# Black-Scholes Model and Risk Neutral Pricing

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1. See Benth's book: Exercise 4.4, page 137.
2. See Benth's book: page 113.
3. See Benth's book: Exercise 4.8, pages 140-141.
4. See Benth's book: Exercises 4.9 and 4.12, pages 142-143.
5. (a) Consider the process $Z_t = \log(S_t)$. Taking into account that $\partial_x \log(x) = 1/x, \partial_{xx} \log(x) = -1/x^2$and

$$\begin{aligned} dS_t &= S_t\mu(t)dt + S_t\sigma(t)dW_t, \\ (dS_t)^2 &= S_t^2\sigma^2(t)dt, \end{aligned}$$

we can apply Itô's formula to get that

$$Z_t = \log(S_0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW_s - \frac{1}{2}\int_0^t \sigma^2(s)ds.$$

Taking the exponential we obtain the result

$$S_t = \exp(Z_t) = S_0 \exp\left(\int_0^t \mu(s)ds + \int_0^t \sigma(s)dW_s - \frac{1}{2}\int_0^t \sigma^2(s)ds\right).$$

(b) Consider the process

$$M_t = \exp\left(-\int_0^t \frac{\mu(s)-r(s)}{\sigma(s)}dW_s - \frac{1}{2}\int_0^t \left(\frac{\mu(s)-r(s)}{\sigma(s)}\right)^2 ds\right), \qquad t \in [0,T].$$

Note that, as $\min_{t\in[0,T]} \sigma(t) > \sigma^* > 0$, we have that $\frac{\mu(t)-r(t)}{\sigma(t)} \in L^2([0,T])$ and, by exercise 4. e) in List 2, $M_t$ is a martingale under $P$. Alternatively, we can use Novikov's theorem because

$$\begin{aligned} &\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \left(\frac{\mu(t)-r(t)}{\sigma(t)}\right)^2 dt\right)\right] \\ &= \exp\left(\frac{1}{2}\int_0^T \left(\frac{\mu(t)-r(t)}{\sigma(t)}\right)^2 dt\right) < \infty, \end{aligned}$$

due to the fact that $\mu(t), r(t)$ and $\sigma(t)$ are deterministic and $\frac{\mu(t)-r(t)}{\sigma(t)} \in L^2([0,T])$. Therefore, according to Girsanov's theorem, we can define a probability measure $Q$ by considering the following Radon-Nikodym derivative with respect to $P$

$$\frac{dQ}{dP} = M_T = \exp\left(-\int_0^T \frac{\mu(t)-r(t)}{\sigma(t)}dW_t - \frac{1}{2}\int_0^T \left(\frac{\mu(t)-r(t)}{\sigma(t)}\right)^2 dt\right).$$

Note that, as $M_T > 0$, $Q$ is actually equivalent to $P$. In addition, also by Girsanov's theorem, we have that the process

$$\tilde{W}_t = \int_0^t \frac{\mu(s) - r(s)}{\sigma(s)} ds + W_t,$$

is a Brownian motion under $Q$. Rewriting the dynamics of $S_t$ in terms of $\tilde{W}$ we get that

$$\begin{aligned} dS_t &= S_t\mu(t)dt + S_t\sigma(t)dW_t \\ &= S_t\mu(t)dt + S_t\sigma(t)\left(d\tilde{W}_t - \frac{\mu(t) - r(t)}{\sigma(t)}dt\right) \\ &= S_t r(t)dt + S_t\sigma(t)d\tilde{W}_t. \end{aligned}$$

Moreover, $B_t = \exp\left(\int_0^t r(d)ds\right)$ which yields $B_t^{-1} = \exp\left(-\int_0^t r(d)ds\right)$ and

$$dB_t^{-1} = -r(t)B_t^{-1}dt.$$

Hence, by the integration by parts formula, we have

$$\begin{aligned} d(\tilde{S}_t) &= d(B_t^{-1}S_t) = B_t^{-1}dS_t + S_t dB_t^{-1} + \overbrace{\left(dB_t^{-1}\right)(dS_t)}^{=0} \\ &= B_t^{-1}\left\{S_t r(t)dt + S_t\sigma(t)d\tilde{W}_t\right\} + S_t\left\{-r(t)B_t^{-1}dt\right\} = \tilde{S}_t\sigma(t)d\tilde{W}_t. \end{aligned}$$

As $\tilde{W}$ is a B.M. under $Q$, we get that

$$\tilde{S}_t = S_0 \exp\left(\int_0^t \sigma(s)d\tilde{W}_s - \frac{1}{2}\int_0^t \sigma^2(s)ds\right),$$

is a martingale under $Q$. Alternatively, one can prove that $\tilde{S}_t\sigma(t)$ belongs to $L^2_{a,T}$.

(c) By assumption, we have that $\tilde{V}_t(\phi)$ is a martingale under $Q$ and $V_T(\phi) = \max(0, S_T - K)$. Hence,

$$\begin{aligned} \exp\left(-\int_0^t r(s)ds\right)V_t(\phi) &= \tilde{V}_t(\phi) = \mathbb{E}_Q[\tilde{V}_T(\phi)|\mathcal{F}_t] \\ &= \mathbb{E}_Q\left[\max(0, S_T - K)\exp\left(-\int_0^T r(s)ds\right)|\mathcal{F}_t\right], \end{aligned}$$

which yields

$$V_t(\phi) = \mathbb{E}_Q\left[\max(0, S_T - K)\exp\left(-\int_t^T r(s)ds\right)|\mathcal{F}_t\right]. \tag{1}$$

Using the expression for $S_t$ in section a) and rewriting it in terms of $\tilde{W}$ we can write

$$S_T = S_t \exp\left(\int_t^T r(s)ds + \int_t^T \sigma(s)d\tilde{W}_s - \frac{1}{2}\int_t^T \sigma^2(s)ds\right),$$

or, alternatively,

$$S_T \exp\left(-\int_t^T r(s)ds\right) = S_t \exp\left(\int_t^T \sigma(s)d\tilde{W}_s - \frac{1}{2}\int_t^T \sigma^2(s)ds\right).$$

Plugging this expression in equation (1) we obtain

$$V_t(\phi) = \mathbb{E}_Q\left[\max\left(0, S_t\exp\left(\int_t^T \sigma(s)d\tilde{W}_s - \frac{1}{2}\int_t^T \sigma^2(s)ds\right) - K\exp\left(-\int_t^T r(s)ds\right)\right)|\mathcal{F}_t\right].$$

Recall the following general property of the conditional expectation. Let $X$ be a $\mathcal{G}$-measurable random variable and $Y$ be a random variable independent of $\mathcal{G}$, then for any Borel measurable function $\Psi$ such that $\mathbb{E}[|\Psi(X,Y)|] < \infty$ we have that

$$\mathbb{E}\left[\Psi(X,Y)|\mathcal{G}\right] = \mathbb{E}\left[\Psi(x,Y)\right]|_{x=X}.$$

Applying this property to $\mathcal{G} = \mathcal{F}_t, X = S_t, Y = \exp\left(\int_t^T \sigma(s)d\tilde{W}_s\right)$ and

$$\Psi(x,y) = \max\left(0, x\exp\left(y - \frac{1}{2}\int_t^T \sigma^2(s)ds\right) - K\exp\left(-\int_t^T r(s)ds\right)\right),$$

we get $V_t(\phi) = F(t,S_t)$ where

$$F(t,x) = \mathbb{E}_Q\left[\max\left(0, x\exp\left(\int_t^T \sigma(s)d\tilde{W}_s - \frac{1}{2}\int_t^T \sigma^2(s)ds\right) - K\exp\left(-\int_t^T r(s)ds\right)\right)\right].$$

Note that $\mathbb{F}^W = \mathbb{F}^{\tilde{W}}$ and $\int_t^T \sigma(s)d\tilde{W}_s$ is independent of $\mathcal{F}_t$ because $\tilde{W}$ has independent increments under $Q$.

(d) In order to find an explicit expression for $F(t,x)$, note that $\int_t^T \sigma(s)d\tilde{W}_s \sim \mathcal{N}(0, \int_t^T \sigma^2(s)ds)$ under $Q$. Define

$$\sigma^2(t,T) = \int_t^T \sigma^2(s)ds, \qquad r(t,T) = \int_t^T r(s)ds.$$

Then,

$$\begin{aligned} F(t,x) &= \mathbb{E}_Q\left[\max\left(0, x\exp\left(\int_t^T \sigma(s)d\tilde{W}_s - \frac{1}{2}\sigma^2(t,T)\right) - Ke^{-r(t,T)}\right)\right] \\ &= e^{-r(t,T)}\mathbb{E}_Q\left[\max\left(0, \exp\left(\int_t^T \sigma(s)d\tilde{W}_s + \log(x) + r(t,T) - \frac{1}{2}\sigma^2(t,T)\right) - K\right)\right] \\ &= e^{-r(t,T)}\mathbb{E}_Q\left[\max\left(0, Z-K\right)\right], \end{aligned}$$

where

$$\log(Z) \sim \mathcal{N}\left(\log(x) + r(t,T) - \frac{1}{2}\sigma^2(t,T), \sigma^2(t,T)\right),$$

under $Q$. Hence, we can use the Black-Scholes formulae to obtain

$$F(t,x) = x\Phi(d_1) - Ke^{-r(t,T)}\Phi(d_2),$$

with

$$\begin{aligned} d_1 &= \frac{\log\left(\frac{x}{K}\right) + r(t,T) + \frac{1}{2}\sigma^2(t,T)}{\sqrt{\sigma^2(t,T)}}, \\ d_2 &= \frac{\log\left(\frac{x}{K}\right) + r(t,T) - \frac{1}{2}\sigma^2(t,T)}{\sqrt{\sigma^2(t,T)}} \end{aligned}$$

6. By the risk neutral pricing formula we get, as $\mathcal{F}_0 = \{\Omega, \varnothing\}$, that

$$\pi_0(H) = e^{-rT}\mathbb{E}_Q\left[\max\left(\left(\prod_{i=1}^n S_{t_i}\right)^{1/n} - K, 0\right)\right],$$

where $Q$ is the unique risk-neutral probability measure in the Black-Scholes model. That is, the probability measure given by

$$\frac{dQ}{dP} = \exp\left(-\frac{u-r}{\sigma}W_T - \left(\frac{u-r}{\sigma}\right)^2 T\right).$$

Recall that $S_t$ can be written as

$$S_t = S_0 \exp\left(\sigma \tilde{W}_t + \left(r - \frac{\sigma^2}{2}\right) t\right),$$

where $\tilde{W}$ is a Brownian motion under $Q$. Moreover, we have the recursion

$$\begin{aligned} S_{t_i} &= S_0 \exp\left(\sigma \tilde{W}_{t_i} + \left(r - \frac{\sigma^2}{2}\right) t_i\right) \\ &= S_0 \exp\left(\sigma \tilde{W}_{t_{i-1}} + \left(r - \frac{\sigma^2}{2}\right) t_{i-1}\right) \\ &\quad \times \exp\left(\sigma\left(\tilde{W}_{t_i} - \tilde{W}_{t_{i-1}}\right) + \left(r - \frac{\sigma^2}{2}\right)(t_i - t_{i-1})\right) \\ &= S_{t_{i-1}} \exp\left(\Delta \tilde{W}_i + \left(r - \frac{\sigma^2}{2}\right) \Delta t_i\right), \end{aligned}$$

where $\Delta \tilde{W}_i \triangleq \tilde{W}_{t_i} - \tilde{W}_{t_{i-1}}$ and $\Delta t_i \triangleq t_i - t_{i-1}$. Iterating this recursion we get

$$S_{t_i} = S_0 \prod_{k=1}^{i} \exp\left(\Delta \tilde{W}_k + \left(r - \frac{\sigma^2}{2}\right) \Delta t_k\right),$$

and

$$\prod_{i=1}^{n} S_{t_i} = S_0 \prod_{i=1}^{n} \prod_{k=1}^{i} \exp\left(\Delta \tilde{W}_k + \left(r - \frac{\sigma^2}{2}\right) \Delta t_k\right).$$

Taking logarithm we obtain

$$\begin{aligned} Z &\triangleq \log\left(\left(\prod_{i=1}^{n} S_{t_i}\right)^{1/n}\right) = \log(S_0) + \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{i} \left\{\Delta \tilde{W}_k + \left(r - \frac{\sigma^2}{2}\right) \Delta t_k\right\} \\ &= \log(S_0) + \frac{1}{n} \sum_{i=1}^{n} (n-i+1) \Delta \tilde{W}_i + \frac{1}{n}\left(r - \frac{\sigma^2}{2}\right) \sum_{i=1}^{n} (n-i+1) \Delta t_k. \end{aligned}$$

Obviously, under $Q$, $Z$ has normal distribution with

$$\mathbb{E}[Z] = \log(S_0) + \frac{1}{n}\left(r - \frac{\sigma^2}{2}\right) \sum_{i=1}^{n} (n-i+1) \Delta t_k$$

and

$$\operatorname{Var}[Z] = \operatorname{Var}\left[\frac{1}{n} \sum_{i=1}^{n} (n-i+1) \Delta \tilde{W}_i\right] = \frac{1}{n^2} \sum_{i=1}^{n} (n-i+1)^2 \Delta t_i,$$

where to compute the variance we have used that $\Delta \tilde{W}_i$ are independent of each other. Hence, we have reduced the problem to compute

$$\pi_0(H) = e^{-rT} \mathbb{E}\left[\max\left(e^Z - K, 0\right)\right],$$

where

$$Z \sim \mathcal{N}\left(\log(S_0) + \frac{1}{n}\left(r - \frac{\sigma^2}{2}\right) \sum_{i=1}^{n} (n-i+1) \Delta t_k, \frac{1}{n^2} \sum_{i=1}^{n} (n-i+1)^2 \Delta t_i\right).$$

A formula for this expectation is given in the solution of exercise 33 in List 1.

7. These contingent claims are examples of the so called *packages*, which are linear combinations of simpler options and positions in cash. Let $C(t, S_t; K)$ denote the arbitrage free price at time $t$ of a call option with strike $K$ (and exercise time $T$).

(a) The payoff $H_1 = \min\left(\max(S_T, K_1), K_2\right)$ with $K_2 > K_1 > 0$ can be rewritten as

$$H_1 = K_1 + \max(0, S_T - K_1) - \max(0, S_T - K_2).$$

Using the risk-neutral pricing formula we get that

$$\begin{aligned}
\pi_t(H_1) &= e^{-r(T-t)}\mathbb{E}_Q[H_1|\mathcal{F}_t] \\
&= e^{-r(T-t)}\mathbb{E}_Q[K_1 + \max(0, S_T - K_1) - \max(0, S_T - K_2)|\mathcal{F}_t] \\
&= e^{-r(T-t)}K_1 + e^{-r(T-t)}\mathbb{E}_Q[\max(0, S_T - K_1)|\mathcal{F}_t] \\
&\quad -e^{-r(T-t)}\mathbb{E}_Q[\max(0, S_T - K_2)|\mathcal{F}_t] \\
&= e^{-r(T-t)}K_1 + C(t, S_t; K_1) - C(t, S_t; K_2).
\end{aligned}$$

(b) The payoff $H_2 = \max(S_T, S_0e^{rT}) - K$, with $K > 0$ can be rewritten as

$$H_2 = \max(0, S_T - S_0e^{rT}) + S_0e^{rT} - K.$$

Using the risk-neutral pricing formula we get that

$$\begin{aligned}
\pi_t(H_2) &= e^{-r(T-t)}\mathbb{E}_Q[H_2|\mathcal{F}_t] \\
&= e^{-r(T-t)}\mathbb{E}_Q[\max(0, S_T - S_0e^{rT}) + S_0e^{rT} - K|\mathcal{F}_t] \\
&= e^{-r(T-t)}\mathbb{E}_Q[\max(0, S_T - S_0e^{rT})|\mathcal{F}_t] + e^{-r(T-t)}\left\{S_0e^{rT} - K\right\} \\
&= C(t, S_t; S_0e^{rT}) + S_0e^{rt} - e^{-r(T-t)}K.
\end{aligned}$$

8. See See Benth's book: pages 79-81.