = \inf \left\{ t \geq 0 : \int_0^t h_s^2 ds \geq n \right\},$$

and, by convention, $\tau_n = T$ if $\int_0^T h_s^2 ds < n$. The sequence $\{\tau_n\}_{n\geq 1}$ is nondecreasing and $\tau_n \uparrow T, P$-a.s. thanks to (3) . Note that

$$\tau_n > t \Longleftrightarrow \int_0^t h_s^2 ds < n,$$

and, hence, the process $h^n := \{h_t \mathbf{1}_{[0,\tau_n]}(t)\}$ belongs to $L^2_{a,T}$ since $\mathbb{E}\left[\int_0^T h_t^n dt\right] \leq n$ and the measurability and adaptedness requirements hold due to the fact that τ_n is a stopping time. In general, if $g \in L^2_{a,T}$ and τ is a stopping time, then the process $g\mathbf{1}_{[0,\tau]} \in L^2_{a,T}$ and

$$\int_0^\tau g_t dW_t := \int_0^T g_t \mathbf{1}_{[0,\tau]}(t) dW_t.$$

Hence, if $h \in L^0_{a,T}$ we can define

$$\int_0^t h_s dW_s = \int_0^t h_s^n dW_s = \int_0^t h_s \mathbf{1}_{[0,\tau_n]} dW_s, \qquad 0 \leq t \leq \tau_n,$$

and since $\tau_n \uparrow T, P$-a.s. we have an integral defined for every $t \in [0,T]$. We need to check that this definition is consistent. That is, if $m \geq n$, then on the set $\{t \leq \tau_n\}$ the integral with respect to h^m must coincide with the integral with respect to h^n. As $\tau_n \leq \tau_m, P$-a.s. we have that

$$\int_0^t h_s^m \mathbf{1}_{[0,\tau_n]} dW_s = \int_0^t h_s \mathbf{1}_{[0,\tau_m]} \mathbf{1}_{[0,\tau_n]} dW_s = \int_0^t h_s \mathbf{1}_{[0,\tau_n]} dW_s = \int_0^t h_s^n dW_s,$$

and we can conclude.

This integral is linear and has continuous paths, but it may have infinite expectation and variance. Instead of the isometry property there is a continuity property in probability. Namely, if $\{h^n\}_{n\geq 1}$ is sequence of processes in $L^0_{a,T}$ which converges to $h \in L^0_{a,T}$ in probability,i.e.

$$P\left(\int_0^T |h_t^n - h_t|^2 dt > \varepsilon \right) \underset{n\to\infty}{\longrightarrow} 0, \quad \forall \varepsilon > 0,$$

then

$$\int_0^T h_t^n dW_t \xrightarrow[n\to\infty]{P} \int_0^T h_t dW_t,$$

i.e,

$$P\left(\left| \int_0^T h_t^n dW_s - \int_0^T h_t dW_s \right| > \varepsilon \right) \underset{n\to\infty}{\longrightarrow} 0, \quad \forall \varepsilon > 0.$$

The proof is a trivial application of the following result.

Proposition 15 (Lenglart's Inequality) *Suppose that $h \in L^0_{a,T}$. For all $\varepsilon, \delta > 0$ we have*

$$P\left(\left| \int_0^T h_t dW_t \right| \geq \varepsilon \right) \leq P\left(\int_0^T h_t^2 dt \geq \delta \right) + \frac{\delta}{\varepsilon^2}.$$

Alternatively, we may have used Lenglart's inequality to construct the Itô integral of $h \in L^0_{a,T}$. One can show that for $h \in L^0_{a,T}$ there exists a sequence $\{h^n\}_{n\geq 1}$ of processes in $\mathcal{S}_T$ such that $\int_0^T h^n_t dt$ converges in probability to $\int_0^T h_t dt$. Using Lenglart's inequality one may define the $\int_0^T h_t dW_t$ as the limit in probability of $\int_0^T h^n_t dW_t$ and check that this limit does not depend on the approximating sequence.

Remark 16 *We have constructed the Itô integral for processes in $h \in L^0_{a,T}$ by localising h in $L^2_{a,T}$, i.e. by considering $h^n = h\mathbf{1}_{[0,\tau_n]}$. A by-product of this approach is that there exists a increasing sequence of stopping times $\{\tau_n\}_{n\geq 1}$ converging to T such that the stopped integral*

$$\int_0^{t\wedge\tau_n} h_s dW_s = \int_0^t h_s \mathbf{1}_{[0,\tau_n]} dW_s, \qquad t \in [0,T].$$

is a martingale. This is precisely the definition of local martingale.

References

[1] I. Karatzas and S.E. Shreve. *Brownian Motion and Stochastic Calculus.* Second Edition. Springer, New York. 1998.

[2] D. Nualart. *Stochastic Processes.* Unpublished notes. You can find it at https://www.math.ku.edu/~nualart.

[3] B. Øksendal. *Stochastic Differential Equations. An Introduction with Applications.* Springer, Berlin. 2010.