Stochastic Calculus

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1 Itô Processes

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Throughout this lecture W will denote a Wiener process (possibly d_W-dimensional) defined on $(\Omega, \mathcal{F}, P)$. The reference filtration $\mathbb{F}$ will be the minimal augmented filtration generated by W, i.e. $\mathbb{F}^W$.

Definition 1 *A stochastic process $X = \{X_t\}_{t\in[0,T]}$ that can be written as*

$$X_t = X_0 + \int_0^t g_s ds + \int_0^t h_s dW_s, \tag{1}$$

where $h, g \in L^2_{a,T}$ and $X_0 \in L^2(\Omega, \mathcal{F}_0, P)$ is called a L^2-Itô process.

Remark 2 *Note that, actually, as $\mathcal{F}_0 = \{\varnothing, \Omega\}$ the random variable X_0 is a constant. Moreover, one can prove that the representation (1) is unique, in the sense that if X can also be written as*

$$X_t = X_0' + \int_0^t g_s' ds + \int_0^t h_s' dW,$$

where $h', g' \in L^2_{a,T}$ and $X_0' \in L^2(\Omega, \mathcal{F}_0, P)$, then $h_t(\omega) = h_t'(\omega)$ and $g_t(\omega) = g_t'(\omega), \lambda \otimes P$-a.e and $X_0 = X_0'$.

An obvious extension of the previous definition is when W is a d_W-dimensional Brownian motion, X_0 is a d_X-dimensional random vector with components in $L^2(\Omega, \mathcal{F}_0, P)$, h is a $d_X \times d_W$ matrix and g is a d_X vector with components in $L^2_{a,T}$. Then, X is a d_X-dimensional vector with components given by

$$X_t^i = X_0^i + \int_0^t g_s^i ds + \sum_{j=1}^{d_W} \int_0^t h_s^{i,j} dW_s^j, \qquad i = 1, ..., d_X. \tag{2}$$

From a notational point of view, sometimes it is convenient to write the previous expressions in differential form

$$dX_t^i = g_s^i ds + \sum_{j=1}^{d_W} h_s^{i,j} dW_s^j, \qquad i = 1, ..., d_X.$$

However, keep in mind that the differential form is only a handy notation for the the integral form, which is the one with a sound mathematical meaning.

Remark 3 *The definition of an L^2-Itô process can be extended to $X_0 \in L^0(\Omega, \mathcal{F}_0, P), h \in L^0_{a,T}$ and g measurable and adapted process such that $\int_0^T |g_t| dt < \infty, P$-a.s.. We call such a process an Itô process.*

2 The Itô Formula

If we only could compute Itô integrals using their definition, the concept will be of limited applicability. The Itô integral has been a success because one can develop a calculus for it. That is, given an Itô process X and a regular function f, one can prove that the process $Y_t = f(t, X_t)$ is still an Itô process and we can find the explict representation of Y as an Itô process.

Theorem 4 (Itô Formula) *Let $f \in C^{1,2}([0,T] \times \mathbb{R})$, that is, $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$ are continuous functions and*

$$X_t = X_0 + \int_0^t g_s ds + \int_0^t h_s dW_s, \quad 0 \le t \le T.$$

an L^2-Itô process. Assume that

$$\mathbb{E}\left[\int_0^T \left\{ \left(\frac{\partial f}{\partial t}(t, X_t)\right)^2 + \left(g_t \frac{\partial f}{\partial x}(t, X_t)\right)^2 + \left(h_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t)\right)^2 \right\} dt\right] < \infty, \tag{3}$$

and

$$\mathbb{E}\left[\int_0^T \left(h_t \frac{\partial f}{\partial x}(t, X_t)\right)^2 dt\right] < \infty. \tag{4}$$

Then,

$$f(t, X_t) = f(0, X_0) + \int_0^t \left\{ \frac{\partial f}{\partial t}(s, X_s) + g_s \frac{\partial f}{\partial x}(s, X_s) + \frac{1}{2} h_s^2 \frac{\partial^2 f}{\partial x^2}(s, X_s) \right\} ds + \int_0^t h_s \frac{\partial f}{\partial x}(s, X_s) dW_s,$$

for $0 \le t \le T$ and it is also an L^2-Itô process.

Proof. We just give the heuristics for the case $X_t = W_t$. Let $\pi_n = \{0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t\}$ be a sequence of partitions of the interval $[0,t]$. From a Taylor expansion of f around the points in the partition π_n we get

$$\begin{aligned}
f(W_t) - f(W_0) &= \sum_{i=0}^{n-1} f(W_{t_{i+1}}) - f(W_{t_i}) \\
&= \sum_{i=0}^{n-1} f'(W_{t_i})(W_{t_{i+1}} - W_{t_i}) + \frac{1}{2} \sum_{i=0}^{n-1} f''(W_{t_i})(W_{t_{i+1}} - W_{t_i})^2 \\
&+ \frac{1}{2} \sum_{i=0}^{n-1} \left(f''(\xi_i) - f''(W_{t_i})\right)(W_{t_{i+1}} - W_{t_i})^2 \\
&=: A_1 + A_2 + A_3.
\end{aligned}$$

where ξ_i is a random variable in the random interval $[\min(W_{t_i}, W_{t_{i+1}}), \max(W_{t_i}, W_{t_{i+1}})]$. The term A_1 converges in L^2 to the Itô integral, the term A_2 converges in L^2 to $\int_0^t f''(W_s) ds$ and the term A_3 converges in L^2 to zero. ■

Some remarks are in order:

Remark 5 *1. Sometimes we will write the Itô formula in differential form, which reads*

$$\begin{aligned}
df(t, X_t) &= \left\{ \frac{\partial f}{\partial t}(t, X_t) + g_t \frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2} h_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) \right\} dt + h_t \frac{\partial f}{\partial x}(t, X_t) dW_t \\
&=: \frac{\partial f}{\partial t}(t, X_t) dt + \frac{\partial f}{\partial x}(t, X_t) dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t) (dX_t)^2,
\end{aligned}$$

where to write the last line we have used the so called Itô's product rule

$\times$	dt	dW_t
dt	0	0
dW_t	0	dt

.

This last expression is useful for mnemonic reasons and can be given a rigourous meaning using the concept of covariation between processes.

2. If we consider the stochastic integral for processes in $L^0_{a,T}$, that is, for an Itô process the previous theorem holds without requiring the integrability conditions (3) *and* (4)*.*

Example 6 *Let $f(x)=x^n, n\geq 2$ and $X_t=W_t$. Then, we obtain*

$$W_t^n = n\int_0^t W_s^{n-1}dW_s + \frac{n(n-1)}{2}\int_0^t W_s^{n-2}ds.$$

Theorem 7 *Let W be a d_W-dimensional Brownian motion and X a d_X-dimensional Itô process such as in* (2)*. Let f be a $\mathbb{R}^{d_Y}$- valued function with components in $C^{1,2}([0,T],\mathbb{R}^{d_X})$. Then the process $Y_t=f(t,X_t)$ has the following differential expression*

$$dY_t^k = f^i(0,X_0) + \frac{\partial f^k}{\partial t}(t,X_t)dt + \sum_{i=1}^{d_X}\frac{\partial f}{\partial x_i}(t,X_t)dX_t^i + \frac{1}{2}\sum_{i,j=1}^{d_X}\frac{\partial^2 f}{\partial x_i \partial x_j}(t,X_t)dX_t^i dX_t^j, \quad k=1,...,d_Y,$$

with the Itô product rules $dW_t^i dW_t^j=\delta_{ij}dt$ and $(dt)^2=dW_t^i dt = dtdW_t^i=0$.

As a consequence of this multidimensional Itô formula we can deduce the following useful result.

Proposition 8 (Integration by parts formula) *Let X and Y be two Itô processes.Then,*

$$d(X_tY_t) = X_sdY_s + Y_sdX_s + dX_sdY_s.$$

Example 9 *Let's find the Itô expression for the process tW_t. Thanks to the integration by parts formula we get*

$$d(tW_t) = tdW_t + W_tdt + dtdW_t = tdW_t + W_tdt,$$

which written in integral form reads

$$tW_t = 0W_0 + \int_0^t sdW_s + \int_0^t W_sds = \int_0^t sdW_s + \int_0^t W_sds.$$

3 The Martingale Representation Theorem

We assume that the sigma algebra $\mathcal{F}$ is equal to $\mathcal{F}_T=\mathcal{F}_T^W$. This assumption is crucial and the results presented in this section do not hold true, in general, if we allow $\mathcal{F}$ to contain more information that the one generated by W up to time T. The martingale representation theorem is one of the cornerstones of stochastic calculus and essentially says that any martingale, having enough integrability, with respect to the Brownian filtration can be expressed in an unique way as an stochastic integral with respect to this Brownian motion. This result paves the way of the martingale methods in option pricing.

The following is a technical lemma that we state without proof.

Lemma 10 *The linear span of random variables of the Dóleans exponential (or stochastic exponential) type, i.e., random variables of the form*

$$\exp\left(\int_0^T h_tdW_t - \frac{1}{2}\int_0^T h_t^2dt\right),$$

for a deterministic process $h\in L^2([0,T])$, is dense in $L^2(\Omega,\mathcal{F}_T,P)$.

Theorem 11 (Itô Representation Theorem) *Let $F\in L^2(\Omega,\mathcal{F}_T,P)$. Then, there exists a unique $h\in L^2_{a,T}$ such that*

$$F = \mathbb{E}[F] + \int_0^T h_tdW_t. \tag{5}$$

Proof. The idea is to prove first the result for F being of Dóleans exponential type and then extend it to any square integrable F by a density argument. Let $h \in L^2([0,T])$ deterministic. Assume that

$$F = \exp\left(\int_0^T h_t dW_t - \frac{1}{2}\int_0^T h_t^2 dt\right),$$

and consider the process

$$X_t = \exp\left(\int_0^t h_s dW_s - \frac{1}{2}\int_0^t h_s^2 ds\right).$$

Applying Itô's formula to the exponential function we obtain

$$dX_t = Y_t h_t dW_t + \left(-\frac{1}{2}Y_t h_t^2 + \frac{1}{2}Y_t h_t^2\right) dt = Y_t h_t dW_t,$$

or

$$X_t = 1 + \int_0^t X_s h_s dW_s, \quad t \in [0,T].$$

In particular,

$$F = 1 + \int_0^T X_t h_t dW_t,$$

and taking expectations we get that $\mathbb{E}[F] = 1$ because

$$\begin{aligned}\mathbb{E}\left[\int_0^T |X_t h_t|^2 dt\right] &= \int_0^T \mathbb{E}\left[|X_t|^2\right] |h_t|^2 dt = \int_0^T \exp\left(\int_0^t |h_s|^2 ds\right) |h_t|^2 dt \\ &\le \exp\left(\int_0^t |h_s|^2 ds\right) \int_0^T |h_t|^2 dt < \infty.\end{aligned}$$

and, therefore, the expectation of the Itô integral is zero (note that we have used that $\mathbb{E}\left[|X_t|^2\right] = \exp\left(\int_0^t |h_s|^2 ds\right)$, see exercises of List 2). For arbitrary F let $\{F_n\}_{n\ge1}$ be a sequence of linear combinations of Dóleans exponentials approximating F in $L^2(\Omega, \mathcal{F}_T, P)$. By linearity, the property (5) also holds for $\{F_n\}_{n\ge1}$. Then, for each n we have that there exists $h^n \in L^2_{a,T}$ such that

$$F_n = \mathbb{E}[F_n] + \int_0^T h_t^n dW_t. \tag{6}$$

Using (6), the linearity of the expectation, that the integrals has zero mean and the Itô isometry we get that

$$\begin{aligned}\mathbb{E}[(F_n - F_m)^2] &= \mathbb{E}\left[\left(\mathbb{E}[F_n] - \mathbb{E}[F_m] + \int_0^T (h_t^n - h_t^m) dW_t\right)^2\right] \\ &= \mathbb{E}[F_n - F_m]^2 + \mathbb{E}\left[\int_0^T |h_t^n - h_t^m|^2 dt\right] \\ &\ge \mathbb{E}\left[\int_0^T |h_t^n - h_t^m|^2 dt\right],\end{aligned}$$

As $\{F_n\}_{n\ge1}$ is a convergent sequence in $L^2(\Omega, \mathcal{F}_T, P)$ we have that $\{F_n\}_{n\ge1}$ is Cauchy and, hence,

$$\left[\int_0^T |h_t^n - h_t^m|^2 dt\right] \le \mathbb{E}[(F_n - F_m)^2] \underset{n,m\to\infty}{\longrightarrow} 0,$$

Therefore, we have proved that the sequence $\{h^n\}_{n\ge1}$ is a Cauchy sequence in $L^2(\Omega \times [0,T])$ which is a complete metric space. Consequently, it will converge to a process $h \in L^2(\Omega \times [0,T])$. We can choose a version of h which is adapted, because there exists a subsequence h^n which converges to h

for almost all $(\omega, t) \in \Omega \times [0,T]$. Therefore, $h_t(\cdot)$ is $\mathcal{F}_t$-measurable for a.e. $t \in [0,T]$ (that is true because the filtration satisfies the usual conditions) and changing h in set of measure zero on $[0,T]$ we can get h adapted. Finally, using Itô's isometry again we get that

$$F = L^2\text{- lim } F_n = L^2\text{- lim} \left(\mathbb{E}[F_n] + \int_0^T h_t^n dW_t \right) = \mathbb{E}[F] + \int_0^T h_t dW_t.$$

The uniqueness is also an easy consequence of Itô isometry and the fact that a positive process, say u, such that $\mathbb{E}[\int_0^T u_s ds] = 0$ must be $P \otimes \lambda$-a.e. zero. ■

From the Itô's representation theorem one can show the martingale representation theorem.

Theorem 12 (Martingale Representation Theorem) *Let $M = \{M_t\}_{t\in[0,T]}$ be a square integrable $\mathbb{F}$-martingale. Then, there exists a unique $h \in L^2_{a,T}$ such that*

$$M_t = \mathbb{E}[M_0] + \int_0^t h_s dW_s.$$

Proof. Note that $M_t = \mathbb{E}[M_T|\mathcal{F}_t]$. Applying Itô's representation theorem to $M_T \in L^2(\Omega, \mathcal{F}_T, P)$ we have that there exists a unique $h \in L^2_{a,T}$ such that $M_T = \mathbb{E}[M_0] + \int_0^T h_s dW_s$. Hence,

$$\begin{aligned} M_t = \mathbb{E}[M_T|\mathcal{F}_t] &= \mathbb{E}\left[\mathbb{E}[M_0] + \int_0^T h_s dW_s | \mathcal{F}_t \right] \\ &= \mathbb{E}[M_0] + \mathbb{E}\left[\int_0^T h_s dW_s | \mathcal{F}_t \right] \\ &= \mathbb{E}[M_0] + \int_0^t h_s dW_s, \end{aligned}$$

where in the last equality we have used that the stochastic integral, as a process, is an $\mathbb{F}$-martingale. ■

Example 13 *Consider the random variable W_T^3. Then, by example* (6) *we have that*

$$W_T^3 = 3\int_0^T W_t^2 dW_t + 3\int_0^T W_t dt.$$

Moreover, by example (9) *we have that*

$$\int_0^T W_t dt = TW_T - \int_0^T t dW_t = \int_0^T (T-t)\, dW_t.$$

Therefore, we can conclude that

$$W_T^3 = \int_0^T 3\left(W_t^2 + (T-t)\right) dW_t$$

All the results of this section also hold when the filtration is generated by a multidimensional Brownian motion. They can even be extended to random variables that are not square integrable using the Itô integral for processes in $L^0_{a,T}$, but then we loss the unicity in the representation.

4 Girsanov's Theorem

Girsanov's theorem basically states that if we add a drift term to a Brownian motion then the new process does not change much in the sense that it is a Brownian motion but under a different probability measure.

The prof of Girsanov's theorem is based on the following characterization of Brownian motion that we state without proof.

Theorem 14 (Lévy's characterization of Brownian motion) *Let $X = \{X_t\}_{t\in\mathbb{R}_+}$ be a real valued process with continuous paths and $\mathbb{F} = \mathbb{F}^X$ the minimal augmented filtration generated by X. The following two statements are equivalent:*

1. *X is a $\mathbb{F}$-Brownian motion.*
2. *X and $X^2 - t$ are both $\mathbb{F}$-martingales.*

The following lemma is a version of the Bayes' theorem. The proof is similar to the proof of exercise 32. in List 1.

Lemma 15 *Let Q and P be two probability measures on some measurable space $(\Omega, \mathcal{F})$ such that $Q \ll P$. Let $X \in L^1(\Omega, \mathcal{F}, Q)$ and $\mathcal{H} \subset \mathcal{F}$ a sub-σ-algebra of $\mathcal{F}$. Then,*

$$\mathbb{E}_Q[X|\mathcal{H}]\mathbb{E}_P\left[\frac{dQ}{dP}|\mathcal{H}\right] = \mathbb{E}_P\left[X\frac{dQ}{dP}|\mathcal{H}\right].$$

Theorem 16 (Girsanov) *Let Y be an Itô process of the form*

$$dY_t = g_t dt + dW_t, \quad Y_0 = 0, \quad 0 \le t \le T.$$

Set

$$M_t = \exp\left(-\int_0^t g_s dW_s - \frac{1}{2}\int_0^t g_s^2 ds\right), \quad 0 \le t \le T$$

Assume that M is a martingale with respect to P and define the measure Q on $\mathcal{F}_T$ by $\frac{dQ}{dP} = M_T$. Then Q is a probability measure on $\mathcal{F}_T$ and Y is a Brownian motion under Q.

Proof. The idea is to use Lévy's characterization theorem. We have to check that Y and $Y^2 - t$ are martingales under Q. Define $K_t = M_t Y_t$. By the integration by parts formula we have that

$$\begin{aligned} dK_t &= M_t dY_t + Y_t dM_t + dM_t dY_t \\ &= M_t(g_t dt + dW_t) - Y_t\left(M_t g_t dW_t\right) - dW_t\left(M_t g_t dW_t\right) \\ &= M_t(1 - Y_t g_t)dW_t. \end{aligned}$$

Hence, K_t is a martingale under P. By Lemma 15, we get that

$$\mathbb{E}_Q[Y_t|\mathcal{F}_s] = \frac{\mathbb{E}_P[M_t Y_t|\mathcal{F}_s]}{\mathbb{E}_P[M_t|\mathcal{F}_s]} = \frac{\mathbb{E}_P[K_t|\mathcal{F}_s]}{M_s} = \frac{K_s}{M_s} = Y_s,$$

so we have proved that Y is a martingale under Q. The proof that $Y^2 - t$ is a martingale under Q is similar. ■

The delicate point in Girsanov's Theorem is the assumption that M is a martingale. In order to check this assumption one can use the following result.

Lemma 17 (Novikov's) *Let M be as in Girsanov's theorem. If*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T |g_t|^2 dt\right)\right] < \infty,$$

then M is a martingale and $\mathbb{E}[M_t] = \mathbb{E}[M_0] = 1$.

5 Stochastic Differential Equations (SDEs)

A particularly useful type of Itô processes are solutions of SDEs. Let $b^i(t,x), \sigma_{ij}(t,x); i = 1,...,d_X, j = 1,...,d_W$, be Borel measurable functions from $\mathbb{R}_+ \times \mathbb{R}^{d_X}$ to $\mathbb{R}$. Define the drift vector $b(t,x) = \{b^i(t,x)\}_{i=1,...,d_X}$ and the dispersion matrix $\sigma(t,x) = \{\sigma_{ij}(t,x)\}_{i=1,...,d_X, j=1,...,d_W}$. The goal of this section is to give a meaning to the stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \tag{7}$$

written componentwise as

$$dX_t^i = b^i(t, X_t)ds + \sum_{j=1}^{d_W} \sigma_{ij}(t, X_t)dW_t^j, \quad i = 1, ..., d_X,$$

where $W = \{W_t^1, ..., W_t^{d_W}\}$ is Brownian motion and $X = \{X_t^1, ..., X_t^{d_X}\}$ is a suitable stochastic process with continuous process with continuous sample paths, the "solution" of the equation. The drift vector $b(t, x)$ and the dispersion matrix $\sigma(t, x)$ are the coefficients of the equation and the $d_X \times d_X$-matrix $a(t, x) := \sigma(t, x)\sigma^T(t, x)$ with elements

$$a_{ij}(t, x) = \sum_{k=1}^{d_W} \sigma_{ik}(t, x)\sigma_{jk}(t, x),$$

is called the diffusion matrix.

Definition 18 *Let $(\Omega, \mathcal{F}, P)$ be a probability space with a d_W-dimensional Brownian motion W defined on it and consider $\mathbb{F}$ the augmented filtration of W. A strong solution, on $(\Omega, \mathcal{F}, P)$ and with respect to the fixed Brownian motion W, of the equation* (7) *is a process $X = \{X_t\}_{t\in[0,T]}$ with continuous sample paths and satisfying*

1. *X is $\mathbb{F}$-adapted.*
2. *$X_0 = x \in \mathbb{R}^d$.*
3. *$P\left(\int_0^t \left|b^i(s, X_s)\right| + \left|\sigma_{ij}(s, X_s)\right|^2 ds < \infty\right) = 1$ holds for every $i = 1, ..., d_X, j = 1, ..., d_W$ and $t \in [0, T]$.*
4. *The integral version of* (7) *holds, i.e,*

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad P\text{-a.s.,}.$$

The following theorem gives sufficient conditions on b and σ to guarantee the existence and uniqueness of a solution of equation (7)*. From now on $\|\cdot\|$ will denote the euclidean norm of the the appropriate dimension.*

Theorem 19 (Existence and Uniqueness) *Let the coefficients b and σ be continuous functions satisfying*

1. *(Lipschitz coefficients) $\|b(t, x) - b(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \leq K \|x - y\|$,*
2. *(Linear growth) $\|b(t, x)\|^2 + \|\sigma(t, x)\|^2 \leq K^2(1 + \|x\|^2)$, for all $x \in \mathbb{R}$,*

for all $t \in \mathbb{R}_+, x, y \in \mathbb{R}^{d_X}$ and a constant $K > 0$ (where $\|\cdot\|$ denotes the Euclidean norm of suitable dimension). Then, for any $T > 0$ there exists a (unique up to indistinguishability) strong solution X of (7) *in the interval $[0, T]$. Moreover, this solution X satisfies for fixed $m \geq 1$*

$$\mathbb{E}\left[\sup_{0\leq s\leq t} \|X_s\|^{2m}\right] < Ce^{Ct}\left(1 + \|x\|^{2m}\right),$$

for all $t \in [0, T]$ and a suitable constant C.

5.1 The Markov Property of the Solution of a SDE

We shall denote by $X^{s,x} = \{X_t^{s,x}\}_{t\in[s,T]}$ the unique solution of equation (7) starting from x at time s. That is, $X^{s,x}$ satisfies

$$X_t^{s,x} = x + \int_s^t b(u, X_u^{s,x})du + \int_s^t \sigma(u, X_u^{s,x})dW_u, \quad t \in [s, T].$$

Moreover, we will denote by $X = X^x = X^{0,x}$ the solution of equation (7) starting from x at time 0. Using the uniqueness of the solution of equation (7) one can deduce the following *flow* property of the solution

$$X_t^x = X_t^{s,X_s^x}, \quad P\text{-a.s.} \quad t \in [s,T].$$

From the flow property one can prove the following theorem.

Theorem 20 (Markov property of SDEs) *Assume that the hypothesis of Theorem 19 hold. Then, X is a Markov process with respect to the filtration $\mathbb{F}$. Furthermore, for any Borel measurable function f such that $\mathbb{E}[|f(X_t)|] < \infty, t \in [0,T]$ we have*

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] = \psi(X_s),$$

where $s \leq t$ and $\psi(x) = \mathbb{E}[f(X_t^{s,x})]$. The previous equality is often written as

$$\mathbb{E}\left[f(X_t)|\mathcal{F}_s\right] = \mathbb{E}\left[f(X_t^{s,x})\right]|_{x=X_s}.$$

Remark 21 *If X is time homogenous, that is, the coefficients $b(t,x) = b(x)$ and $\sigma(t,x) = \sigma(x)$ do not depend on time, then the markov property can be written as*

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] = \mathbb{E}\left[f(X_{t-s}^x)\right]|_{x=X_s}.$$

5.2 The Feynman-Kac Representation

Associated to the coefficients (b,σ) of an SDE we can define the following second order differential operator

$$A_t f(x) = \frac{1}{2}\sum_{i=1}^{d_X}\sum_{j=1}^{d_X} a_{ij}(t,x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d_X} b^i(t,x)\frac{\partial f}{\partial x_i}(x), \qquad f \in C^2(\mathbb{R}^{d_X}),$$

where the matrix a is the diffusion matrix $a = \sigma(t,x)\sigma^T(t,x)$. We call this operator the characteristic operator of (b,σ). We now consider the following Cauchy problem corresponding to A_t.

Problem 22 *Find a function $v(t,x) : [0,T] \times \mathbb{R}^{d_X} \to \mathbb{R}$ with*

$$-\frac{\partial v}{\partial t}(t,x) + k(t,x)v(t,x) = A_t v(t,x), \qquad \text{on } [0,T) \times \mathbb{R}^{d_X}$$
$$v(T,x) = f(x), \qquad \text{for } x \in \mathbb{R}^{d_X},$$

where $k : [0,T] \times \mathbb{R}^{d_X} \to \mathbb{R}_+$ is continuous and $f : \mathbb{R}^{d_X} \to \mathbb{R}$ is also continuous and satisfy

$$|f(x)| \leq L(1 + \|x\|^{2\lambda}), \qquad L > 0, \lambda \geq 1 \qquad \text{or} \qquad f(x) \geq 0.$$

Theorem 23 (Feynman-Kac Representation) *Let $v : [0,T] \times \mathbb{R}^{d_X} \to \mathbb{R}$ be a continuous solution of the Cauchy problem (22) with $v \in C^{1,2}\left([0,T) \times \mathbb{R}^{d_X}\right)$ and satisfying the following polynomial growth condition*

$$\sup_{0 \leq t \leq T} |v(t,x)| \leq M(1 + \|x\|^{2\mu}) \qquad \text{with} \quad M > 0, \mu \geq 1.$$

Assume that A_t is the characteristic operator of (b,σ) satisfying the hyphotesis of Theorem 19. Then $v(t,x)$ admits the following stochastic representation

$$v(t,x) = \mathbb{E}\left[f(X_T^x)\exp\left(-\int_t^T k(s,X_s^x)ds\right)\middle|\mathcal{F}_t\right] = \mathbb{E}\left[f(X_T^{t,x})\exp\left(-\int_t^T k(s,X_s^{t,x})ds\right)\right]$$

on $[0,T] \times \mathbb{R}^{d_X}$. In particular, such a solution is unique.

Proof. We sketch the main idea. As $v(t,x)$ is $C^{1,2}$ we can apply Itô's formula to

$$\exp\left(-\int_0^t k(s,X_s)ds\right)v(t,X_t)$$

to get that the process

$$\begin{aligned} M_t &:= \exp\left(-\int_0^t k(s,X_s)ds\right)v(t,X_t) \\ &\quad -\int_0^t \exp\left(-\int_0^s k(r,X_r)dr\right)\left(\frac{\partial v}{\partial t}(s,X_s)+A_s v(s,X_s)-k(s,X_s)v(s,X_s)\right)ds, \end{aligned}$$

is equal to the stochastic integral

$$\sum_{j=1}^{d_W}\int_0^t \exp\left(-\int_0^s k(r,X_r)dr\right)\sum_{i=1}^{d_X}\frac{\partial v}{\partial x_i}(s,X_s)\sigma_{ij}(s,X_s)dW_s^j.$$

As a consequence, if

$$\mathbb{E}\left[\int_0^T\left|\exp\left(-\int_0^s k(r,X_r)dr\right)\frac{\partial v}{\partial x_i}(s,X_s)\sigma_{ij}(s,X_s)\right|^2 ds\right]<\infty,\qquad i=1,...,d_X, j=1,...,d_W,$$

then M_t is a martingale and if $v(t,x)$ satisfies the Cauchy problem we get that

$$M_t=\exp\left(-\int_0^t k(s,X_s)ds\right)v(t,X_t),$$

is a martingale. Hence, by the martingale property of M_t we can write

$$\exp\left(-\int_0^t k(s,X_s)ds\right)v(t,X_t)=\mathbb{E}\left[\exp\left(-\int_0^T k(s,X_s)ds\right)v(T,X_T)\middle|\mathcal{F}_t\right],$$

which yields

$$\begin{aligned} v(t,X_t) &= \mathbb{E}\left[\exp\left(-\int_t^T k(s,X_s)ds\right)v(T,X_T)\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[f(X_T^{t,x})\exp\left(-\int_t^T k(s,X_s^{t,x})ds\right)\right], \end{aligned}$$

by the Markov property of $X=X^x$. ■

Remark 24 *Note that in the previous theorem we have assumed that there exists a solution to the Cauchy problem satisfying certain regularity conditions and polynomial growth. If we can show that such solution exists then it is given by the above expectation and it is unique. This is done by calculating the above expectation, which depends on the parameters (t,x), and showing that it solves the PDE and it satisfies the required regularity and growth conditions. However, the previous theorem does not resolve, in general, wether or not a solution to the Cauchy problem actually exists because the previous approach may fail, that is, the above expectation may fail to have the required regularity or growth condition. Another interesting use of this stochastic representation formula is to numerically compute the solution (when it does exist) by Monte Carlo methods.*

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