Introduction and Techniques
in Financial Mathematics
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Exercises
List 2
Solutions and Hints

# Brownian Motion and Stochastic Calculus

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Recall first some definitions given in class.

**Definition 1 (Def. Class)** *A standard Brownian motion is a process satisfying*

1. *$W$ has continuous paths $P$-a.s.,*
2. *$W_0 = 0, P$-a.s.,*
3. *$W$ has independent increments,*
4. *For all $0 \leq s < t$, the law of $W_t - W_s$ is a $\mathcal{N}(0,(t-s))$.*

**Definition 2** *$X$ is a Gaussian process if for any $t_1, t_2, \cdots t_n, \in \mathbb{R}_+, n \in \mathbb{N}$ the vector*

$$(X_{t_1}, X_{t_2}, ..., X_{t_n}),$$

*is multivariate normal.*

A useful criterion to check if a vector is multivariate normal is the following

**Proposition 3** *A vector $(X_1, ..., X_n)$ is multivariate normal if and only if for all $\lambda_i \in \mathbb{R}, i = 1, ..., n$ one has that the random variable $\sum_{i=1}^n \lambda_i X_i$ is (univariate) normal.*

**Remark 4** *Note that, by proposition 3, we may assume that the times $\{t_i\}_{i=1,...,n}$ in definition 2 are ordered, i.e., $0 \leq t_1 < t_2 < \cdots < t_n$.*

In Exercise 1 the following alternative definition of Brownian motion is introduced.

**Definition 5 (Def. Gaussian)** *A standard Brownian motion is a process satisfying*

**a)** *$W$ has continuous paths $P$-a.s.,*

**b)** *$W$ is a Gaussian process,*

**c)** *$W$ is centered ($\mathbb{E}[W_t] = 0$) and the covariance function*

$$K(s,t) \triangleq \mathbb{E}[(W_t - \mathbb{E}[W_t])(W_s - \mathbb{E}[W_s])] = \mathbb{E}[W_t W_s] = \min(s,t).$$

Recall also the definition of $\mathbb{F}$-Brownian motion.

**Definition 6** *A $\mathbb{F}$-Brownian motion $W$ is a real stochastic process adapted to $\mathbb{F}$ satisfying*

1. *$W$ has continuous paths $P$-a.s.,*
2. *$W_0 = 0, P$-a.s,*
3. *For all $0 \leq s < t$, the random variable $W_t - W_s$ is independent of $\mathcal{F}_s$.*
4. *For all $0 \leq s < t$, the law of $W_t - W_s$ is a $\mathcal{N}(0,(t-s))$.*

1. In this exercise we have to show that Def. Class is equivalent to Def. Gaussian

   Def. Class $\Longrightarrow$ Def. Gaussian.: Clearly 1. $\Rightarrow$ $a$). Properties 3. and 4. yields that for any $0 \leq t_1 < t_2 < \cdots < t_n, n \in \mathbb{N}$ the vector

   $$(W_{t_n} - W_{t_{n-1}}, W_{t_{n-1}} - W_{t_{n-2}}, ..., W_{t_1})$$

   has multivariate normal distribution and, by a linear transform, we get that

   $$(W_{t_1}, ..., W_{t_{n-1}}, W_{t_n})$$

   has a multivariate normal distribution. Alternatively, for all $\lambda_i \in \mathbb{R}, i = 1, ..., n$ we have that

   $$\sum_{i=1}^{n} \lambda_i W_{t_i} = \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{i} \left(W_{t_j} - W_{t_{j-1}}\right) = \sum_{j=1}^{n} \left(\sum_{i=j}^{n} \lambda_i\right) \left(W_{t_j} - W_{t_{j-1}}\right),$$

   which is a univariate normal by properties 3. and 4. Hence, we can conclude that $W$ is a Gaussian process. By 4. we get that $\mathbb{E}[W_t] = 0$ for all $t \geq 0$. Moreover, if $s < t$ we have that

   $$\begin{aligned} K(s,t) &= \mathbb{E}[W_s W_t] = \mathbb{E}[W_s(W_t - W_s) + W_s^2] = \mathbb{E}[W_s(W_t - W_s)] + \mathbb{E}[W_s^2] \\ &= \mathbb{E}[W_s]\mathbb{E}[(W_t - W_s)] + s = s, \end{aligned}$$

   where we have used 3. and 4. A similar reasoning can be done if $t < s$, so we get that $K(s,t) = \min(s,t)$.

   Def. Gaussian $\Longrightarrow$ Def. Class.: Clearly $a$) $\Rightarrow$ 1. By property $c$) we have that $\mathbb{E}[W_0] = 0$ and $\mathbb{E}[W_0^2] = K(0,0) = 0$, which yields that $W_0 = 0, P$-a.s. and, hence, property 2. is satisfied. Note that if $(Z_1, Z_2)$ is bivariate Gaussian then $Z_1$ is independent of $Z_2$ if and only if $\mathbb{E}[Z_1 Z_2] = \mathbb{E}[Z_1]\mathbb{E}[Z_2]$. In order to prove 3., we have to show that $W_t - W_s$ is independent of $W_v - W_u$ for any $0 \leq u \leq v \leq s \leq t$. As $W$ is a Gaussian process by $b$), we have that $(W_t - W_s, W_v - W_u)$ is bivariate Gaussian and it suffices to prove that

   $$\mathbb{E}[(W_t - W_s)(W_v - W_u)] = 0,$$

   but using property $c$) we get that

   $$\begin{aligned} \mathbb{E}[(W_t - W_s)(W_v - W_u)] &= K(t,v) - K(t,u) - K(s,v) + K(s,u) \\ &= v - u - v + u = 0. \end{aligned}$$

   Finally, by $b$) again, we have that for all $0 \leq s < t$ the law of $W_t - W_s$ is Gaussian. And using $c$) we get that

   $$\begin{aligned} \mathbb{E}[W_t - W_s] &= \mathbb{E}[W_t] - \mathbb{E}[W_s] = 0, \\ Var[W_t - W_s] &= \mathbb{E}\left[(W_t - W_s)^2\right] \\ &= K(t,t) - K(t,s) - K(s,t) + K(s,s) = t - s, \end{aligned}$$

   which yields property 4.

2. $W$ is a Brownian motion, $a > 0$ and $\mathbb{F}$ the minimal augmented filtration generated by $W$.

   (a) $X_t = -W_t, t \in \mathbb{R}_+ : X$ has continuous paths $P$-a.s. because $W$ has continuous paths $P$-a.s.. For any $0 \leq t_1 < t_2 < \cdots < t_n, n \in \mathbb{N}$ the vector

   $$(X_{t_1}, ..., X_{t_{n-1}}, X_{t_n}) = (-W_{t_1}, ..., -W_{t_{n-1}}, -W_{t_n})$$

is Gaussian because $(W_{t_1},...,W_{t_{n-1}},W_{t_n})$ is Gaussian, as $W$ is a Gaussian process. Moreover

$$\begin{aligned}
\mathbb{E}[X_t] &= \mathbb{E}[-W_t] = -\mathbb{E}[W_t] = 0, \\
K(s,t) &= \mathbb{E}[X_s X_t] = \mathbb{E}[(-W_s)(-W_t)] = \mathbb{E}[W_s W_t] = \min(s,t).
\end{aligned}$$

Therefore, by the previous exercise we can conclude that $X$ is a Brownian motion. Note also that $\mathbb{F} = \mathbb{F}^X$. This is because $X_t = \varphi(W_t)$, where $\varphi(x) = -x$ is a bijection. This yields that $\sigma(X_t) = \sigma(W_t)$ and, therefore, $\mathcal{F}_t^X = \sigma(X_s : 0 \le s \le t) = \sigma(W_s : 0 \le s \le t) = \mathcal{F}_t$ for any $t$. Hence, we can conclude that $X$ is also an $\mathbb{F}$-Brownian motion.

(b) $X_t = W_{a+t} - W_a, t \in \mathbb{R} : X$ has continuous paths $P$-a.s. because $W$ has continuous paths $P$-a.s.. For any $0 < t_1 < t_2 < \cdots < t_n, n \in \mathbb{N}$ the vector

$$(W_a, W_{a+t_1}, ..., W_{a+t_{n-1}}, W_{a+t_n}),$$

is multivariate normal because $W$ is a Gaussian process. By a linear transformation of the previous vector we get that

$$\left(W_{a+t_1} - W_a, ..., W_{a+t_{n-1}} - W_a, W_{a+t_n} - W_a\right),$$

is also multivariate normal and, hence, the process $X$ is Gaussian. We also have that

$$\begin{aligned}
\mathbb{E}[X_t] &= \mathbb{E}[W_{a+t} - W_a] = 0, \\
K(s,t) &= \mathbb{E}[X_s X_t] = \mathbb{E}[(W_{a+s} - W_a)(W_{a+t} - W_a)] \\
&= K(a+s, a+t) - K(a+s, a) - K(a, a+t) + K(a,a) \\
&= \min(a+s, a+t) - \min(a+s, a) - \min(a, a+t) + \min(a,a) \\
&= \min(a+s, a+t) - a = \min(s,t).
\end{aligned}$$

Therefore, we can conclude that $X$ is a Brownian motion. However, $X$ is not an $\mathbb{F}$-Brownian motion because $X$ is not adapted to $\mathbb{F}$, note that $X_t = W_{a+t} - W_a$ depends on $W_{a+t}$ which is not $\mathcal{F}_t$ measurable.

(c) $X_t = W_{at^2}, t \in \mathbb{R}_+ : X$ is not a Brownian motion because, although has continuous paths, is Gaussian and centered, one has that

$$K(s,t) = \mathbb{E}[X_s X_t] = \mathbb{E}[X_{as^2} W_{at^2}] = \min(as^2, as^2) \neq \min(s,t).$$

As $X$ is not a Brownian motion, it cannot be an $\mathbb{F}$-Brownian motion.

3. Let $f \in L^2([0,T])$. First we will show that $X_t = \int_0^t f(s)dW_s \sim \mathcal{N}\left(0, \int_0^t |f(s)|^2 ds\right)$ for every $t \in [0,T]$. Define $f_t(s) \triangleq f(s)\mathbf{1}_{[0,t]}(s)$ and note that $|f_t(s)| \le |f(s)|$ for all $s \in [0,T]$. As $f$ is deterministic, we have that for all $t \in [0,T]$ the process $f_t$ it is also measurable and adapted and

$$\mathbb{E}\left[\int_0^T |f_t(s)|^2 ds\right] = \int_0^T |f_t(s)|^2 ds \le \int_0^T |f(s)|^2 ds < \infty.$$

Hence, $f_t \in L^2_{a,T}$ and $X_t = \int_0^t f(s)dW_s = \int_0^T f_t(s)dW_s$. Consider a sequence of partitions

$$\pi^n = \{0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = T\}, n \ge 1,$$

with $\|\pi^n\| \triangleq \max_{1 \le i \le n} |t_i - t_{i-1}|$ converging to zero when $n$ tends to infinity. We can consider the sequence of function $f_{t,n}(s) = \sum_{i=1}^{n-1} f_t(t_i)\mathbf{1}_{[t_i,t_{i+1})}(s)$. We have that $f_{t,n}$ converges $\lambda$-a.e. to $f_t$ and

$$\mathbb{E}\left[\int_0^T |f_t(s) - f_{t,n}(s)|^2 ds\right] = \int_0^T |f_t(s) - f_{t,n}(s)|^2 ds \underset{n\to\infty}{\longrightarrow} 0,$$

by dominated convergence. Hence, $f_{n,t}(s)$ (as a process in $s$) is a sequence of simple processes approximating $f_t$ in $L^2_{a,T}$ and, by the construction of the Itô integral, we have that

$$\int_0^T f_{t,n}(s)dW_s \overset{L^2}{\underset{n\to\infty}{\longrightarrow}} \int_0^T f_t(s)W_s.$$

In addition, for every $n \geq 1$,

$$\int_0^T f_{t,n}(s)dW_s = \sum_{i=1}^{n-1} f_t(t_i)\left(W_{t_i} - W_{t_i}\right),$$

which is a sum of independent Gaussian random variables with law $\mathcal{N}(0, \sum_{i=1}^{n-1} |f_t(t_i)|^2 (t_i - t_{i-1}))$. We recall the following result that we give without proof (it is easy using the relationship between characteristic functions and weak convergence). Let $Z_n$ be a sequence of random variables with laws $\mathcal{N}(0, \sigma_n^2)$. If $Z_n \xrightarrow[n\to\infty]{L^2} Z_\infty$ then $\sigma_\infty^2 \triangleq \lim_{n\to\infty} \sigma_n^2 < \infty$ and $Z_\infty \sim \mathcal{N}(0, \sigma_\infty^2)$. In our case, we have that $Z_\infty = \int_0^T f_t(s)W_s = \int_0^t f(s)W_s$ and $\sigma_\infty^2 = \int_0^T f_t^2(s)ds = \int_0^t f^2(s)ds$. Therefore we can conclude that $X_t = \int_0^t f(s)dW_s \sim \mathcal{N}(0, \int_0^t f^2(s)ds)$. Finally, to show that $X$ is a Gaussian process, note that for all $\lambda_i \in \mathbb{R}$, and $\{t_i\}_{i=1,\ldots,n} \in \mathbb{R}_+, n \in \mathbb{N}$ we have that

$$\sum_{i=1}^n \lambda_i X_{t_i} = \sum_{i=1}^n \lambda_i \int_0^{t_i} f(s)dW_s = \int_0^T \left(\sum_{i=1}^n \lambda_i f_{t_i}(s)\right) dW_s,$$

which is an univariate Gaussian because $\sum_{i=1}^n \lambda_i f_{t_i}(s) \in L^2([0,T])$. By the properties of the Itô integral, we get that $\mathbb{E}[X_t] = 0$ and $\mathbb{E}[X_s X_t] = \int_0^{\min(s,t)} f^2(u)du$.

4. Let $W$ be a Brownian motion and $\mathbb{F} = \mathbb{F}^W$. Show that the following processes are $\mathbb{F}$-martingales. A general remark on this kind of problems: If you are given a process that is a regular/smooth transformation of a Brownian motion, the straightforward way of checking that the process is a martingale is to use Itô's formula.

   (a) $X_t = \exp\left(\theta W_t - \frac{\theta^2}{2}t\right), t \in [0,T]$ : $X$ is $\mathbb{F}$-adapted because $X_t = \varphi_t(W_t)$ where $\varphi_t$ is a Borel measurable function for every $t \in [0,T]$. This means that $X_t$ is $\sigma(W_t)$-measurable and, in particular, $\mathcal{F}_t$-measurable because $\sigma(W_t) \subset \mathcal{F}_t$ for every $t \in [0,T]$. $X_t \in L^1(\Omega, \mathcal{F}, P)$,i.e., $\mathbb{E}[|X_t|] < \infty$ because $W_t \sim \mathcal{N}(0,t)$ and, hence,

   $$\mathbb{E}[\exp(\theta W_t)] = \exp(\frac{\theta^2}{2}t),$$

   which yields $\mathbb{E}[|X_t|] < \infty$. To check the martingale property $E[X_t|\mathcal{F}_s] = X_s$, we can check that $\mathbb{E}\left[\frac{X_t}{X_s}|\mathcal{F}_s\right] = 1$, because $X_s$ is $\mathcal{F}_s$-measurable and can go inside the conditional expectation. We have that

   $$\begin{aligned}
   \mathbb{E}\left[\frac{X_t}{X_s}|\mathcal{F}_s\right] &= \mathbb{E}\left[\exp\left(\theta(W_t - W_s) - \frac{\theta^2}{2}(t-s)\right)|\mathcal{F}_s\right] \\
   &= \mathbb{E}\left[\exp\left(\theta(W_t - W_s)\right)|\mathcal{F}_s\right] \exp\left(-\frac{\theta^2}{2}(t-s)\right) \\
   &= \mathbb{E}\left[\exp\left(\theta(W_t - W_s)\right)\right] \exp\left(-\frac{\theta^2}{2}(t-s)\right) \\
   &= \exp\left(\frac{\theta^2}{2}(t-s)\right) \exp\left(-\frac{\theta^2}{2}(t-s)\right) = 1,
   \end{aligned}$$

   where we have used that $\exp\left(-\frac{\theta^2}{2}(t-s)\right)$ is deterministic, that $\exp\left(\theta(W_t - W_s)\right)$ is independent of $\mathcal{F}_s$ and the moment generating function of a normal distribution, i.e., for $Z \sim \mathcal{N}(\mu, \sigma^2)$ we have $\mathbb{E}[\exp(\theta Z)] = \exp\left(\mu\theta + \frac{\theta^2}{2}\sigma^2\right)$.

   (b) $Y_t = e^{t/2}\cos(W_t), t \in [0,T]$ : $Y$ is $\mathbb{F}$-adapted by the same kind of reasoning as in section *a.*. That, $Y_t \in L^1(\Omega, \mathcal{F}, P)$ follows from the fact that $|\cos(x)| \leq 1$, we have

   $$\mathbb{E}\left[|Y_t|\right] = \mathbb{E}\left[\left|e^{t/2}\cos(W_t)\right|\right] = e^{t/2}\mathbb{E}\left[|\cos(W_t)|\right] \leq e^{t/2} < \infty.$$

To check the martingale property we will use Itô's formula. Note that $Y_t = f(t, W_t)$ where $f(t,x) = e^{t/2}\cos(x) \in C^{1,2}([0,T]\times\mathbb{R})$ and $\partial_t f = \frac{1}{2}f, \partial_x f(t,x) = -e^{t/2}\sin(x)$ and $\partial_{xx} f = -f$. Hence,

$$\begin{aligned} Y_t &= f(t,W_t) = 1 + \int_0^t \{\partial_t f(s,W_s) + \frac{1}{2}\partial_{xx} f(s,W_s)\} ds + \int_0^t \partial_x f(s,W_s) dW_s \\ &= 1 - \int_0^t e^{s/2}\sin(W_s) dW_s. \end{aligned}$$

As $e^{s/2}\sin(W_s)$ is measurable and adapted and

$$\mathbb{E}\left[\int_0^T \left|e^{t/2}\sin(W_t)\right|^2 dt\right] \leq \mathbb{E}\left[\int_0^T e^t dt\right] \leq e^T T < \infty,$$

we have that $e^{t/2}\sin(W_t) \in L^2_{a,T}$ and the Itô integral is a martingale.

(c) $Z_t = W_t^2 - t, t \in [0,T]$ : $Z$ is $\mathbb{F}$-adapted by the same kind of reasoning as in section $a$.. $Z$ is integrable because

$$\mathbb{E}[|Z_t|] \leq \mathbb{E}[W_t^2] + t = 2t < \infty.$$

To check the martingale property we could use Itô's formula or the property that a Brownian motion has independent increments.

$$\begin{aligned} \mathbb{E}[Z_t|\mathcal{F}_s] &= \mathbb{E}[W_t^2 - t|\mathcal{F}_s] = \mathbb{E}[(W_t - W_s + W_s)^2|\mathcal{F}_s] - t \\ &= \mathbb{E}[(W_t - W_s)^2|\mathcal{F}_s] - \mathbb{E}[2(W_t - W_s)W_s|\mathcal{F}_s] + \mathbb{E}[W_s^2|\mathcal{F}_s] - t \\ &= \mathbb{E}[(W_t - W_s)^2] - 2W_s\mathbb{E}[(W_t - W_s)|\mathcal{F}_s] + W_s^2 - t \\ &= t - s - 2W_s\mathbb{E}[(W_t - W_s)] + W_s^2 - t = W_s^2 - s = Z_s, \end{aligned}$$

where we have used that $W_t - W_s$ is independent of $\mathcal{F}_s$, that if $V$ is independent of $\mathcal{G}$ one has that $\mathbb{E}[V|\mathcal{G}] = \mathbb{E}[V]$, that $W_s$ is $\mathcal{F}_s$-measurable and that $W$ is a centered process.

(d) $G_t = e^{W_t} - 1 - \frac{1}{2}\int_0^t e^{W_s} ds, t \in [0,T]$ : $G$ is $\mathbb{F}$-adapted because $e^{W_t}$ is $\sigma(W_t)$-measurable and $\sigma(W_t) \subset \mathcal{F}_t$. Moreover, $\int_0^t e^{W_s} ds$ is $\mathcal{F}_t$ measurable because (using the definition of Riemann integrable function) is a $P$-a.s. limit of $\mathcal{F}_t$-measurable random variables. In other words, $G_t$ only depends on the values of $W$ up to time $t$. $G$ is integrable because $W_t$ (and $W_s$) has moments exponential moments of all orders. The martingale property follows by applying Itô's formula to $g(W_t)$ with the function $g(x) = e^x$. Note, $\partial_t g = 0, \partial_x g = \partial_{xx} g = g$. Hence,

$$g(W_t) = e^{W_t} = 1 + \int_0^t e^{W_s} dW_s + \frac{1}{2}\int_0^t e^{W_s} ds,$$

which yields that $G_t = \int_0^t e^{W_s} dW_s$. As $e^{W_t}$ is measurable, adapted and

$$\mathbb{E}\left[\int_0^T \left|e^{W_t}\right|^2 dt\right] = \int_0^T \mathbb{E}\left[e^{2W_t}\right] dt = \int_0^T e^{\frac{4t}{2}} dt \leq e^{2T} T < \infty.$$

Hence, $e^{W_t} \in L^2_{a,T}$ and $G_t$ is a martingale (because the Itô integral of a process in $L^2_{a,T}$ is a martingale).

(e) $H_t = \exp\left(\int_0^t f_s dW_s - \frac{1}{2}\int_0^t f_s^2 ds\right), t \in [0,T], f \in L^2([0,T])$ deterministic: $H$ is $\mathbb{F}$-adapted because $\exp(\int_0^t f_s dW_s)$ is $\mathcal{F}_t$-measurable as the Itô integrals is an $\mathbb{F}$-adapted process. By exercise 3., the law of $\int_0^t f_s dW_s$ is $\mathcal{N}\left(0, \int_0^t f_s^2 ds\right)$ because $f$ is deterministic and square integrable. This yields that $H$ is integrable. To check the martingale property we can repeat the same arguments as in section $a$. or use Itô's formula to $h(x) = \exp(x)$ applied to the Itô process $\int_0^t f_s dW_s - \frac{1}{2}\int_0^t f_s^2 ds$.

5. The square integrability of $M$ ensure that the conditional expectation exists. The result follows using the basic properties of the conditional expectation and expanding $(M_t - M_s)^2$. We have that

$$\begin{aligned}
\mathbb{E}[(M_t - M_s)^2|\mathcal{F}_s] &= \mathbb{E}\left[M_t^2 - 2M_sM_t + M_s^2|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[M_t^2|\mathcal{F}_s\right] - 2M_s\mathbb{E}\left[M_t|\mathcal{F}_s\right] + M_s^2 \\
&= \mathbb{E}\left[M_t^2|\mathcal{F}_s\right] - 2M_s^2 + M_s^2 \\
&= \mathbb{E}\left[M_t^2 - M_s^2|\mathcal{F}_s\right],
\end{aligned}$$

where we have used that $M_s$ and $M_s^2$ are $\mathcal{F}_s$-measurable and $\mathbb{E}\left[M_t|\mathcal{F}_s\right] = M_s$, the martingale property of $M$.

6. $Y_t = \varphi(M_t)$ where $\varphi$ is a convex function and $M$ is an $\mathbb{F}$-martingale. $Y$ is integrable by assumption. Note that as $\varphi$ is convex it is continuous and, hence, Borel measurable. Therefore, $Y_t$ is $\sigma(M_t)$-measurable and, as $M$ is $\mathbb{F}$-adapted, we have that $Y_t$ is $\mathcal{F}_t$-measurable, which yields that $Y$ is $\mathbb{F}$-adapted. Finally the submartingale property follows from the conditional Jensen's inequality and the fact that $M$ is a martingale, i.e., if $h(V) \in L^1(\Omega, \mathcal{F}, P)$ and $\mathcal{G}$ is a sub-$\sigma$-algebra of $\mathcal{F}$ we have that $\mathbb{E}[h(V)|\mathcal{G}] \geq h(\mathbb{E}[V)|\mathcal{G}])$. Let's write it:

$$\mathbb{E}[Y_t|\mathcal{F}_s] = \mathbb{E}[\varphi(M_t)|\mathcal{F}_s] \geq \varphi(\mathbb{E}[M_t|\mathcal{F}_s]) = \varphi(M_s) = Y_s.$$

7. Let $\mathbb{F}^X = \{\mathcal{F}_t^X = \sigma(X_s : 0 \leq s \leq t)\}_{t\in\mathbb{R}_+}$ and $\mathbb{G}^X = \{\mathcal{F}_t^X = \sigma(X_v - X_u : 0 \leq u \leq v \leq t)\}_{t\in\mathbb{R}_+}$ the natural filtrations generated by the process $X$ and by the increments of the process $X$. These filtrations are actually the same because for any $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \in \mathbb{R}_+, n \in \mathbb{N}$, we have that $\sigma(X_{t_1}, X_{t_2}, ..., X_{t_n}) = \sigma(X_{t_1}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}})$ (note that there is a bijection between these two vectors). The hypothesis in this problem is that the process $X$ has constant mean ($\mathbb{E}[X_t] = m, t \in \mathbb{R}_+$) and independent increments. The independent increments property can be written as follows: for all $s \leq t$ the random variable $X_t - X_s$ is independent of all random variables $X_v - X_u$ with $0 \leq u \leq v \leq s$. Therefore, we have that $X_t - X_s$ is independent of $\mathcal{G}_s^X$. Hence,

$$\mathbb{E}[X_t - X_s|\mathcal{F}_s^X] = \mathbb{E}[X_t - X_s|\mathcal{G}_s^X] = \mathbb{E}[X_t - X_s] = \mathbb{E}[X_t] - \mathbb{E}[X_s] = 0,$$

and we can conclude that $X$ is a martingale (with respect to its natural filtration $\mathcal{F}_t^X$).

8. Let $X \in L^p(\Omega, \mathcal{F}, P)$ for some $p \geq 1$ and $\mathbb{F}$ be a filtration in the probability space $(\Omega, \mathcal{F}, P)$. Then $M_t = \mathbb{E}[X|\mathcal{F}_t]$ is a $\mathbb{F}$-martingale. $M$ is $\mathbb{F}$-adapted by construction. We prove the $p$-th integrability of $M$ (which implies the integrability of $M$). It follows by the conditional Jensen's inequality applied to the convex function $\varphi(x) = |x|^p$, the conservation of expectation property of the conditional expectation and the hypothesis $X \in L^p(\Omega, \mathcal{F}, P)$,

$$\mathbb{E}[|M_t|^p] = \mathbb{E}[|\mathbb{E}[X|\mathcal{F}_t]|^p] \leq \mathbb{E}[\mathbb{E}[|X|^p|\mathcal{F}_t]] = \mathbb{E}[|X|^p] < \infty.$$

The martingale property of $M$ follows from the tower property of the conditional expectation

$$\mathbb{E}[M_t|\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[X|\mathcal{F}_s] = M_s.$$

9. For this exercise I only give the solution. $W = (W_t^1, W_t^2, W_t^3)_{t\in\mathbb{R}_+}$ is a 3-dimensional Brownian motion.

   (a) $u_1(t, W_t) = 5 + 4t + \exp\left(3W_t^1\right)$ has the following Itô differential

$$\begin{aligned}
du_1(t, W_t) &= 3\exp\left(3W_t^1\right)dW_t^1 + \left\{4 + \frac{9}{2}\exp\left(3W_t^1\right)\right\}dt, \\
u_1(0,0) &= 5.
\end{aligned}$$

   (b) $u_2(t, W_t) = (W_t^2)^2 + (W_t^3)^2$ has the following Itô differential

$$\begin{aligned}
du_2(t, W_t) &= 2W_t^2dW_t^2 + 2W_t^3dW_t^3 + 2dt, \\
u_2(0,0) &= 0.
\end{aligned}$$

(c) $u_3(t,W_t) = \log(u_1(t,W_t)u_2(t,W_t))$ has the following Itô differential

$$\begin{aligned} du_3(t,W_t) &= \frac{3\exp\left(3W_t^1\right)}{5+4t+\exp\left(3W_t^1\right)}dW_t^1 + \frac{2W_t^2}{(W_t^2)^2+(W_t^3)^2}dW_t^2 \\ &+\frac{2W_t^3}{(W_t^2)^2+(W_t^3)^2}dW_t^3 + \left\{ \frac{4+\frac{9}{2}\exp\left(3W_t^1\right)}{5+4t+\exp\left(3W_t^1\right)} \right. \\ &+\frac{2}{(W_t^2)^2+(W_t^3)^2} - \frac{9}{2}\frac{\exp\left(6W_t^1\right)}{\left(5+4t+\exp\left(3W_t^1\right)\right)^2} \\ &\left. -\frac{2(W_t^2)^2+2(W_t^3)^2}{\left((W_t^2)^2+(W_t^3)^2\right)^2} \right\} dt \end{aligned}$$

10. Recall that a discrete time, integrable and $\{\mathcal{H}_n\}_{n\geq 0}$-adapted process $\{Z_n\}_{n\geq 0}$ is a $\{\mathcal{H}_n\}$-martingale if $\mathbb{E}[Z_n|\mathcal{H}_{n-1}] = Z_{n-1}$ for all $n \geq 1$. $G_n$ is $\{\mathcal{F}_n\}$-adapted because $H_i(M_i - M_{i-1})$ is $\mathcal{F}_i$-measurable for $0 \leq i \leq n$ and, hence, $\mathcal{F}_n$-measurable. Next step is to check that $\mathbb{E}[|G_n|] < \infty$ for all $n \geq 1$. We have, using the triangular inequality and the fact that $M$ is integrable because it is a martingale, that

$$\begin{aligned} \mathbb{E}[|G_n|] &= \mathbb{E}\left[\left|\sum_{i=1}^n H_i(M_i - M_{i-1})\right|\right] \leq \sum_{i=1}^n \mathbb{E}\left[|H_i(M_i - M_{i-1})|\right] \\ &\leq \sum_{i=1}^n C_i \mathbb{E}\left[|(M_i - M_{i-1})|\right] \leq \sum_{i=1}^n C_i \left\{\mathbb{E}\left[|M_i|\right] + \mathbb{E}\left[|M_{i-1}|\right]\right\} \\ &\leq 2\sup_{0\leq i\leq n} \mathbb{E}\left[|M_i|\right] \sum_{i=1}^n C_i < \infty, \end{aligned}$$

To check the martingale property we can write

$$\begin{aligned} \mathbb{E}[G_n|\mathcal{F}_{n-1}] &= \mathbb{E}\left[\sum_{i=1}^n H_i(M_i - M_{i-1})|\mathcal{F}_{n-1}\right] \\ &= \mathbb{E}\left[G_{n-1} + H_n(M_n - M_{n-1})|\mathcal{F}_{n-1}\right] \\ &= G_{n-1} + H_n\mathbb{E}\left[M_n - M_{n-1}|\mathcal{F}_{n-1}\right] \\ &= G_{n-1} + \mathbb{E}\left[M_n|\mathcal{F}_{n-1}\right] - M_{n-1} \\ &= G_{n-1} + M_{n-1} - M_{n-1} = G_{n-1}, \end{aligned}$$

where we have used that $G_{n-1}, H_n$ and $M_{n-1}$are $\mathcal{F}_{n-1}$-measurable and $M$ is a martingale.

11. In this exercise we have to find the Itô representation of some square integrable random variables, that is, if $F \in L^2(\Omega, \mathcal{F}_T, P)$ there exists a process $f \in L^2_{a;T}$ such that $F = \mathbb{E}[F] + \int_0^T f_s dW_s$.

(a) $F_1 = W_T$. We can write

$$W_T = \int_0^T dW_t = 0 + \int_0^T dW_t = \mathbb{E}[W_T] + \int_0^T 1 dW_t.$$

(b) $F_2 = W_T^2$. We can use Itô's formula to get that

$$\begin{aligned} W_T^2 &= W_0 + \int_0^T \frac{2}{2}dt + \int_0^T 2W_t dW_t \\ &= 0 + T + \int_0^T 2W_t dW_t = \mathbb{E}[W_T^2] + \int_0^T 2W_t dW_t. \end{aligned}$$

(c) $F_3 = e^{W_T}$. Here, the idea is to use that we know that $f(t,W_t) = \exp(W_t - \frac{t}{2})$ is a martingale so we can write it as an stochastic integral of some process in $L^2_{a,T}$. To find such a process we use Itô's formula. We have that $\partial_t f(t,x) = -\frac{1}{2} f(t,x)$ and $\partial_x f(t,x) = \partial_{xx} f(t,x) = f(t,x)$. Therefore,

$$\exp\left(W_T - T/2\right) = 1 + \int_0^T \exp\left(W_t - t/2\right) dW_t,$$

which yields that

$$\begin{aligned}
\exp\left(W_T\right) &= e^{T/2} + e^{T/2} \int_0^T \exp\left(W_t - t/2\right) dW_t \\
&= \mathbb{E}[\exp\left(W_T\right)] + \int_0^T \exp\left(W_t + (T-t)/2\right) dW_t.
\end{aligned}$$

Note that $e^{T/2}$ can go inside the stochastic integral because is deterministic.

(d) $F_4 = \int_0^T W_t dt$. By the integration by parts formula one can write

$$TW_T = \int_0^T W_t dt + \int_0^T t dW_t,$$

which yields

$$\begin{aligned}
\int_0^T W_t dt &= TW_T - \int_0^T t dW_t = T \int_0^T dW_t - \int_0^T t dW_t \\
&= \int_0^T (T-t) dW_t = \mathbb{E}\left[\int_0^T W_t dt\right] + \int_0^T (T-t) dW_t.
\end{aligned}$$

Note that $\mathbb{E}\left[\int_0^T W_t dt\right] = \int_0^T \mathbb{E}\left[W_t\right] dt = \int_0^T 0 dt = 0$.

(e) $F_6 = \int_0^T t^2 W_t^2 dt$. The idea is to consider the process $Y_t = W_t^2 - t$ that we know it is a martingale. By Itô's formula we have that $dY_t = 2W_t dW_t$. On the other hand consider the process given by $dX_t = t^2 dt$ which is equal to $X_t = \frac{t^3}{3}$. Now, we can apply integration by parts formula to the process $X_t Y_t$, taking into account that $d(X_t)d(Y_t) = 0$, to get

$$\begin{aligned}
\frac{T^3}{3}(W_T^2 - T) &= X_t Y_t = X_0 Y_0 + \int_0^T X_t dY_t + \int_0^T Y_t dX_t \\
&= 0 + \int_0^T \frac{2}{3} t^3 W_t dW_t + \int_0^T t^2 (W_t^2 - t) dt \\
&= \int_0^T \frac{2}{3} t^3 W_t dW_t + \int_0^T t^2 W_t^2 dt - \frac{T^4}{4},
\end{aligned}$$

but note that

$$\frac{T^3}{3}(W_T^2 - T) = \frac{T^3}{3} \int_0^T 2W_t dW_t = \int_0^T \frac{2}{3} T^3 W_t dW_t.$$

Hence, we get that

$$\begin{aligned}
\int_0^T t^2 W_t^2 dt &= \frac{T^4}{4} + \int_0^T \frac{2}{3} T^3 W_t dW_t - \int_0^T \frac{2}{3} t^3 W_t dW_t \\
&= \mathbb{E}\left[\int_0^T t^2 W_t^2 dt\right] + \int_0^T \frac{2}{3}(T^3 - t^3) W_t dW_t,
\end{aligned}$$

because

$$\mathbb{E}\left[\int_0^T t^2 W_t^2 dt\right] = \int_0^T t^2 \mathbb{E}\left[W_t^2\right] dt = \int_0^T t^3 dt = \frac{T^4}{4}.$$

12. $P|_{\mathcal{F}_t}$ and $Q|_{\mathcal{F}_t}$ are probabilities on $(\Omega, \mathcal{F}_t)$ that are constructed by considering the restriction of $P$ and $Q$, respectively, to $\mathcal{F}_t$, that is, $P|_{\mathcal{F}_t}(A) = P(A)$ and $Q|_{\mathcal{F}_t}(A) = Q(A), A \in \mathcal{F}_t$. Hence,

$$A \in \mathcal{F}_t \text{ such that } 0 = P|_{\mathcal{F}_t}(A) = P(A) \underset{Q \ll P}{\Longrightarrow} 0 = Q(A) = Q|_{\mathcal{F}_t}(A),$$

and we get that $Q|_{\mathcal{F}_t} \ll P|_{\mathcal{F}_t}$. Define $Z_T = \frac{dQ}{dP}$ and $Z_t = \mathbb{E}_P[Z_T|\mathcal{F}_t]$. By exercise 8., we know that $Z$ is a $\mathbb{F}$-martingale. Moreover, for all $A \in \mathcal{F}_t$ we have that

$$\begin{aligned} Q|_{\mathcal{F}_t}(A) &= Q(A) = \mathbb{E}_Q[\mathbf{1}_A] = \mathbb{E}_P[Z_T \mathbf{1}_A] = \mathbb{E}_P[E[Z_T \mathbf{1}_A|\mathcal{F}_t]] \\ &= \mathbb{E}_P[E[Z_T|\mathcal{F}_t]\mathbf{1}_A] = \mathbb{E}_P[Z_t \mathbf{1}_A] = \mathbb{E}_{P|_{\mathcal{F}_t}}[Z_t \mathbf{1}_A], \end{aligned}$$

which yields that $Z_t = \frac{dQ|_{\mathcal{F}_t}}{dP|_{\mathcal{F}_t}}, P|_{\mathcal{F}_t}$-a.s. Note that we have used that $\mathbb{E}_P[X] = \mathbb{E}_{P|_{\mathcal{F}_t}}[X]$ for any $X$ that is $\mathcal{F}_t$-measurable. One can check this property by using the definitions of Lebesgue integral and the fact that $P|_{\mathcal{F}_t}$ coincides with $P$ on any $\mathcal{F}_t$-measurable set. Finally, we have to prove that $Y$ is a martingale under $Q \iff ZY$ is a martingale under $P$.

$\Rightarrow$) By exercise 32 in List 1 (see the solution of the optional exercise) We have that

$$\mathbb{E}_Q[Y_t|\mathcal{F}_s]\mathbb{E}_P[Z_t|\mathcal{F}_s] = \mathbb{E}_P[Z_t Y_t|\mathcal{F}_s], \quad s < t. \tag{1}$$

Note that, as $Y$ is a martingale under $Q$ we get $\mathbb{E}_Q[Y_t|\mathcal{F}_s] = Y_s$ and as $Z$ is a martingale under $P$ we get $\mathbb{E}_P[Z_t|\mathcal{F}_s] = Z_s$. Hence, the left hand side of equation (1) is equal to $Z_s Y_s$ and we can conclude that $ZY$ is a martingale under $P$.

$\Leftarrow$) As $ZY$ and $Z$ are martingales under $P$, we get that $\mathbb{E}_P[Z_t Y_t|\mathcal{F}_s] = Z_s Y_s$ and $\mathbb{E}_P[Z_t|\mathcal{F}_s] = Z_s$ and equation (1) is equal to

$$\mathbb{E}_Q[Y_t|\mathcal{F}_s]Z_s = Z_s Y_s \iff \mathbb{E}_Q[Y_t|\mathcal{F}_s] = Y_s.$$

Therefore, we can conclude that $Y$ is a martingale under $Q$.