Introduction and Techniques
in Financial Mathematics
UiO-STK4510
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Exercises
List 1
Solutions and Hints

Probability and Measure Theory

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1. If $\#\Omega < \infty$ the answer is yes. If $\#\Omega = \infty$ the answer is no. Consider $\Omega = \mathbb{N}$. Then $A = \{\text{even numbers}\} \notin \mathcal{F}$ but is a countable union of $A_n = \{2n\} \in \mathcal{F}$, for all $n \geq 1$. Hence, $\mathcal{F}$ is not closed by countable unions.

2. Yes. You have to check the properties that define a σ-algebra. It is useful to separate the cases where an arbitrary set $A \in \mathcal{F}$ (or a sequence of sets when checking the σ-additivity) is countable or uncountable.

3. We have that

   (a) $\{0\} = \left(\cup_{n\geq 1}[\frac{1}{n+1}, \frac{1}{n}]\right)^c$.

   (b) $\{\frac{1}{n} : n \geq 2\} = \cup_{n\geq 1}\left([\frac{1}{n+1}, \frac{1}{n}] \cap [\frac{1}{n}, \frac{1}{n-1}]\right)$.

   (c) $(\frac{1}{n}, 1] = \left(\cup_{1\leq i\leq n-1 1}[\frac{1}{i+1}, \frac{1}{i}]\right) \cap [\frac{1}{n+1}, \frac{1}{n}]$.

   (d) $(0, \frac{1}{n}] = \cup_{i\geq n}[\frac{1}{i+1}, \frac{1}{i}]$.

4. It is a consequence of De Morgan's law $(\cap_{n\geq 1}A_n)^c = \cup_{n\geq 1}A_n^c$.

5. It is a consequence of the fact that the mapping X^{-1} preserves all set operations. That is $X^{-1}(\cup_{n\geq 1}A_n) = \cup_{n\geq 1}X^{-1}(A_n), X^{-1}(A^c) = \left(X^{-1}(A)\right)^c, etc..$

6. Define $X : A \to \Omega$ by $X(\omega) = \omega$, that is, X is the injection of the set A into Ω. Then, $A \cap B = X^{-1}(B)$ for any $B \in \mathcal{F}$. By exercise 5, we get that $\mathcal{F}_A = \{X^{-1}(B) : B \in \mathcal{F}\}$ is a σ-algebra.

7. It follows easily from the definition of σ-algebra.

8. That if $\{A_n\}_{n\geq 1} \subset \mathcal{F}$, then $\limsup_{n\to\infty} A_n \in \mathcal{F}$ and $\liminf_{n\to\infty} A_n \in \mathcal{F}$ follows from the fact that $\mathcal{F}$ is closed by countable unions (definition of σ-algebra) and closed by countable intersections (exercise 4.). That

   $$\limsup_{n\to\infty} A_n = \{A_n \text{ occurs for infinitely many } n\},$$

   and

   $$\liminf_{n\to\infty} A_n = \{A_n \text{ occurs for all but finitely many } n\},$$

   is just writing in words the iterated intersection and union in the definitions of $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$. That $\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$ is obvious.

9. To get $P(A) \geq 0$ for all $A \in \mathcal{F} = \mathcal{P}(\mathbb{N})$ we must impose $\alpha_i \geq 0, i \in \mathbb{N}$. To get $P(\Omega) = 1$ we must impose $\sum_{i=1}^{\infty} \alpha_i = 1$. The convergence of the series implies that $\alpha_i \to 0$ when $i \to +\infty$, which rules out the possibility of having all α_i's equal. For any set $A \in \mathcal{F}$ define

   $$P(A) = \sum_{i\in A} P(\{i\}) = \sum_{i\in A} \alpha_i.$$

   To show the σ-additivity one uses that the series $\sum_{i=1}^{\infty} \alpha_i$ is absolutely convergent and, hence, the terms of the series can be reordered without changing the value of its sum, which yields that $\sum_{i\in\cup_k^{\infty} A_k} \alpha_i = \sum_{k=1}^{\infty}\sum_{i\in A_k} \alpha_i$.

10. Let $(\Omega, \mathcal{F}, P)$ be a probability space.

    (a) First prove that if $B \subset A$ then $P(B) \leq P(A)$. This is done by considering the decomposition $A = (A \cap B) \biguplus (A \cap B^c)$ using the additivity and the non-negativity of P and that $A \cap B = B$.

    (b) To prove that
    $$P\left(\bigcup_{n=1}^{N} A_n\right) \leq \sum_{n=1}^{N} P(A_n),$$
    one proceeds by induction. The base case is trivial. In order to prove the general case define $B_n = A_n \backslash (\cup_{i=1}^{n-1} A_i) = A_n \cap \left((\cup_{i=1}^{n-1} A_i)^c\right), 1 \leq n \leq N$. The sets B_n are pairwise disjoint and satisfy $\cup_{i=1}^{n} A_i = \biguplus_{i=1}^{n} B_i$. Then,
    $$\begin{aligned} P\left(\bigcup_{n=1}^{N} A_n\right) &= P\left(\bigcup_{n=1}^{N} B_n\right) = P\left(\bigcup_{n=1}^{N-1} B_n\right) + P(B_N) \\ &\leq P\left(\bigcup_{n=1}^{N-1} A_n\right) + P(A_N) \leq \sum_{n=1}^{N} P(A_n), \end{aligned}$$
    where in the first inequality we have used the monotonicity property with $B_N \subset A_N$ and in the second inequality we have used the induction hypothesis.

11. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{A_n\}_{n\geq 1} \subset \mathcal{F}$ a sequence of events.

    (a) Define $B_1 = A_1, B_k = A_k \backslash A_{k-1} = A_k \cap A_{k-1}^c, k = 2, 3, ...$ The sets B_k are pairwise disjoint and $\cup_{k=1}^{\infty} A_k = \biguplus_{k=1}^{\infty} B_k$ and $A_n = \biguplus_{k=1}^{n} B_k$. The result now follows easily from the σ-additivity of P and the definition of the sum of a series.

    (b) Note that $1 = P(\Omega) = P(B \biguplus B^c) = P(B) + P(B^c)$ which yields $P(B^c) = 1 - P(B)$. Furthermore, the sequence of sets $\{A_k^c\}_{k=1}^{\infty}$ is an increasing sequence $A_k^c \subset A_{k+1}^c$. Using de Morgan's law $(\cap_{k=1}^{\infty} A_k)^c = \cup_{k=1}^{\infty} A_k^c$ and the conclusion in 11.(a) the result follows.

    (c) Define $B_n = \cup_{k=1}^{n} A_n$. $\{B_n\}_{n=1}^{\infty}$ is a sequence of increasing processes. The result follows by 11.(a), 10.(b) and the definition of sum of a series.

    (d) Follows from 11.(c).

    (e) Take complements using de Morgan's law and reduce the problem to the case 11.(d).

    (f) Note that $\sum_{n=1}^{\infty} P(A_n) < \infty$ implies that $\lim_{k\to\infty} \sum_{n=k}^{\infty} P(A_n) = 0$. Recall that
    $$\limsup_{n\to\infty} A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_n$$
    and that $B_n = \cup_{k=n}^{\infty} A_n$ is a decreasing sequence $B_{n+1} \subset B_n$. The result follows by using 11.(b) and 11.(c).

12. It follows from exercise 13.

13. X takes values in $\{x_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$. Consider $A_i = \{\omega : X(\omega) = x_i\} = X^{-1}(\{x_i\})$. The family of sets $\mathcal{P} = \{A_i\}_{i\in\mathbb{N}}$ is a countable partition of Ω,i.e., $A_i \cap A_j = \varnothing$ if $i \neq j$ and $\Omega = \biguplus_{i\in\mathbb{N}} A_i$. By exercise 5. $\sigma(X) = \{X^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\}$ but $X^{-1}(B) = \biguplus_{\{i:x_i\in B\}} A_i$. Hence, we can conclude that the elements of $\sigma(X)$ are countable (or finite) unions of elements of $\mathcal{P}$. That is $B \in \sigma(X)$ if and only if there exists $J \subset \mathbb{N}$ such that $B = \biguplus_{i\in J} A_i$. This is the general structure of the σ-algebra generated by a countable partition of Ω. A function Y is measurable with respect to $\sigma(X)$ if and only if Y is constant over the elements of the partition $\mathcal{P}$.

14. No. The proof of this fact and an example follows form exercise 13, because $\mathcal{F}$ is generated by a finite partition of Ω.

15. $Y = g \circ X$, where $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable and $g : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by $g(x) = x^2$ is measurable because is continuous. Hence, Y is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable because is the composition of measurable functions. To check if X is $\sigma(Y)$-measurable we can use a corollary of the factorization theorem (Corollary 36 in Lecture 3). We need to find a Borel measurable function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $X = \varphi(Y)$. As $Y = g(X)$, the desired φ must be equal to g^{-1} and g^{-1} only exists if g is injective on $X(\Omega) \subset \mathbb{R}$, the image of X. Therefore, X is $\sigma(Y)$-measurable iff g is injective on $X(\Omega)$. For instance, if $X(\Omega) = \mathbb{R}_+$ then X is $\sigma(Y)$-measurable, but if $X(\Omega) = \mathbb{R}$ then X is not $\sigma(Y)$-measurable.

16. Consider the measurable sets $A = \{\omega : X(\omega) > 0\}$ and $A_n = \{\omega : X(\omega) > 1/n\}, n \geq 1$. A_n is an increasing sequence that converges to A. Note that $X \geq X\mathbf{1}_{A_n} \geq \frac{1}{n}\mathbf{1}_{A_n}$. Using the monotonicity of the Lebesgue integral (if $f \geq g, P$-a.s. $\Rightarrow \int_\Omega f dP \geq \int_\Omega g dP$) show that $P(A_n) = 0$ and using exercise 11.(a) conclude that $P(A) = 0$, which yields that $X = 0, P$-a.s.

17. Consider the measurable sets $A_+ = \{\omega : X(\omega) > 0\}$ and $A_- = \{\omega : X(\omega) < 0\}$. As $\{\omega : X(\omega) \neq 0\} = A_- \cup A_+$, we can conclude if we show that $P(A_-) = P(A_+) = 0$. Note that $X\mathbf{1}_{A_+} \geq 0, P$-a.s. and by hypothesis $\mathbb{E}[X\mathbf{1}_{A_+}] = 0$. Then, by exercise 16., we have that $X\mathbf{1}_{A_+} = 0, P$-a.s.. But as $X(\omega) > 0$ for $\omega \in A_+$, the only possibility to have $X\mathbf{1}_{A_+} = 0, P$-a.s. is that $P(A_+) = 0$. The reasoning for A_- is similar.

18. First, note that $Q \sim P$ is equivalent to require that $P(A) = 0$ iff $Q(A) = 0$ for all $A \in \mathcal{F}$. This equivalence follows from the definition of $Q \ll P$ and $P \ll Q$. Then, if we assume that $Q \sim P$ we have that $Q \ll P$ and by the Radon-Nikodym theorem we get that

$$\begin{aligned} Q\left(\left\{\frac{dQ}{dP} = 0\right\}\right) &= \int_{\left\{\frac{dQ}{dP}=0\right\}} \frac{dQ}{dP} dP = \int_\Omega \mathbf{1}_{\left\{\frac{dQ}{dP}=0\right\}} \frac{dQ}{dP} dP \\ &= \int_\Omega 0 dP = 0, \end{aligned}$$

where we have used that $\mathbf{1}_{\left\{\frac{dQ}{dP}=0\right\}} \frac{dQ}{dP} = 0, P$-a.s. and that the value of the integral does not change when interchanging P-a.s. equal integrands. But, $Q\left(\left\{\frac{dQ}{dP} = 0\right\}\right) = 0$ implies $P\left(\left\{\frac{dQ}{dP} = 0\right\}\right) = 0$, because we have assumed that $Q \sim P$. Assume now that $Q \ll P$ and that $P\left(\left\{\frac{dQ}{dP} = 0\right\}\right) = 0$. We must show that if $Q(A) = 0$ then $P(A) = 0$. First note that $\forall B \in \mathcal{F}$ we have that $A \cap B \in \mathcal{F}$ and $Q(A \cap B) = 0$, by the monotonicity of Q. By the Radon-Nikodym

$$0 = Q(A \cap B) = \int_{A \cap B} \frac{dQ}{dP} dP = \int_B \mathbf{1}_A \frac{dQ}{dP} dP,$$

which yields that $\mathbf{1}_A \frac{dQ}{dP} = 0, P$-a.s.. This last P-a.s. equality combined with $P\left(\left\{\frac{dQ}{dP} = 0\right\}\right) = 0$ implies that $P(A) = 0$. As $P\left(\left\{\frac{dQ}{dP} = 0\right\}\right) = 0$ we can define P-a.s. the random variable $Z = \left(\frac{dQ}{dP}\right)^{-1}$. Let's check that Z is P-a.s. equal to $\frac{dP}{dQ}$. Using Proposition 55 in Lecture 2, we have that, for all $B \in \mathcal{F}$, we can write

$$\int_\Omega \mathbf{1}_B Z dQ = \int_\Omega \mathbf{1}_B Z \frac{dQ}{dP} dP = \int_\Omega \mathbf{1}_B \left(\frac{dQ}{dP}\right)^{-1} \frac{dQ}{dP} dP = \int_\Omega \mathbf{1}_B dP = P(B),$$

which shows that $\frac{dP}{dQ} = \left(\frac{dQ}{dP}\right)^{-1}, P$-a.s.

19. Define $\phi(z) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2})$. First we have to check that $\int_{-\infty}^{+\infty} \phi(z) dz = 1$. This is a classical computation that follows from computing the double integral $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \phi(x)\phi(y) dx dy$ using Fubini's theorem and a change of variables to polar coordinates,i.e., $x = r\cos\theta, y = r\sin\theta, r \in (0, +\infty)$ and $\theta \in [0, 2\pi]$. Next, consider $(\Omega, \mathcal{F}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $Z : \Omega \to \mathbb{R}$ defined by $Z(\omega) = \omega$, which is clearly $(\mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$-measurable. By the definition of P_Z is easy to check that $P_Z = Q$. Using the image measure theorem one gets that

$$\mathbb{E}[Z] = \int_{-\infty}^{+\infty} z\phi(z) dz, \qquad \mathbb{E}[Z^2] = \int_{-\infty}^{+\infty} z^2 \phi(z) dz.$$

As $\phi'(z) = -z\phi(z)$ and $\lim_{z\to-\infty} z^n\phi(z) = \lim_{z\to+\infty} z^n\phi(z) = 0, n \in \mathbb{N}\cup\{0\}$ we get that

$$\mathbb{E}[Z] = -\int_{-\infty}^{+\infty} \phi'(z)dz = \lim_{z\to-\infty}\phi(z) - \lim_{z\to+\infty}\phi(z) = 0.$$

To compute $\mathbb{E}[Z^2]$ one uses the integration by parts formula to get

$$\int_{-\infty}^{+\infty} z^2\phi(z)dz = [-z\phi(z)]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty}\phi(z)dz = 0 + 1 = 1.$$

Hence, $\mathrm{Var}[Z] = \mathbb{E}[(Z-\mathbb{E}[Z])^2] = \mathbb{E}[Z^2] = 1$. Note that $X = g(Z) = g\circ Z$ where $g(z) = \sigma z + \mu$ is a continuous (and hence measurable) function. Therefore, X is a random variable because it is the composition of measurable functions, because Z is a random variable. The distribution function of X is given by $F_X(x) = Q(X\le x) = Q(\sigma Z + \mu \le x) = Q(Z \le \frac{x-\mu}{\sigma}) = \int_{-\infty}^{\frac{x-\mu}{\sigma}}\phi(z)dz$ and after the change of variable $z = (y-\mu)/\sigma$ one gets that

$$F_X(x) = \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)dy.$$

By the linearity of the expectation one has that $\mathbb{E}[X] = \mu + \sigma\mathbb{E}[Z] = \mu$ and

$$\mathbb{E}[X^2] = \mathbb{E}[(\mu+\sigma Z)^2] = \mu^2 + 2\mu\mathbb{E}[Z] + \sigma^2\mathbb{E}[Z^2] = \mu^2+\sigma^2.$$

Moreover, if Y is an arbitrary random variable with $E[Y^2] < +\infty$ then, using again the linearity of the expectation, we get that

$$\mathrm{Var}[Y] = \mathbb{E}[(Y-\mathbb{E}[Y])^2] = E[Y^2] - 2\mathbb{E}[Y]\mathbb{E}[Y] + \mathbb{E}[Y]^2 = E[Y^2] - \mathbb{E}[Y]^2.$$

Hence, $\mathrm{Var}[X] = \sigma^2$.

20. To be handed in.

21. X is measurable because it is a random variable. $Y = \exp(X) = g\circ X$ where $g(x) = \exp(x)$ continuous and hence measurable. Therefore, Y is a random variable because it is the composition of measurable functions. The law of Y is given by

$$\begin{aligned} P_Y((-\infty, y]) &= P(Y\le y) = P(\exp(X)\le y) = \mathbf{1}_{\{y>0\}}P(X \le \log(y)) \\ &= \mathbf{1}_{\{y>0\}}\int_{-\infty}^{\log(y)}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx \\ &= \mathbf{1}_{\{y>0\}}\int_{0}^{y}\frac{1}{z\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(\log(z)-\mu)^2}{2\sigma^2}\right)dz. \end{aligned}$$

This shows that $dP_Z \ll d\lambda$ and

$$\frac{dP_Z}{d\lambda}(z) = \frac{1}{z\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(\log(z)-\mu)^2}{2\sigma^2}\right).$$

Note that $\mathbb{E}[Y^n] = \mathbb{E}[\exp(nX)] = \psi(n)$ and $Var[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \psi(2) - \psi(1)^2$, where $\psi(\theta)$ is the function defined in exercise 20.

22. It follows from the monotonicity of the expectation (integral with respect to dP) and the inequality $f(X) \ge f(X)\mathbf{1}_{\{X\ge a\}} \ge f(a)\mathbf{1}_{\{X\ge a\}}, P$-a.s..

23. Let $\{t_n\}_{n\ge1}\subset I\backslash\{t_0\}$ be an arbitrary sequence of numbers converging to t_0. Apply, the dominated convergence theorem to $\frac{X_{t_n}-X_{t_0}}{t_n-t_0}$. To check the hypothesis in the dominated convergence theorem is useful to consider the mean value theorem if f is $C^1(I)$ then $f(t_1) - f(t_2) = f(\xi)(t_1-t_2)$ for some $t_1, t_2, \xi \in I$.

24. By the independence of X and Y, the joint density of (X,Y) is given by $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ where f_X and f_Y are the densities of X and Y, respectively, which exists by assumption. Let $Z = X+Y$. Compute $P(Z \le z)$ integrating $f_{X,Y}(x,y)$ over the set $\{x+y \le z\}$. Rewrite the double integral, using Fubini's theorem and a change of variable, as a double integral where the outer integral goes from $-\infty$ to z. Taking derivatives with respect to z you obtain the desired density, which is

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(u)f_Y(z-u)du = \int_{-\infty}^{+\infty} f_X(z-u)f_Y(u)du.$$

25. As X and Y are i.i.d. with law $\mathcal{N}(\mu,\sigma^2)$ we have that (X,Y) is multivariate normal with density given by

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{(x-\mu)^2+(y-\mu)^2}{2\sigma^2}\right).$$

Next, use Theorem 18 in Lecture 3 with $S_0 = \varnothing, S_1 = \mathbb{R}^2$ and $g : S_1 \to g(S_1) = \mathbb{R}^2$ given by $(u,v) = g(x,y) = (x+y, x-y)$. Note that $g_1^{-1} : g(S_1) \to S_1$ is given by $g_1^{-1}(u,v) = (\frac{u+v}{2}, \frac{u-v}{2}) = (x,y)$ and $\det J_{g_1^{-1}}(u,v) = -\frac{1}{2}$. Hence,

$$f_{U,V}(u,v) = \frac{1}{4\pi\sigma^2}\exp\left(-\frac{\frac{u^2}{2}+\frac{u^2}{2}+2\mu^2-2\mu u}{2\sigma^2}\right)\mathbf{1}_{\mathbb{R}^2}(u,v).$$

(U,V) are independent iff $f_{U,V}(u,v) = f_U(u)f_V(v)$ for some densities $f_U(u)$ and $f_V(v)$. This happens iff $\mu = 0$ and, in this case, $f_U(u)$ and $f_V(v)$ are the densities of a normal random variable with zero mean and variance $2\sigma^2$.

26. As X and Y are i.i.d. with law $\mathcal{N}(0,\sigma^2)$ we have that (X,Y) is multivariate normal with density given by

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right).$$

Next, use Theorem 18 in Lecture 3 with $S_0 = \mathbb{R}\times\{0\}, S_1 = (0,+\infty)\times\mathbb{R}, S_2 = (-\infty,0)\times\mathbb{R}$ and $g : S_i \to g(S_i) = [0,+\infty)\times\mathbb{R}, i = 1,2$, given by $(u,v) = g(x,y) = (\sqrt{x^2+y^2}, x/y)$. Note that $g_i^{-1} : g(S_i) \to S_i, i = 1,2$ are given by $g_1^{-1}(u,v) = (\frac{uv}{\sqrt{1+v^2}}, \frac{u}{\sqrt{1+v^2}})$ and $g_2^{-1}(u,v) = (\frac{-uv}{\sqrt{1+v^2}}, \frac{-u}{\sqrt{1+v^2}})$ and $\det J_{g_1^{-1}}(u,v) = \det J_{g_2^{-1}}(u,v) = -\frac{u}{1+v^2}$. Hence,

$$f_{U,V}(u,v) = u\exp\left(-\frac{u^2}{2\sigma^2}\right)\mathbf{1}_{(0,+\infty)}(u)\frac{1}{\pi\sigma^2}\left(\frac{1}{1+v^2}\right)\mathbf{1}_{\mathbb{R}}(v),$$

and, as the joint density of (U,V) factorizes, we have that U and V are independent.

27. We can write $\det Q$ in terms of ρ. We have that $\det Q = (1-\rho^2)\sigma_X^2\sigma_Y^2$. As $\det Q \ge 0$ we must have $\rho \in [-1,1]$. As $\sigma_X^2\sigma_Y^2 > 0$, we have that $|\rho| < 1$ iff $\det Q > 0$, which implies that $P_{X,Y} \ll \lambda^2$ by Theorem 31 in Lecture 3. Moreover,

$$Q^{-1} = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}^{-1} = \frac{1}{(1-\rho^2)\sigma_X^2\sigma_Y^2}\begin{pmatrix} \sigma_Y^2 & -\rho\sigma_X\sigma_Y \\ -\rho\sigma_X\sigma_Y & \sigma_X^2 \end{pmatrix}.$$

Also by Theorem 31 in Lecture 3 we have that

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi\det Q}}\exp\left(\frac{1}{2(1-\rho^2)\sigma_X^2\sigma_Y^2}\begin{pmatrix} x-\mu_Y \\ y-\mu_y \end{pmatrix}' Q^{-1}\begin{pmatrix} x-\mu_Y \\ y-\mu_y \end{pmatrix}\right)$$

After a little bit of algebra in the terms of the exponential one gets the desired expression for $f_{X,Y}(x,y)$. As $\sigma_X^2\sigma_Y^2 > 0$, we have that $|\rho| = 1$ iff $\det Q = 0$, which implies that the distribution is degenerated or singular, which means that is concentrated in lower dimensional subspace of $\mathbb{R}^2$. This yields that the distribution of (X,Y) is not absolutely continuous with respect to λ^2. A more formal proof is as follows: Define $Z = Y - \mu_Y - \rho\frac{\sigma_Y}{\sigma_X}(X - \mu_X)$ and check that $\mathbb{E}[Z^2] = (1-\rho^2)\sigma_Y^2$, that is, $\mathbb{E}[Z^2] = 0$ iff $|\rho| = 1$. Thanks to exercise 16, we deduce that

$$Y = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(X - \mu_X), \qquad P\text{-a.s} \tag{1}$$

iff $|\rho| = 1$. This shows precisely that the values of (X, Y) are concentrated on the line given by equation (1), which has two dimensional Lebesgue measure zero. This fact contradicts the very definition of $P_{X,Y}$ being absolutely continuous with respect to λ^2. To compute the law of Y conditioned to X, assuming that $|\rho| < 1$, we use the formula in Example 45 in Lecture 3. We get, after a little bit of algebra, that

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1}{\sqrt{2\pi\sigma_Y^2\,(1-\rho^2)}}\exp\left(-\frac{\left(y - \left(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)\right)\right)^2}{2\sigma_Y^2(1-\rho^2)}\right).$$

which yields that $Y|X = x$ is $\mathcal{N}(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X), \sigma_Y^2\left(1-\rho^2\right))$. The case $|\rho| = 0$ yields that Y is the constant $\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X)$.

28. The idea is to check that $P(Z \le z) = P(Y \le z)$ for all $z \in \mathbb{R}$. Set $\phi(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$. Then,

$$\begin{aligned}
P(Z &\le z) = P(Y\mathbf{1}_{\{|Y|\le a\}} - Y\mathbf{1}_{\{|Y|>a\}} \le z)\\
&= \mathbb{E}[\mathbf{1}_{\{Y\mathbf{1}_{\{|Y|\le a\}} - Y\mathbf{1}_{\{|Y|>a\}} \le z\}}]\\
&= \int_{-\infty}^{+\infty} \mathbf{1}_{\{Y\mathbf{1}_{\{|Y|\le a\}} - Y\mathbf{1}_{\{|Y|>a\}} \le z\}}\phi(y)dy\\
&= \int_{-\infty}^{-a} \mathbf{1}_{\{-y\le z\}}\phi(y)dy + \int_{-a}^{a} \mathbf{1}_{\{y\le z\}}\phi(y)dy + \int_{a}^{+\infty} \mathbf{1}_{\{-y\le z\}}\phi(y)dy\\
&= -\int_{+\infty}^{a} \mathbf{1}_{\{u\le z\}}\phi(-u)du + \int_{-a}^{a} \mathbf{1}_{\{y\le z\}}\phi(y)dy - \int_{-a}^{-\infty} \mathbf{1}_{\{u\le z\}}\phi(-u)du\\
&= \int_{a}^{+\infty} \mathbf{1}_{\{u\le z\}}\phi(u)du + \int_{-a}^{a} \mathbf{1}_{\{y\le z\}}\phi(y)dy + \int_{-\infty}^{-a} \mathbf{1}_{\{u\le z\}}\phi(u)du\\
&= \int_{-\infty}^{+\infty} \mathbf{1}_{\{y\le z\}}\phi(y)dy = P(Y \le z),
\end{aligned}$$

where we have used the change of variable $-y = u$ and the fact that $\phi(-x) = \phi(x)$. On the other hand, (Y, Z) is multivariate Gaussian iff any lineal combination of X and Y is a (one dimensional) Gaussian random variable. Consider $W := Y + Z = 2Y\mathbf{1}_{\{|Y|\le a\}}$. Clearly W is not Gaussian because $P(W > b) = 0$ for any $b > a$, while for any Gaussian random variable this probability is strictly positive. Hence, we can conclude that (Y, Z) is NOT multivariate Gaussian.

29. The idea is to check that $\mathbb{E}[(Y - X)^2] = 0$, which yields that $Y = X, P$-a.s. by exercise 16. When checking that $\mathbb{E}[(Y - X)^2] = 0$ one uses the hypothesis in the exercise, the conservation of the expectation property of the conditional expectation and "what is measurable goes out" property of the conditional expectation.

30. We call the conditional expectation defining property (CEDP) the following property:

$$\forall B \in \mathcal{G} \qquad \mathbb{E}[X\mathbf{1}_B] = \mathbb{E}[Z\mathbf{1}_B]. \tag{2}$$

In order to prove that $\mathbb{E}[X|\mathcal{G}]$ is equal to some given random variable Z we always have to check two things. First, that the candidate Z is $\mathcal{G}$-measurable and second that the candidate Z satisfies (2). Then, we can conclude that $\mathbb{E}[X|\mathcal{G}] = Z, P$-a.s.

(a) It follows from the CEDP taking $B = \Omega$ and the fact that a constant, in particular $\mathbb{E}[X]$, is measurable with respect to any σ-algebra, in particular $\mathcal{G}$.

(b) Set $A_- = \{\mathbb{E}[X|\mathcal{G}] < 0\}$, which is $\mathcal{G}$-measurable, and assume that $P(A_-) > 0$. Then, using the monotonicity property for the ordinary expectation we get that $\mathbb{E}[X\mathbf{1}_{A_-}] \geq 0$ because by hypothesis $X \geq 0, P$-a.s.. However, by the CEDP we get that

$$\mathbb{E}[X\mathbf{1}_{A_-}] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]\mathbf{1}_{A_-}] < 0,$$

which is a contradiction and we can conclude that $P(A_-) = 0$, which yields that $\mathbb{E}[X|\mathcal{G}] \geq 0, P$-a.s.

(c) That $\mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}] = \mathbb{E}[X|\mathcal{H}]$ follows by exercise 30.(d) because $\mathbb{E}[X|\mathcal{H}]$ is also $\mathcal{G}$ measurable. In order to prove that $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$, for all $B \in \mathcal{H} \subset \mathcal{G}$ we have that

$$\mathbb{E}[\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}]\mathbf{1}_B] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]\mathbf{1}_B] = \mathbb{E}[X\mathbf{1}_B] = \mathbb{E}[\mathbb{E}[X|\mathcal{H}]\mathbf{1}_B],$$

where we have used the CEDP with respect to the appropriate σ-algebras.

(d) This property is proved first for indicator functions, then for simple functions, then for positive functions and finally for arbitrary functions. Let $Y = \mathbf{1}_A, A \in \mathcal{G}$. For $B \in \mathcal{G}$ we have that

$$\begin{aligned}\mathbb{E}[\mathbb{E}[XY|\mathcal{G}]\mathbf{1}_B] &= \mathbb{E}[XY\mathbf{1}_B] = \mathbb{E}[X\mathbf{1}_{A\cap B}] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]\mathbf{1}_{A\cap B}] \\ &= \mathbb{E}[\mathbf{1}_A\mathbb{E}[X|\mathcal{G}]\mathbf{1}_B] = \mathbb{E}[Y\mathbb{E}[X|\mathcal{G}]\mathbf{1}_B],\end{aligned}$$

where we have used the CEDP, that $A \cap B \in \mathcal{G}$ and the definition of Y. This proves the result for indicator functions. For simple functions (linear combinations of indicator functions of $\mathcal{G}$ measurable sets) the result follows from the linearity of conditional expectation. If Y is a positive functions, consider

$$Y_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n}\mathbf{1}_{\{\frac{i-1}{2^n} \leq Y < \frac{i}{2^n}\}} + n\mathbf{1}_{\{Y \geq \frac{i}{2^n}\}}, \qquad n \geq 1,$$

which is an increasing sequence of positive, simple and $\mathcal{G}$-measurable functions such that $Y_n \nearrow Y, P$-a.s.. Then, as $XY \in L^1$ the conditional expectation exists and

$$\mathbb{E}[XY|\mathcal{G}] = \mathbb{E}[X \lim_{n\to\infty} Y_n|\mathcal{G}] = \lim_{n\to\infty} \mathbb{E}[XY_n|\mathcal{G}] = \lim_{n\to\infty} Y_n\mathbb{E}[X|\mathcal{G}] = Y\mathbb{E}[X|\mathcal{G}], \quad P\text{-a.s.},$$

where we have used the conditional monotone convergence theorem and that the result holds for simple functions. One could have used the CEDP with the usual monotone convergence theorem to prove the result. For a general $\mathcal{G}$-measurable Y such that $YX \in L^1$, note that also $Y^+X \in L^1$ and $Y^-X \in L^1$ and then we can write

$$\mathbb{E}[XY|\mathcal{G}] = \mathbb{E}[XY^+|\mathcal{G}] - \mathbb{E}[XY^-|\mathcal{G}] = Y^+\mathbb{E}[X|\mathcal{G}] - Y^-\mathbb{E}[X|\mathcal{G}] = Y\mathbb{E}[X|\mathcal{G}].$$

31. By exercise 13., we know that the candidate random variable must be constant on the elements of the partition $\{A_n\}_{n\geq 1}$. This is the case for $\sum_{n\geq 1} \frac{\mathbb{E}[X\mathbf{1}_{A_n}]}{P(A_n)}\mathbf{1}_{A_n}$. Therefore, we only need to prove that it satisfies CEDP. This follows easily from the particular structure of the elements of $\mathcal{G}$, described in exercise 13. We have that $B \in \mathcal{G}$ iff $B = \cup_{i\in J} A_i$ where J is a countable subset of $\mathbb{N}$. Then, on the one hand

$$\mathbb{E}[X\mathbf{1}_B] = \mathbb{E}[X\sum_{i\in J}\mathbf{1}_{A_i}] = \sum_{i\in J}\mathbb{E}[X\mathbf{1}_{A_i}],$$

and, on the other hand

$$\begin{aligned}\mathbb{E}[\left(\sum_{n\geq 1}\frac{\mathbb{E}[X\mathbf{1}_{A_n}]}{P(A_n)}\mathbf{1}_{A_n}\right)\mathbf{1}_B] &= \mathbb{E}[\left(\sum_{n\geq 1}\frac{\mathbb{E}[X\mathbf{1}_{A_n}]}{P(A_n)}\mathbf{1}_{A_n}\right)\left(\sum_{i\in J}\mathbf{1}_{A_i}\right)] \\ &= \mathbb{E}[\sum_{i\in J}\frac{\mathbb{E}[X\mathbf{1}_{A_i}]}{P(A_i)}\mathbf{1}_{A_i}] \\ &= \sum_{i\in J}\mathbb{E}[\frac{\mathbb{E}[X\mathbf{1}_{A_i}]}{P(A_i)}\mathbf{1}_{A_i}] = \sum_{i\in J}\frac{\mathbb{E}[X\mathbf{1}_{A_i}]}{P(A_i)}\mathbb{E}[\mathbf{1}_{A_i}] \\ &= \sum_{i\in J}\mathbb{E}[X\mathbf{1}_{A_i}].\end{aligned}$$

The only delicate point in the previous reasoning is the commutation between taking expectation and the sum of a series. But this can be justified using Fubini's theorem or Dominated convergence (try to write the precise reasoning).

32. To be handed in.

33. Denote by $\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$ the density of a standard normal distribution and $\Phi(z) = \int_{-\infty}^{z} \phi(y)dy$ its cumulative distribution function. Therefore, using the image measure theorem and then the change of variable $x = \mu + \sigma z, v$, one gets

$$
\begin{aligned}
\mathbb{E}[\max(0, \exp(X) - K)] &= \int_{\mathbb{R}} \max(0, \exp(x) - K)\, dP_X \\
&= \int_{-\infty}^{+\infty} \max(0, \exp(x) - K) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\
&= \int_{-\infty}^{+\infty} \max(0, \exp(\mu + \sigma z) - K)\, \phi(z)dz \\
&= \int_{-\infty}^{+\infty} (\exp(\mu + \sigma z) - K)\, \mathbf{1}_{\{\exp(\mu+\sigma z) \geq K\}} \phi(z)dz \\
&= \int_{\frac{\log(K)-\mu}{\sigma}}^{+\infty} \exp(\mu + \sigma z)\phi(z)dz - K \int_{\frac{\log(K)-\mu}{\sigma}}^{+\infty} \phi(z)dz \\
&= A_1 + A_2
\end{aligned}
$$

Note that $\Phi(z) + \Phi(-z) = 1$ for all $z \in \mathbb{R}$. Hence,

$$
A_2 = -K \int_{\frac{\log(K)-\mu}{\sigma}}^{+\infty} \phi(z)dz = -K\left(1 - \Phi\left(\frac{\log(K)-\mu}{\sigma}\right)\right) = -K\Phi\left(\frac{\mu - \log(K)}{\sigma}\right).
$$

For the term A_1, first note that

$$
\exp(\mu + \sigma z)\phi(z) = \exp\left(\mu + \frac{\sigma^2}{2}\right)\phi(z - \sigma),
$$

and, therefore, making the change of variable $y = z - \sigma$ we get

$$
\begin{aligned}
A_1 &= \exp\left(\mu + \frac{\sigma^2}{2}\right) \int_{\frac{\log(K)-\mu}{\sigma}}^{+\infty} \phi(z - \sigma)dz \\
&= \exp\left(\mu + \frac{\sigma^2}{2}\right) \int_{\frac{\log(K)-\mu-\sigma^2}{\sigma}}^{+\infty} \phi(y)dy = \exp\left(\mu + \frac{\sigma^2}{2}\right)\left(1 - \Phi\left(\frac{\log(K) - \mu - \sigma^2}{\sigma}\right)\right) \\
&= \exp\left(\mu + \frac{\sigma^2}{2}\right)\Phi\left(\frac{\mu + \sigma^2 - \log(K)}{\sigma}\right).
\end{aligned}
$$

Therefore, we get that

$$
\mathbb{E}[\max(0, \exp(X) - K)] = \exp\left(\mu + \frac{\sigma^2}{2}\right)\Phi\left(\frac{\mu + \sigma^2 - \log(K)}{\sigma}\right) - K\Phi\left(\frac{\mu - \log(K)}{\sigma}\right).
$$