Introduction and Techniques
in Financial Mathematics
UiO-STK4510
Autumn 2015
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Solution Compulsory Exercise

Task 2

In this task you will do some computations on one of the first models for stochastic interest rates. Consider the process $\{r_t\}_{t\in\mathbb{R}_+}$ given by the stochastic differential equation (s.d.e)

$$dr_t = k(\theta - r_t)dt + \sigma dW_t, \quad r_0 \in \mathbb{R}_+, \tag{1}$$

where k, θ and σ are strictly positive constants and W is a standard Brownian motion.

a) First find the solution $\{Z_t\}_{t\in\mathbb{R}_+}$ of the s.d.e.

$$dZ_t = -aZ_t dt + \sigma dW_t, \quad Z_0 \in \mathbb{R},$$

as a function to the initial condition Z_0, where a and σ are strictly positive constants.

b) For any $\beta \in \mathbb{R}$ define the process $Y_t = Z_t + \beta$. Find the s.d.e. satisfied by Y. Choosing appropriately the parameters $a \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$, find the solution $\{r_t\}_{t\in\mathbb{R}_+}$ of the s.d.e. (1).

c) Compute $\mathbb{E}[r_t]$ and $\mathrm{Var}[r_t]$. What is the law of r_t?

d) r_t takes negative values with strict probability and, therefore, it is not a good model for interest rates. A possible solution is to consider $R_t = \exp(r_t)$. Compute the dynamics of R_t, i.e., find the s.d.e. describing the evolution of R_t

Solution Task 2

a) We consider the integrating factor e^{at}, that is, let $X_t = e^{at}Z_t$. By the integration by parts formula, and taking into account that $(de^{at})(dZ_t) = 0$, we get that

$$\begin{aligned} dX_t &= d(e^{at}Z_t) = e^{at}dZ_t + Z_t d(e^{at}) + \left(de^{at}\right)(dZ_t) \\ &= e^{at}\{-aZ_t dt + \sigma dW_t\} + Z_t a e^{at} dt + 0 \\ &= \{-ae^{at}Z_t + ae^{at}Z_t\}dt + e^{at}\sigma dW_t \\ &= e^{at}\sigma dW_t, \end{aligned}$$

or, in integral form,

$$e^{at}Z_t = X_t = Z_0 + \int_0^t \sigma e^{as} dW_s,$$

which yields the following explicit expression for Z_t

$$Z_t = Z_0 e^{-at} + e^{-at}\int_0^t \sigma e^{as} dW_s.$$

b) $Y = \{Y_t\}_{t\in\mathbb{R}_+}$ satifies the following s.d.e.

$$\begin{aligned} dY_t &= d(Z_t+\beta) = dZ_t = -aZ_t dt + \sigma dW_t \\ &= -a(Y_t-\beta)dt + \sigma dW_t \\ &= a(\beta - Y_t)dt + \sigma dW_t, \\ Y_0 &= Z_0 + \beta. \end{aligned}$$

If we choose $a = k$ and $\beta = \theta$ and $Z_0 = r_0 - \beta = r_0 - \theta$, we get that the s.d.e. satisfied by Y is the same as the s.d.e. satisfied by $r = \{r_t\}_{t\in\mathbb{R}_+}$. As this s.d.e. satisfies the conditions to get a unique strong solution we get that

$$\begin{aligned} r_t &= Y_t = Z_t + \beta = \beta + Z_0 e^{-at} + e^{-at}\int_0^t \sigma e^{as} dW_s \\ &= \theta + (r_0-\theta)e^{-kt} + e^{-kt}\int_0^t \sigma e^{ks} dW_s \\ &= r_0 e^{-kt} + \theta(1-e^{-kt}) + e^{-kt}\int_0^t \sigma e^{ks} dW_s. \end{aligned}$$

c) As for all $t \geq 0$ we have that $\sigma e^{ks} \in L^2_{a,t}$, we get that

$$\mathbb{E}\left[\int_0^t \sigma e^{ks} dW_s\right] = 0,$$

and by the Itô isometry

$$\mathbb{E}\left[\left(\int_0^t \sigma e^{ks} dW_s\right)^2\right] \mathbb{E}\left[\int_0^t \sigma^2 e^{2ks} ds\right] = \frac{\sigma^2}{2k}\left(e^{2kt}-1\right).$$

Therefore,

$$\begin{aligned} \mathbb{E}[r_t] &= r_0 e^{-kt} + \theta(1-e^{-kt}) + e^{-kt}\mathbb{E}\left[\int_0^t \sigma e^{ks} dW_s\right] \\ &= r_0 e^{-kt} + \theta(1-e^{-kt}), \end{aligned}$$

and

$$\begin{aligned} \mathrm{Var}\left[r_t\right] &= \mathbb{E}[(r_t - \mathbb{E}[r_t])^2] = \mathbb{E}\left[\left(e^{-kt}\int_0^t \sigma e^{ks} dW_s\right)^2\right] \\ &= e^{-2kt}\frac{\sigma^2}{2k}\left(e^{2kt}-1\right) = \frac{\sigma^2}{2k}\left(1-e^{-2kt}\right). \end{aligned}$$

d) Applying Itô formula to $R_t = f(r_t) = \exp(r_t)$, taking into account that,

$$r_t = \log(R_t),$$

$$f'(x) = f''(x) = f(x),$$

and that

$$(dr_t)^2 = (k(\theta - r_t)dt + \sigma dW_t)^2 = \sigma^2 dt,$$

by the product rule of Itô differentials, we obtain

$$\begin{aligned} dR_t &= df(r_t) = f'(r_t)dr_t + \frac{1}{2}f''(r_t)(dr_t)^2 \\ &= \exp(r_t)dr + \frac{1}{2}\exp(r_t)(dr_t)^2 \\ &= R_t\left\{k(\theta - r_t)dt + \sigma dW_t\right\} + \frac{\sigma^2}{2}R_t dt \\ &= R_t\left\{k(\theta - \log(R_t)) + \frac{\sigma^2}{2}\right\}dt + \sigma R_t dW_t. \end{aligned}$$

Task 3

In this task you will work on the relationship between a call option and a put option on the same asset, with the same strike and with the same exercise time. Recall that a call option is a contingent claim H_1 with $H_1 = h_1(S_T) = \max(0, S_T - K)$. A put option is a contingent claim H_2 with $H_2 = h_2(S_T) = \max(0, K - S_T)$.

a) Let C_t and P_t denote the price at time t of a call option and a put option with the same strike price K and exercise time T. Assume that those contingent claims are traded in the market in addition to the riskless asset and risky asset S_t. Assuming only that there are no arbitrage oportunities in the market (we do not specify any particular dynamics for S), deduce that

$$C_t - P_t = S_t - e^{-r(T-t)}K, \quad 0 \leq t \leq T. \tag{2}$$

b) Assume now that we are in the Black-Scholes model. Use the Black-Scholes formula for pricing contingent claims of the form $h(S_T)$ to compute the price of a put option P_t and its hedging strategy. Compute also the greeks.

c) Check that formula (2) holds in the Black-Scholes model.

Solution Task 3

a) If at time t we buy a put option, sell a call option and buy one unit of the risky asset (one share of the stock) we have a portfolio with value

$$P_t - C_t + S_t.$$

This portfolio, will have the following value at time T,

$$\max(K - S_T, 0) - \max(S_T - K, 0) + S_T = \begin{cases} 0 - (S_T - K) + S_T & \text{if} \quad S_T \geq K \\ K - S_T - 0 + S_T & \text{if} \quad S_T < K \end{cases} = K.$$

Hence, we have a replicating portolio made of calls, puts and risky asset for the constant claim K at time T. But on the other hand, we can construct a replicating portfolio for the constant claim K at time T by investing solely on the riskless asset at time t the amount $Ke^{-r(T-t)}$. As we have assumed that there are no arbitrage opportunities in the market the t arbitrage free price of both porfolios (that replicate the same contingent claim K) must coincide and this yields that

$$P_t - C_t + S_t = Ke^{-r(T-t)},$$

and reordering the terms in the equation we get

$$C_t - P_t = S_t - Ke^{-r(T-t)}.$$

b) The price process $\pi_t(H)$ of a contingent claim $H = h(S_T)$ is given by

$$\pi_t(H) = f(t, S_t),$$

and the hedging strategy is given by

$$(\phi_t^0, \phi_t^1) = \left(e^{-rt} \left\{ f(t, S_t) - S_t \frac{\partial f}{\partial x}(t, S_t) \right\}, \frac{\partial f}{\partial x}(t, S_t) \right), \qquad t \in [0, T],$$

where

$$f(t, x) = e^{-r(T-t)} \mathbb{E}[h(Z_T^{t,x})],$$

and $\log Z_T^{t,x} \sim \mathcal{N}\left(\log(x) + \left(r - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right)$. In what follows, let

$$\begin{aligned}
\phi(y) &\triangleq \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right), \\
\Phi(y) &\triangleq \int_{-\infty}^{y} \phi(x)dx, \\
d_1(x) &\triangleq \frac{\log(x/K) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \\
d_2(x) &\triangleq \frac{\log(x/K) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.
\end{aligned}$$

The following relations will be useful afterwards

$$\begin{aligned}
\phi(y) &= \phi(-y), \\
\Phi(y) + \Phi(-y) &= 1, \\
d_1(x) - d_2(x) &= \sigma\sqrt{T-t}, \\
d_1'(x) &= d_2'(x) = \frac{1}{x\sigma\sqrt{T-t}}, \\
\frac{\phi(d_1(x))}{\phi(d_2(x))} &= \frac{Ke^{-r(T-t)}}{x}
\end{aligned}$$

In the case of a put option we have $H_2 = h_2(S_T) = \max(0, K - S_T)$ and, therefore,

$$\begin{aligned}
f(t,x) &= e^{-r(T-t)}\mathbb{E}[h_2(Z_T^{t,x})] = e^{-r(T-t)}\mathbb{E}[\max(0, K - Z_T^{t,x})] = e^{-r(T-t)}\mathbb{E}[(K - Z_T^{t,x})\mathbf{1}_{\{Z_T^{t,x} \leq K\}}] \\
&= Ke^{-r(T-t)}\mathbb{E}[\mathbf{1}_{\{Z_T^{t,x} \leq K\}}] - e^{-r(T-t)}\mathbb{E}[Z_T^{t,x}\mathbf{1}_{\{Z_T^{t,x} \leq K\}}] \triangleq A_1 - A_2
\end{aligned}$$

In order to express the results in terms of the distribution function of the standard normal law, we note that the random variable

$$Y \triangleq \frac{\log(Z_T^{t,x}) - \log(x) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$

has a $\mathcal{N}(0,1)$ law. To compute A_1 note that $Z_T^{t,x} \leq K$ if and only if

$$Y = \frac{\log(Z_T^{t,x}) - \log(x) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \leq -d_2(x).$$

Hence,

$$\begin{aligned}
A_1 &= Ke^{-r(T-t)}\mathbb{E}[\mathbf{1}_{\{Z_T^{t,x} \leq K\}}] = Ke^{-r(T-t)}\mathbb{E}[\mathbf{1}_{\{Y \leq -d_2(x)\}}] \\
&= Ke^{-r(T-t)} \int_{-\infty}^{+\infty} \mathbf{1}_{\{Y \leq -d_2(x)\}}\phi(y)dy = Ke^{-r(T-t)} \int_{-\infty}^{-d_2(x)} \phi(y)dy \\
&= Ke^{-r(T-t)}\Phi(-d_2(x)),
\end{aligned}$$

and

$$\begin{aligned}
A_2 &= e^{-r(T-t)}\mathbb{E}[\exp\left(\log(Z_T^{t,x})\right)\mathbf{1}_{\{Z_T^{t,x} \leq K\}}] \\
&= e^{-r(T-t)}\mathbb{E}\left[\exp\left(\log(x) + \left(r - \frac{\sigma^2}{2}\right)(T-t) + Y\sigma\sqrt{T-t}\right)\mathbf{1}_{\{Y \leq -d_2(x)\}}\right] \\
&= x\int_{-\infty}^{+\infty} \exp\left(-\frac{\sigma^2}{2}(T-t) + y\sigma\sqrt{T-t}\right)\mathbf{1}_{\{y \leq -d_2(x)\}}\phi(y)dy \\
&= x\int_{-\infty}^{-d_2(x)} \phi(y - \sigma\sqrt{T-t})dy = x\int_{-\infty}^{-d_2(x)-\sigma\sqrt{T-t}} \phi(z)dz \\
&= x\int_{-\infty}^{-d_1(x)} \phi(z)dz = x\Phi(-d_1(x)),
\end{aligned}$$

and we obtain

$$f(t,x) = Ke^{-r(T-t)}\Phi(-d_2(x)) - x\Phi(-d_1(x)).$$

Therefore, the price of a put at time t is equal to

$$\pi_t(H_2) = f(t,S_t) = Ke^{-r(T-t)}\Phi(-d_2(S_t)) - S_t\Phi(-d_1(S_t)).$$

Moreover, note that if (ϕ^0,ϕ^1) is the hedging strategy then

$$\phi_t^0 e^{rt} + \phi_t^1 S_t = V_t(\phi) = \pi_t(H) = f(t,S_t) = Ke^{-r(T-t)}\Phi(-d_2(S_t)) - S_t\Phi(-d_1(S_t)),$$

and equating the coefficients for S_t and e^{rt} we get

$$\begin{aligned}\phi_t^0 &= -\Phi(-d_1(S_t)),\\ \phi_t^1 &= Ke^{-rT}\Phi(-d_2(S_t)).\end{aligned}$$

Alternatively, we can compute the derivative of $f(t,x)$ with respect to x,

$$\begin{aligned}\frac{\partial f}{\partial x}(t,x) &= -Ke^{-r(T-t)}\phi(-d_2(x))d_2'(x) - \{\Phi(-d_1(x)) - x\phi(-d_1(x))d_1'(x)\}\\ &= -\frac{Ke^{-r(T-t)}\phi(d_2(x))}{x\sigma\sqrt{T-t}} - \Phi(-d_1(x)) + \frac{\phi(d_1(x))}{\sigma\sqrt{T-t}}\\ &= -\Phi(-d_1(x)) + \frac{1}{\sigma\sqrt{T-t}}\left\{\phi(d_1(x)) - \frac{Ke^{-r(T-t)}\phi(d_2(x))}{x}\right\}\\ &= -\Phi(-d_1(x)),\end{aligned}$$

which yields $\phi_t^1 = -\Phi(-d_1(S_t))$ and

$$\phi_t^0 = e^{-rt}\left\{f(t,S_t) - S_t\frac{\partial f}{\partial x}(t,S_t)\right\} = Ke^{-rT}\Phi(-d_2(S_t)).$$

The computation of the Greeks is an exercise on taking derivatives. The results are:

$$\begin{aligned}\Delta &= \frac{\partial f}{\partial S_t}(t,S_t) = -\Phi(-d_1(S_t)),\\ \Gamma &= \frac{\partial^2 f}{\partial S_t^2}(t,S_t) = \frac{\phi(d_1)}{S_t\sigma\sqrt{T-t}},\\ \Theta &= \frac{\partial f}{\partial t}(t,S_t) = re^{-r(T-t)}K\Phi(-d_2) - \frac{\sigma S_t\phi(d_1)}{2\sqrt{T-t}}\\ &= Ke^{-r(T-t)}\left\{r\Phi(-d_2) - \phi(d_2)\frac{\sigma}{2\sqrt{T-t}}\right\},\\ \rho &= \frac{\partial f}{\partial r}(t,S_t) = -(T-t)e^{-r(T-t)}K\Phi(-d_2),\\ \frac{\partial f}{\partial \sigma}(t,S_t) &= \sqrt{T-t}S_t\phi(d_1)\\ &= Ke^{-r(T-t)}\phi(d_2)\sqrt{T-t}.\end{aligned}$$

c) In the Black-Scholes model, the price of a call is

$$C_t = S_t\Phi(d_1(S_t)) - e^{-r(T-t)}K\Phi(d_2(S_t)),$$

and the price of a put is

$$P_t = Ke^{-r(T-t)}\Phi(-d_2(x)) - S_t\Phi(-d_1(S_t)).$$

Hence,

$$\begin{aligned}C_t - P_t &= S_t\{\Phi(d_1(S_t)) + \Phi(-d_1(S_t)\} - e^{-r(T-t)}K\{\Phi(d_2(S_t) + \Phi(-d_2(x))\}\\ &= S_t - e^{-r(T-t)}K,\end{aligned}$$

because for any $x \in \mathbb{R}$ we have that $\Phi(x) + \Phi(-x) = 1$.