The Black-Scholes Model

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1 Modeling Assumptions

The Black-Scholes (BS) model consists in a financial market where there are two assets, one risky asset (the stock) and one riskless asset (the bank account). The investors in this model can trade continuously in this market within an investment horizon $[0, T_*]$. Assume that we introduce a third asset, called contingent claim, that pays the owner $H \geq 0$ at a future time $T \leq T^*$ depends on the risky asset. The goal is to give a "reasonable" price $\Pi_t(H)$ for this new asset for any $t \in [0, T]$. We will show that an investor can replicate the value of the contingent claim by continuously rebalancing a certain type of portfolios involving the riskless and risky assets. This replicating portfolio naturally gives the price of the contingent claim by non-arbitrage arguments.

1.1 Prices

The price of the riskless asset, denoted by $B = \{B_t\}_{t \in [0,T]}$, is modeled by a continuous time deterministic process satisfying the following ODE.

$$\begin{aligned} dB_t &= rB_t dt, \\ B_0 &= 1, \end{aligned}$$

where $r \in \mathbb{R}_+$ is called the risk-free interest rate. Note that, $B_t = e^{rt}, 0 \leq t \leq T$. The riskless asset represents savings account, which continuously compound in value at rate r.

The price of the risky asset, denoted by $S = \{S_t\}_{t \in [0,T]}$, is modeled by a continuous time stochastic process satisfying the SDE

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t, \qquad t \in [0, T], \\ S_0 &= S_0 > 0. \end{aligned} \tag{1}$$

Applying Itô's formula, one can check that the process

$$S_t = f(t, W_t) = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right),$$

satisfies the SDE (1) . Therefore, S_t is a geometric Brownian motion with drift $\mu - \frac{\sigma^2}{2}$ and volatility σ. Let us summarize the underlying hypothesis of the BS model on the prices of assets.

- The assets are traded continuously and their prices have continuous paths.
- The risk-free interest rate r is constant.
- The logreturns of the risky asset S_t follows a Brownian motion with drift:

  $$\log\left(\frac{S_t}{S_u}\right) = \left(\mu - \frac{\sigma^2}{2}\right)(t - u) + \sigma\left(W_t - W_u\right),$$

  and, hence, they are stationary.
- The risky asset does not give dividends.

1.2 Trading Strategies

Definition 1 *A portfolio or trading strategy is a measurable and adapted stochastic process* $\phi = \{(\phi_t^0, \phi_t^1)\}_{t\in[0,T]}$, *where* ϕ_t^0 *and* ϕ_t^1 *are the number of units invested at time* t *in the assets* B *and* S, *respectively. The value of the portfolio is the stochastic process* $V(\phi) = \{V_t(\phi)\}_{t\in[0,T]}$ *given by*

$$V_t(\phi) = \phi_t^0 B_t + \phi_t^1 S_t = \phi_t^0 e^{rt} + \phi_t^1 S_t. \tag{2}$$

Note that a portfolio is adapted to the filtration generated by the driving Brownian motion W, which for this model coincides with the filtration generated by the price process S. Hence, the investor is not allowed to have knowledge of future prices when taking its investment decisions. Also note that if $\phi = \{\phi^0, \phi^1\}$ is a portfolio, once we have set a value for ϕ^1, the value for ϕ^0 is automatically determined by the expression

$$\phi^0 = B_t^{-1}(V_t(\phi) - \phi_t^1 S_t) = e^{-rt}(V_t(\phi) - \phi_t^1 S_t),$$

which is adapted. Finally, there is no positivity restriction on ϕ^0 and ϕ^1, which means that we can borrow money or short-sell the risky asset.

Definition 2 *A portfolio is called self-financing if its value is an Itô process satisfying*

$$dV_t(\phi) = \phi_t^0 dB_t + \phi_t^1 dS_t \tag{3}$$

or

$$\begin{aligned} V_t(\phi) &= \phi_0^0 + \phi_0^1 S_0 + \int_0^t \phi_s^0 dB_s + \int_0^t \phi_s^1 dW_s \\ &=: \phi_0^0 + \phi_0^1 S_0 + G_t(\phi), \end{aligned}$$

where $G_t(\phi)$ *is called the gains process.*

Formula (3) says that the instantaneous changes in the portfolio value are completely determined by the price changes of the underlying assets. Financially, this means that we do not withdrawn or add any funds to the portfolio during the trading period. We just set an initial investment $\phi_0^0 + \phi_0^1 S_0$ and from then on the portfolio value $V_t(\phi)$ only changes due to the trading gains, given by the gains process $G_t(\phi)$.

Definition 3 *A self-financing trading strategy* ϕ *is an arbitrage opportunity if* $V_0(\phi) \le 0$ *and* $V_T(\phi) \ge 0$ *with* $P(V_T(\phi) > 0) > 0$ *(or equivalently* $\mathbb{E}[V_T(\phi)] > 0$.*)*

Financially speaking an arbitrage is the opportunity of earning money from a zero (or even negative) investment without taking risk. In liquid markets such opportunities do not exist or only exist temporarily, because market participants eliminate them by trading. Hence, we must ensure that our mathematical model does not allow for arbitrages. If we just consider self-financing portfolios there exists some portfolios, called doubling strategies, that produce arbitrage opportunities. The following requirement eliminates these portfolios.

Definition 4 *A self-financing trading strategy* ϕ *is called admissible if the discounted value of the portfolio* $\tilde{V}_t(\phi) \triangleq e^{-rt}V_t(\phi)$ *is a martingale under the measure* Q *defined by the following Radon-Nykodim derivative with respect to* P

$$\frac{dQ}{dP} = \exp\left(-\frac{\mu - r}{\sigma}W_T - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T\right).$$

In the next lecture, we will clarify this choice for the class of admissible portfolios as we will prove that in the Black-Scholes market there are no arbitrage opportunities within this class.

Remark 5 *Using Hölder's inequality and the fact that a normal random variable has exponential moments of all orders one can prove that a sufficient condition for* ϕ^1 *to be admissible is that* $\phi^1 \in L_{a,T}^{2+\varepsilon}$, *for some* $\varepsilon > 0$. *That is,* ϕ^1 *is measurable, adapted and*

$$\mathbb{E}\left[\int_0^T \left|\phi_t^1\right|^{2+\varepsilon} dt\right] < \infty,$$

where the expectation is under the measure P.

Let's summarize some of the assumption of the Black-Scholes model regarding the trading strategies:

- Trading is done continuously in time
- We can buy or sell any fraction of the riskless and risky assets.
- The portfolios are self-financing.
- The portfolios must satisfy some credit constraint.
- The market is arbitrage free.
- There are no transaction costs.

1.3 Contingent Claims

Definition 6 *A T-contingent claim is a financial contract that pays the holder a nonnegative random amount H at time T, which is called the exercise time. The random variable H is assumed to be $\mathcal{F}_T$-measurable, with $\mathbb{E}[H^{2+\varepsilon}] < \infty$ for some $\varepsilon > 0$.*

Definition 7 *A contingent claim H is replicable or attainable if there exists a self-financing admissible trading strategy ϕ such that $V_T(\phi) = H, P$-a.s.. We call such ϕ the replicating or hedging strategy of H.*

Our goal is to give an arbitrage free price for T-contingent claims. The following result is crucial.

Theorem 8 (Arbitrage-free pricing) *Assume that we can buy or sell in the market, at any time $0 \leq t < T$, a replicable contingent claim H. Then, the only arbitrage free price of H at time t, denoted by $\pi_t(H)$, is given by $V_t(\phi)$, the value of the replicating portfolio at time t.*

Proof. Suppose that $\pi_t(H) < V_t(\phi)$, then you can buy the contingent claim, sell the replicating portfolio and invest the remaining amount on the risk-free asset. If $\pi_t(H) > V_t(\phi)$ you can sell the contingent claim, buy the replicating portfolio and invest the remaining amount on the risk-free asset. At time T you will have a risk-free profit of $|\pi_t(\phi) - V_t(\phi)|e^{r(T-t)}$. Hence, the only possibility to avoid arbitrage is to have $\pi_t(H) = V_t(\phi)$,P-a.s.,$0 \leq t \leq T$. ∎

Definition 9 *We say that a financial market is complete if any contingent claim is replicable.*

It can be proved that the Black-Scholes model is a complete market. We will discuss further on this issue the next lecture. Note also that in complete markets contingent claims are redundant assets. This means that there is no economic reason to include them as an additional asset in the market. Of course, they are redundant because one always can consider a replicating portfolio instead of the contingent claim.

Let's summarize the hypothesis on the Black-Scholes model regarding contingent claims:

- The market is complete.

2 The Black-Scholes PDE

We derive the partial differential equation (PDE) that must satisfy the price process $\pi_t(H)$ of a replicable contingent claim H of the form $H = h(S_T)$. By Theorem 8, $\pi_t(H)$ must be equal to $V_t(\phi)$, the value of the replicating portfolio.

Theorem 10 *Let $H = h(S_T)$ be a replicable contingent claim and $\phi = (\phi^0, \phi^1)$ its replicating portfolio. Assume that the value of the portfolio takes the form $V_t(\phi) = f(t, S_t)$ for some $f \in C^{1,2}([0,T) \times \mathbb{R}_+)$. Then, the function $f(t,x)$ satisfies the Black-Scholes PDE*

$$\frac{\partial f}{\partial t}(t,x) + rx\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(t,x) = rf(t,x), \qquad (t,x) \in [0,T) \times \mathbb{R}_+ \tag{4}$$
$$f(T,x) = h(x), \qquad x \in \mathbb{R},$$

and ϕ is given by

$$(\phi_t^0, \phi_t^1) = \left(e^{-rt}\left\{f(t,S_t) - S_t\frac{\partial f}{\partial x}(t,S_t)\right\}, \frac{\partial f}{\partial x}(t,S_t)\right), \qquad t\in[0,T].$$

Proof. First we find the representation of $V(\phi)$ as an Itô process using the self-financing property. We get that

$$\begin{aligned} dV_t(\phi) &= \phi_t^0 dB_t + \phi_t^1 dS_t = \phi_t^0 rB_t dt + \phi_t^1\left(\mu S_t dt + \sigma S_t dW_t\right) \\ &= \left(\phi_t^0 re^{rt} + \phi_t^1 \mu S_t\right)dt + \phi_t^1 \sigma S_t dW_t. \end{aligned}$$

Secondly, we apply Itô's formula to $f(t,S_t)$ to get an alternative representation of $V(\phi)$ as an Itô process. We have that

$$\begin{aligned} df(t,S_t) &= \frac{\partial f}{\partial t}(t,S_t)dt + \frac{\partial f}{\partial x}(t,S_t)dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,S_t)(dS_t)^2 \\ &= \left(\frac{\partial f}{\partial t}(t,S_t) + \mu S_t\frac{\partial f}{\partial x}(t,S_t)dt + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma S_t\frac{\partial f}{\partial x}(t,S_t)dW_t. \end{aligned}$$

As the representation of a process as an Itô process is unique, we can identify the terms multiplying dW_t in the expressions for $dV_t(\phi)$ and $df(t,S_t)$ to get that

$$\sigma S_t\frac{\partial f}{\partial x}(t,S_t) = \phi_t^1\sigma S_t,$$

which yields that $\phi_t^1 = \frac{\partial f}{\partial x}(t,S_t), t\in[0,T]$. Note that, given the previous expression for ϕ_t^1, we can write ϕ_t^0 in terms of $S_t, f(t,S_t)$ and $\frac{\partial f}{\partial x}(t,S_t)$, namely

$$\phi_t^0 = B_t^{-1}\left(V_t(\phi) - \phi_t^1 S_t\right) = e^{-rt}\left(f(t,S_t) - S_t\frac{\partial f}{\partial x}(t,S_t)\right).$$

The next step is to identify the terms multiplying dt and using the previous expressions for ϕ_t^1 and ϕ_t^0 in terms of $S_t, f(t,S_t)$ and $\frac{\partial f}{\partial x}(t,S_t)$.We obtain that

$$\begin{aligned} \frac{\partial f}{\partial t}(t,S_t) + \mu S_t\frac{\partial f}{\partial x}(t,S_t)dt + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial x^2} &= \phi_t^0 re^{rt} + \phi_t^1\mu S_t \\ &= \frac{f(t,S_t) - S_t\frac{\partial f}{\partial x}(t,S_t)}{e^{rt}}re^{rt} + \frac{\partial f}{\partial x}(t,S_t)\mu S_t, \end{aligned}$$

and carrying out the simplifications we get

$$\frac{\partial f}{\partial t}(t,S_t) + rS_t\frac{\partial f}{\partial x}(t,S_t)dt + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial x^2} = rf(t,S_t). \tag{5}$$

As the law of S_t is absolutely continuous with respect to the Lebesgue measure and its support is $(0,+\infty)$ we can conclude that equation (5) holds for all $x\in(0,+\infty)$. ■

3 Solution of the Black-Scholes PDE

At first sight, solving the Black-Scholes PDE (BS-PDE) seems a quite hard task. However, by a suitable change of variables the BS-PDE can be reduced to the heat equation, possibly the most well known PDE around. Nevertheless, our approach for solving the BS-PDE will be based on the Feynman-Kac representation formula. This representation will reveal an interesting feature of the solution, namely, that it can be obtained as an expectation of a geometric Brownian motion with drift given by the riskless rate r instead of the drift μ. This suggests the martingale approach or risk-neutral pricing presented on the next lecture.

Consider the SDE

$$\begin{aligned} dZ_t &= rZ_t dt + \sigma Z_t dW_t, \\ Z_0 &= S_0, \end{aligned}$$

which has the characteristic operator

$$A_t f(x) = Af(x) = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(x) + rx\frac{\partial f}{\partial x}(x).$$

We can set the following Cauchy problem

$$\begin{aligned} -\frac{\partial f}{\partial t}(t,x) + k(t,x)f(t,x) = A_t f(x), \qquad (t,x) \in [0,T) \times \mathbb{R}, \\ f(T,x) = h(x), \qquad x \in \mathbb{R}, \end{aligned}$$

which setting $k(t,x) = r$ and reordering terms yields the BS-PDE

$$\begin{aligned} \frac{\partial f}{\partial t}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(x) + rx\frac{\partial f}{\partial x}(x) = rf(t,x) \qquad \text{on } [0,T) \times \mathbb{R} \\ f(T,x) = h(x), \qquad \text{for } x \in \mathbb{R}, \end{aligned}$$

Then, according to the Feynman-Kac representation theorem a good candidate for solving the BS-PDE is

$$f(t,x) = \mathbb{E}[e^{-r(T-t)}h(Z_T^{t,x})].$$

As $Z^{t,x}$ is a geometric Brownian motion with drift $r - \frac{\sigma^2}{2}$ and volatility σ starting at x at time t, we have that

$$Z_s^{t,x} = x\exp\left(\left(r - \frac{\sigma^2}{2}\right)(s-t) + \sigma(W_s - W_t)\right), \qquad s \in [t,T].$$

Therefore, $\log Z_T^{t,x}$ is a $\mathcal{N}\left(\log(x) + \left(r - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right)$ and we can write

$$f(t,x) = \int_{-\infty}^{+\infty} h(\exp(z))\,\phi\left(z; \log(x) + \left(r - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right) dz, \tag{6}$$

where $\phi(z;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{(z-\mu)^2}{2\sigma^2})$ or

$$f(t,x) = \int_{-\infty}^{+\infty} h\left(\exp\left(\log(x) + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\sqrt{T-t}z\right)\right)\phi(z)dz, \tag{7}$$

where $\phi(z) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{z^2}{2})$. The expression (7) is more convenient to compute $f(t,x)$ for some simple functions h (call and put options) because the solution can be expressed in terms of the cumulative normal distribution function. As the normal distribution has exponential moments of any order (Exercise 20, List 1) we get that if h has polynomial growth then $f(t,x)$ has polynomial growth. To show that $f(t,x)$ is smooth in (t,x) is more delicate. To justify the interchange between the derivative and the expectation one can use the dominated convergence theorem. However, to compute derivatives with respect to t,x or other parameters, one must use expression (6) . This is due to the fact that h does not need to be regular. This is known as the density approach for computing the sensitivity parameters, because we have transferred the problem of differentiating a possibly non-smooth h to differentiating the density of $\log Z_T^{t,x}$, which is smoother in this case, you can compare with Exercise 23 in List 1. After some tedious computations one can show that $f(t,x)$ satisfies the BS-PDE. As an important example, let us compute the derivative of $f(t,x)$ with respect to x using the density approach. First, define

$$\begin{aligned} \psi(z;x) &:= \phi\left(z; \log(x) + \left(r - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right) \\ &= \frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\exp\left(-\frac{\left(z - \log(x) - \left(r - \frac{\sigma^2}{2}\right)(T-t)\right)^2}{2\sigma^2(T-t)}\right), \end{aligned}$$

and note that

$$\begin{aligned}\frac{\partial\psi}{\partial x}(z;x) &= \psi(z;x)\left(-\frac{z-\log(x)-\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma^2(T-t)}\right)\left(-\frac{1}{x}\right)\\ &= \psi(z;x)\left(\frac{z-\log(x)-\left(r-\frac{\sigma^2}{2}\right)(T-t)}{x\sigma^2(T-t)}\right)\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial x}f(t,x) &= \frac{\partial}{\partial x}\mathbb{E}[e^{-r(T-t)}h(Z_T^{t,x})]\\ &= \frac{\partial}{\partial x}\int_{-\infty}^{+\infty} e^{-r(T-t)}h\left(\exp\left(z\right)\right)\psi(z;x)dz\\ &= \int_{-\infty}^{+\infty} e^{-r(T-t)}h\left(\exp\left(z\right)\right)\frac{\partial}{\partial x}\psi(z;x)dz\\ &= \int_{-\infty}^{+\infty} e^{-r(T-t)}h\left(\exp\left(z\right)\right)\left(\frac{z-\log(x)-\left(r-\frac{\sigma^2}{2}\right)(T-t)}{x\sigma^2(T-t)}\right)\psi(z;x)dz\\ &= \mathbb{E}[e^{-r(T-t)}h(Z_T^{t,x})\left(\frac{\log(Z_T^{t,x})-\log(x)-\left(r-\frac{\sigma^2}{2}\right)(T-t)}{x\sigma^2(T-t)}\right)]\end{aligned}$$

We have sketched the proof of the following theorem.

Theorem 11 *The price process $\pi_t(H)$ of a contingent claim $H = h(S_T)$ is given by*

$$\pi_t(H) = f(t, S_t),$$

and the hedging strategy is given by

$$(\phi_t^0, \phi_t^1) = \left(e^{-rt}\left\{f(t,S_t) - S_t\frac{\partial f}{\partial x}(t,S_t)\right\}, \frac{\partial f}{\partial x}(t,S_t)\right), \qquad t \in [0,T],$$

where

$$\begin{aligned}f(t,x) &= e^{-r(T-t)}\mathbb{E}[h(Z_T^{t,x})],\\ \frac{\partial f}{\partial x}(t,x) &= e^{-r(T-t)}\mathbb{E}\left[h(Z_T^{t,x})\left(\frac{\log(Z_T^{t,x})-\log(x)-\left(r-\frac{\sigma^2}{2}\right)(T-t)}{x\sigma^2(T-t)}\right)\right],\end{aligned} \tag{8}$$

and $\log Z_T^{t,x} \sim \mathcal{N}\left(\log(x) + \left(r-\frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right)$.

Remark 12 *When one can obtain a explicit formula for $f(t,x)$ in terms of functions that are differentiable (for instance for call and put options), one usually computes $\frac{\partial f}{\partial x}(t,x)$ by usual differentiation instead of using formula* (8).

Remark 13 *Note that we obtain the price of a contingent claim as*

$$\pi_t(H) = e^{-r(T-t)}\mathbb{E}[h(Z_T^{t,x})],$$

where the dynamics of $Z_T^{t,x}$ it is that of a geometric Brownian motion with parameters r and σ instead of μ and σ. Using the Girsanov's theorem we have that under the probability measure

$$\begin{aligned}\frac{dQ}{dP} &= \exp\left(-\int_0^T\frac{\mu-r}{\sigma}dW_s - \frac{1}{2}\int_0^T\left(\frac{\mu-r}{\sigma}\right)^2dt\right)\\ &= \exp\left(-W_T - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2T\right),\end{aligned}$$

the process

$$d\tilde{W}_t = \frac{\mu - r}{\sigma} dt + dW_t$$

is a Q-Brownian motion. We can rewrite the dynamics of S_t in terms of the process $\tilde{W}$, that is,

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ &= \mu S_t dt + \sigma S_t (d\tilde{W}_t - \frac{\mu - r}{\sigma} dt) \\ &= r S_t dt + \sigma S_t d\tilde{W}_t. \end{aligned}$$

As $\tilde{W}$ is a Brownian motion under Q, we get that S is a geometric Brownian motion with parameters r and σ under Q. Hence, we can write

$$\pi_t(H) = e^{-r(T-t)} \mathbb{E}_Q[h(S_T^{t,x})],$$

because $S^{t,x}$ under Q has the same law as $Z^{t,x}$ under P. The probability measure Q is called the risk-neutral pricing measure because we can obtain the price of a contingent claim as its discounted expected payoff under this measure. The term risk-neutral is due to the fact that the risky asset under Q has the same mean growth rate as the riskless asset r. So the price does not depend on μ the real rate at which the stock price S grows. So you can imagine that under Q we are in a world where nobody values risk at all and all risky assets grow at the riskless rate on average. The quotient $\frac{\mu - r}{\sigma}$ is called the market price of risk and it is the return in excess of the riskless rate (risk adjusted) that the market wants as a compensation for taking risk.

4 Pricing and Hedging of Call Options

One of the most traded contingent claims are call options with strike price $K > 0$. This claim has payoff function

$$H = \max(0, S(T) - K),$$

at exercise time T. We can prove the following result.

Theorem 14 *The price of a call is given by*

$$C(t, S_t) = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2),$$

where

$$\begin{aligned} d_1 &= \frac{\log(S_t/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= \frac{\log(S_t/K) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \end{aligned}$$

and

$$\Phi(x) = \int_{-\infty}^{x} \phi(z) dz = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz.$$

Note also that $d_1 = d_2 + \sigma\sqrt{T-t}$.

Proof. We have that

$$C(t, x) = \mathbb{E}[\max\left(0, Z_T^{t,x} - K\right)] = \mathbb{E}[\max\left(0, \exp(\log Z_T^{t,x}) - K\right)],$$

where $\log Z_T^{t,x} \sim \mathcal{N}\left(\log(x) + \left(r - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right)$. Hence,

$$C(t, x) = \mathbb{E}[\max\left(0, \exp\left\{\log(x) + (r - \frac{\sigma^2}{2})(T-t) + \sigma\sqrt{T-t}Y\right\} - K\right)],$$

where $Y \sim \mathcal{N}(0,1)$. The random variable inside the expectation is zero when

$$\log(x) + (r - \frac{\sigma^2}{2})(T-t) + \sigma\sqrt{T-t}Y` < \log(K),$$

or, equivalently, when $Y < -d_2$. Therefore,

$$\begin{aligned}
C(t,x) &= \int_{-d_2}^{+\infty} \left(\exp\left\{ \log(x) + (r - \frac{\sigma^2}{2})(T-t) + \sigma\sqrt{T-t}z \right\} - K \right) \phi(z)dz \\
&= xe^{r(T-t)} \int_{-d_2}^{+\infty} \exp\left\{ (-\frac{\sigma^2}{2}(T-t) + \sigma\sqrt{T-t}z \right\} \phi(z)dz \\
&- K \int_{-d_2}^{+\infty} \phi(z)dz.
\end{aligned}$$

Note that, by the symmetry of $\phi(z)$, we can write

$$\int_{-d_2}^{+\infty} \phi(z)dz = \int_{-\infty}^{d_2} \phi(z)dz = \Phi(d_2).$$

Hence the second integral is equal to $-K\Phi(d_2)$. For the first integral we do the change of variable $y = z - \sigma\sqrt{T-t}$, which yields

$$\begin{aligned}
&\int_{-d_2}^{+\infty} \exp\left\{ (-\frac{\sigma^2}{2}(T-t) + \sigma\sqrt{T-t}z \right\} \phi(z)dz \\
&= \int_{-d_2}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z-\sigma\sqrt{T-t})^2}{2} \right) dz \\
&= \int_{-d_2-\sigma\sqrt{T-t}}^{+\infty} \phi(y)dy = \Phi(d_2 + \sigma\sqrt{T-t}).
\end{aligned}$$

After multiplying by the discounting factor $e^{-r(T-t)}$ we can conclude. ■

4.1 The Greeks

Note that the price of a call option $C(t, S_t)$ actually depends on other variables (that we can consider as parameters)

$$C(t, S_t) = C(t, S_t; r, \sigma, K).$$

The derivatives with respect to these parameters are known as the Greeks and are relevant for risk-management purposes.

Here, there is a list of the most important:

- Delta:
$$\Delta = \frac{\partial C}{\partial S}(t, S_t) = \Phi(d_1).$$
- Gamma:
$$\Gamma = \frac{\partial C}{\partial S^2} = \frac{\Phi'(d_1)}{\sigma S_t\sqrt{T-t}} = \frac{\phi(d_1)}{\sigma S_t\sqrt{T-t}}$$
- Theta:
$$\begin{aligned}
\Theta = \frac{\partial C}{\partial t} &= -\frac{\sigma S_t \Phi'(d_1)}{2\sqrt{T-t}} - rKe^{-r(T-t)}\Phi(d_2) \\
&= -\frac{\sigma S_t \phi(d_1)}{2\sqrt{T-t}} - rKe^{-r(T-t)}\Phi(d_2).
\end{aligned}$$
- Rho:
$$\rho = \frac{\partial C}{\partial r} = K(T-t)e^{-r(T-t)}\Phi(d_2).$$
- Vega:
$$\frac{\partial C}{\partial \sigma} = S_t\sqrt{T-t}\Phi'(d_1) = S_t\sqrt{T-t}\phi(d_1).$$