Martingale Approach to Pricing and Hedging

1 Risk Neutral Pricing

Assume that we are in the basic Black-Scholes model. We have one riskless asset and one risky asset with prices given, respectively, by $B = \{B_t = e^{rt}\}_{t\in[0,T]}$ and $S_t = \{S_t\}_{t\in[0,T]}$ satisfying

$$dS_t = \mu S_t + \sigma S dW_t, \quad S_0 > 0.$$

We can setup strategies $\phi = \{\phi_t^0, \phi_t^1\}$ with value $V_t(\phi) = \phi_t^0 B_t + \phi_t^1 S_t$ and that satisfy the self-financing condition

$$dV_t(\phi) = \phi_t^0 dB_t + \phi_t^1 dS_t = re^{rt}\phi_t^0 dt + \phi_t^1 dS_t,$$

or equivalently

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi_s^0 dB_s + \int_0^t \phi_s^1 dS_s.$$

1.1 Change of Numeraire

As the price processes are strictly positive, in particular $B_t > 0$, one can always normalize the market by considering

$$\tilde{B}_t = B_t^{-1} B_t = 1 \text{ and } \tilde{S}_t = B_t^{-1} S_t = e^{-rt} S_t.$$

Thus normalization corresponds to regarding the price B_t of the safe investment (riskless asset) as the unit of price (the *numeraire*) and computing the other prices in terms of this unit. Alternatively, one can look at the normalized market as a discounted market where all assets are quoted (priced) in terms of its the present value. Moreover, we can consider the discounted portfolio

$$\tilde{V}_t(\phi) = B_t^{-1} V_t(\phi) = e^{-rt}\left(\phi_t^0 B_t + \phi_t^1 S_t\right) = \phi_t^0 + \phi_t^1 \tilde{S}_t,$$

and, applying integration by parts, one has that

$$\begin{aligned} d\tilde{V}_t(\phi) &= B_t^{-1} dV_t(\phi) - re^{-rt} V_t(\phi) dt + \underbrace{\left(dB_t^{-1}\right)\left(dV_t(\phi)\right)}_{=0} \\ &= e^{-rt} dV_t(\phi) - r\tilde{V}_t(\phi) dt. \end{aligned} \tag{1}$$

If we assume that $dV_t(\phi) = \phi_t^0 dB_t + \phi_t^1 dS_t$, i.e., ϕ is self-financing, then

$$\begin{aligned} d\tilde{V}_t(\phi) &= e^{-rt}\{r\phi_t^0 e^{rt} dt + \phi_t^1 dS_t\} - r\left\{\phi_t^0 + \phi_t^1 \tilde{S}_t\right\} dt \\ &= \phi_t^1 e^{-rt} dS_t - r\phi_t^1 e^{-rt} S_t dt = \phi_t^1\left\{e^{-rt} dS_t + d(e^{-rt}) S_t\right\} = \phi_t^1 d\tilde{S}_t, \end{aligned}$$

which yields that $\tilde{V}_t(\phi)$ is self-financing because $d\tilde{B}_t = d(1) = 0$. Using formula (1) and assuming that $d\tilde{V}_t(\phi) = \phi_t^1 d\tilde{S}_t$ (i.e. ϕ is self-financing in the normalized market) one gets that ϕ is self-financing in the unnormalized market. Hence, self-financing portfolios are invariant by a change of numeraire or discounting, in other words, a portfolio is self-financing if and only if is self-financing in the normalized market. From a financial point of view this makes sense because the self-financing property, that is, the fact that changes in the value of the portfolio are due only to changes in the asset prices, does not depend on which unit we measure the prices. Note that, in the discounted market, a self-financing portfolio is written in integral form as

$$\tilde{V}_t(\phi) = V_0(\phi) + \int_0^t \phi_s^1 d\tilde{S}_s.$$

1.2 Equivalent Martingale Measures and Arbitrage

The process

$$
\begin{aligned}
M_t &= \exp\left(-\int_0^t \frac{\mu - r}{\sigma} dW_s - \frac{1}{2}\int_0^t \left(\frac{\mu - r}{\sigma}\right)^2 ds\right) \\
&= \exp\left(-\frac{\mu - r}{\sigma} W_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t\right),
\end{aligned}
$$

is a martingale with respect to P. By Girsanov's theorem, we can define a probability measure Q by setting $\frac{dQ}{dP} = M_T$ and the process

$$
\tilde{W}_t = \frac{\mu - r}{\sigma} t + W_t,
$$

is a Brownian motion under Q. In addition, as $M_T > 0$, we have that $Q \sim P$. In the previous lecture, we showed that the dynamics of S under Q is that of a geometric Brownian motion with drift $r - \frac{\sigma^2}{2}$ and volatility σ, i.e.,

$$
dS_t = rS_t dt + \sigma S_t d\tilde{W}_t.
$$

If we now compute $d\tilde{S}_t$, we get that

$$
\begin{aligned}
d\tilde{S}_t &= d(e^{-rt}S_t) = -re^{-rt}S_t dt + e^{-rt} dS_t \\
&= -r\tilde{S}_t dt + e^{-rt}\left(rS_t dt + \sigma S_t d\tilde{W}_t\right) \\
&= -r\tilde{S}_t dt + r\tilde{S}_t dt + \sigma \tilde{S}_t d\tilde{W}_t \\
&= \sigma \tilde{S}_t d\tilde{W}_t, \qquad (2)
\end{aligned}
$$

or in explicit form

$$
\tilde{S}_t = \tilde{S}_0 \exp\left(\sigma \tilde{W}_t - \frac{\sigma^2}{2} t\right),
$$

which is a martingale under Q. This motivates the following general definition.

Definition 1 *An equivalent martingale measure (EMM) is a probability measure Q equivalent to P ($P \sim Q$) such that the discounted price of any asset in the market is a martingale under Q.*

Remark 2 *We just have shown that in the basic Black-Scholes model there is at least one equivalent probability measure Q, given by*

$$
\frac{dQ}{dP} = \exp\left(-\frac{\mu - r}{\sigma} W_T - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T\right).
$$

Note also that the discounted price of the riskless asset is constant and, hence, a martingale under any probability measure.

Moreover, we have that

$$
\tilde{V}_t(\phi) = \tilde{V}_0(\phi) + \int_0^t \phi_s^1 d\tilde{S}_s = \tilde{V}_0(\phi) + \int_0^t \phi_s^1 \sigma \tilde{S}_s d\tilde{W}_t,
$$

and $\tilde{V}_t(\phi)$ is a stochastic integral with respect to a Brownian motion under Q. Hence, under the integrability condition

$$
E_Q\left[\int_0^T \left|\phi_t^1 \sigma \tilde{S}_t\right|^2 dt\right] < \infty,
$$

we have that $\tilde{V}_t(\phi)$ is a martingale under Q. This property motivates the following definition of admissibility.

Definition 3 *A self-financing trading strategy ϕ is called admissible if $\tilde{V}_t(\phi)$ is a martingale under Q.*

The next proposition shows that the class of admissible strategies is a good class in terms of arbitrage.

Proposition 4 *The Black-Scholes model is free of arbitrage in the sense that there exists no admissible arbitrage portfolios.*

Proof. Assume that ϕ is an arbitrage portfolio, i.e. $V_0(\phi) \leq 0, V_T(\phi) \geq 0, P$-a.s. and $P(V_T(\phi) > 0) > 0$. As $Q \sim P$, we also have that $V_T(\phi) \geq 0, Q$-a.s. and $Q(V_T(\phi) > 0) > 0$. Then,

$$\begin{aligned}\mathbb{E}_Q\left[\tilde{V}_T(\phi)\right] &= \mathbb{E}_Q\left[V_0(\phi) + \int_0^T \phi_s^1 d\tilde{S}_s\right] \\ &= V_0(\phi) + \mathbb{E}_Q\left[\int_0^T \phi_t^1 d\tilde{S}_t\right] = V_0(\phi) \leq 0,\end{aligned}$$

because the integral is a martingale with zero expectation, under Q. This is a contradiction, because $V_T(\phi) \geq 0, Q$-a.s. and $Q(V_T(\phi) > 0) > 0$ yields that $\mathbb{E}_Q\left[\tilde{V}_T(\phi)\right] > 0$. ■

The following, imprecisely stated, theorem is one of the cornerstones of mathematical finance.

Theorem 5 (First Fundamental Theorem of Asset Pricing) *A market model is free of arbitrage if and only if there exists at least one equivalent martingale measure.*

The difficult part is to show that if the model is free of arbitrage then there exists an equivalent martingale measure.

1.3 Equivalent Martingale Measures and Completeness

Our goal is to give an arbitrage free price to any sensible contingent claim H (a positive $\mathcal{F}_T$-measurable random variable satisfying some integrability constraints) that pays some amount at time T. We just have shown that in the Black-Scholes model there are no arbitrage opportunities because there exists an EMM Q. Moreover, we have seen that if ϕ is admissible then $\tilde{V}(\phi)$ is a martingale under Q. Then, if we combine these facts with the martingale representation theorem we have all the ingredients for pricing and hedging.

Theorem 6 (Risk Neutral Pricing) *Let $H \in L^2(\Omega, \mathcal{F}_T, Q)$ be a contingent claim. Then the arbitrage free price of H is given by*

$$\pi_t(H) = \mathbb{E}_Q[e^{-r(T-t)}H|\mathcal{F}_t], \tag{3}$$

and the price at time 0 is given by

$$\pi_0(H) = \mathbb{E}_Q[e^{-rT}H].$$

Moreover, the hedging strategy ϕ is given by

$$\phi_t^0 = \pi_t(H) - \phi_t^1\tilde{S}_t, \qquad \phi_t^1 = \frac{h_t}{\sigma\tilde{S}_t},$$

where h is the unique process in $L^2_{a,T}$ such that

$$e^{-rT}H = \mathbb{E}_Q[e^{-rT}H] + \int_0^T h_s d\tilde{W}_s.$$

Proof. We have that $\pi_t(H)$, the arbitrage free price at time t of any replicable contingent claim H, is given by $V_t(\phi)$ the value of its admissible hedging portfolio at time t. Hence, if ϕ is a hedging strategy for H we have that

$$H = V_0(\phi) + \int_0^T \phi_t^0 dB_t + \int_0^T \phi_t^1 dS_t,$$

and the discounted portfolio will be a replicating portfolio for the claim $\tilde{H} = e^{-rT}H$, i.e.,

$$\tilde{H} = \tilde{V}_T(\phi) = V_0(\phi) + \int_0^T \phi_t^1 d\tilde{S}_t.$$

It follows from the martingale properties of $\tilde{V}(\phi)$ under Q that

$$\mathbb{E}_Q[e^{-rT}H|\mathcal{F}_t] = \mathbb{E}_Q[\tilde{V}_T(\phi)|\mathcal{F}_t] = \tilde{V}_t(\phi) = e^{-rt}V_t(\phi),$$

which yields

$$V_t(\phi) = \mathbb{E}_Q[e^{-r(T-t)}H|\mathcal{F}_t].$$

The equality $\pi_t(H) = V_t(\phi)$ yields the pricing formula (3). The second step is to prove that a sufficient condition for a claim H to be replicable is that $H \in L^2(\Omega, \mathcal{F}_T, Q)$. Consider the discounted claim $\tilde{H} = e^{-rT}H$, which also belongs to $L^2(\Omega, \mathcal{F}_T, Q)$. Consider the square integrable martingale $M_t = \mathbb{E}_Q[e^{-rT}H|\mathcal{F}_t]$ under Q. As $\tilde{W}$ is a $\mathbb{F}$-Brownian motion under Q we can apply the martingale representation theorem to write

$$M_t = \mathbb{E}_Q[e^{-rT}H] + \int_0^t h_s d\tilde{W}_s,$$

for $h \in L^2_{a,T}$. Define the trading strategy ϕ given by

$$\phi_t^0 = M_t - \phi_t^1 \tilde{S}_t, \qquad \phi_t^1 = \frac{h_t}{\sigma \tilde{S}_t}.$$

The discounted value of this portfolio is

$$\tilde{V}_t(\phi) = \phi_t^0 + \phi_t^1 \tilde{S}_t = M_t,$$

which is a martingale under Q, so ϕ it is admissible. Its final value will be

$$V_T(\phi) = e^{rT}\tilde{V}_T(\phi) = e^{rT}M_T = \mathbb{E}_Q[H|\mathcal{F}_T] = H,$$

therefore $V_t(\phi)$ replicates H. Finally, ϕ is self-financing because

$$d\tilde{V}_t(\phi) = dM_t = h_t d\tilde{W}_t,$$

and, on the other hand, by equation (2) we get

$$\phi_t^1 d\tilde{S}_t = \frac{h_t}{\sigma \tilde{S}_t} d\tilde{S} = h_t d\tilde{W}_t.$$

■

Remark 7 *The previous theorem provides a very general pricing and hedging formulae and it is very useful to prove theoretical results in the field of mathematical finance. However, in practical terms, it may be difficult to use because it involves the computation of a conditional expectation. The computation of the hedging strategy is even more difficult as there are no general formulas for computing the kernels in a martingale representation. If the random variable H is smooth in the sense of Malliavin, then one can use the Clark-Ocone formula for those kernels but even in that case it appears a conditional expectation to compute.*

Remark 8 *A sufficient condition for $H \in L^2(\Omega, \mathcal{F}_T, Q)$ is that $H \in L^{2+\varepsilon}(\Omega, \mathcal{F}_T, P)$ for some $\varepsilon > 0$.*

Remark 9 *In the basic Black-Scholes model the filtration $\mathbb{F}^W = \mathbb{F}^{\tilde{W}}$ and, hence, we can apply directly the martingale representation theorem with $\tilde{W}$. In more general cases, when the drift and volatility of S are random, we only have that $\mathbb{F}^{\tilde{W}} \subseteq \mathbb{F}^W$ and in order to apply the martingale representation theorem with $\tilde{W}$ we would need H to be $\tilde{\mathcal{F}}_T$-measurable. Nevertheless, one can prove that such martingale representation still holds in those cases but it needs additional proof.*

Example 10 *Assume that* $H = h(S_T)$ *then*

$$\pi_t(H) = \mathbb{E}_Q\left[e^{-r(T-t)}h(S_T)|\mathcal{F}_t\right].$$

S_t *solves the following s.d.e.*

$$dS_t = rS_t + \sigma S_t d\tilde{W}_t,$$

under Q. *Hence,* S_t *is a Markov process, and we have that*

$$\mathbb{E}_Q\left[e^{-r(T-t)}h(S_T)|\mathcal{F}_t\right] = \mathbb{E}_Q\left[e^{-r(T-t)}h(S_T^{t,x})\right]|_{x=S_t}.$$

Moreover, by the Feynman-Kac representation, $v(t,x) := \mathbb{E}_Q\left[e^{-r(T-t)}f(S_T^{t,x})\right]$ *solves the Black-Scholes PDE with terminal condition* $h(x)$.

Therefore, the Black-Scholes market is complete for all contingent claims that are square integrable under an equivalent martingale measure. Up till now we have proved that in Black-Scholes model there exists an EMM measure Q given by

$$\frac{dQ}{dP} = \exp\left(-\frac{\mu - r}{\sigma}W_T - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T\right).$$

But, does there exist another $\hat{Q} \sim P$ such that under $\hat{Q}$ the discounted price process $\tilde{S}_t = e^{-rt}S_t$ is a martingale? The answer is no.

Lemma 11 *In the Black-Scholes model* Q *is the unique EMM.*

Proof. I'm going to sketch the proof. Assume that $\hat{Q}$ is another probability measure such that $\hat{Q} \sim P$. Then, we can consider the density process $D_t = \mathbb{E}\left[\frac{d\hat{Q}}{dP}|\mathcal{F}_t\right]$. Assume that $\frac{d\hat{Q}}{dP} \in L^2(P)$. Therefore, by the martingale representation theorem we have that

$$D_t = \mathbb{E}\left[\frac{d\hat{Q}}{dP}\right] + \int_0^t \gamma_s dW_s = 1 + \int_0^t \gamma_s dW_s.$$

By a similar reasonings as in the Girsanov's theorem, one can prove that a process X_t is a martingale under $\hat{Q}$ if and only if $D_t X_t$ is a martingale under P. Assume that $\tilde{S}_t$ is martingale under $\hat{Q}$ then $D_t\tilde{S}_t$ must be a martingale under P. Let us compute the dynamics of $D_t\tilde{S}_t$ under P. First, we have that

$$\begin{aligned} d\tilde{S}_t &= (\mu - r)\tilde{S}_t dt + \sigma\tilde{S}_t dW_t \\ dD_t &= \gamma_t dW_t. \end{aligned}$$

Then, by the integration by parts formula, we get

$$\begin{aligned} d(D_t\tilde{S}_t) &= D_t d\tilde{S}_t + \tilde{S}_t dD_t + dD_t d\tilde{S}_t \\ &= D_t(\mu - r)\tilde{S}_t dt + D_t\sigma\tilde{S}_t dW_t + \tilde{S}_t\gamma_t dW_t + \gamma_t\sigma\tilde{S}_t dt \\ &= \left\{D_t(\mu - r)\tilde{S}_t + \gamma_t\sigma\tilde{S}_t\right\}dt + \left\{D_t\sigma\tilde{S}_t + \tilde{S}_t\gamma_t\right\}dW_t. \end{aligned}$$

Hence, for $D_t\tilde{S}_t$ to be a martingale we must have that

$$D_t(\mu - r)\tilde{S}_t + \gamma_t\sigma\tilde{S}_t = 0, \qquad P \otimes \lambda, \text{ a.e.}$$

which is equivalent to have

$$\gamma_t = -D_t\frac{(\mu - r)}{\sigma}.$$

Thus, D_t has the representation

$$D_t = 1 - \int_0^t D_s\frac{(\mu - r)}{\sigma}dW_s.$$

But the unique solution of this s.d.e. is

$$D_t = \exp\left(-\frac{(\mu - r)}{\sigma}W_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t\right).$$

As $D_T = \frac{dQ}{dP}$ we get that, actually, $\hat{Q} = Q$. The only point left is the assumption that $\frac{d\hat{Q}}{dP} \in L^2(P)$, but this is just a technical point that can be addressed. ■

There exists a deep result that links the uniqueness of a martingale measure and the completeness of a market model.

Theorem 12 (Second Fundamental Theorem of Asset Pricing) *If a market model admits an EMM Q, then the market is complete if and only if Q is the unique EMM.*