Introduction and Techniques
in Financial Mathematics
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Solution Optional Exercise

Problem 1 *Let $X \sim \mathcal{N}(\mu, \sigma^2)$ defined on some probability space $(\Omega, \mathcal{F}, P)$.*

1. *Compute $\psi(\theta) = \mathbb{E}[e^{\theta X}]$ for $\theta \in \mathbb{R}$.*
2. *Define $L(X;\theta) := e^{\theta X}/\psi(\theta)$ and show that $Q_\theta(A) = \mathbb{E}[L(X;\theta)\mathbf{1}_A], A \in \mathcal{F}$ defines a probability measure on $(\Omega, \mathcal{F})$.*
3. *Show that $Q_\theta \ll P$.*
4. *Find the law of X under Q_θ, that is, the law of X as a random variable defined on $(\Omega, \mathcal{F}, Q_\theta)$.*
5. *Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Show that for any $Y \in L^1(\Omega, \mathcal{F}, Q_\theta)$ one has that*

$$\mathbb{E}_{Q_\theta}[Y|\mathcal{G}] = \frac{\mathbb{E}[YL(X;\theta)|\mathcal{G}]}{\mathbb{E}[L(X;\theta)|\mathcal{G}]}, \quad Q_\theta\text{-a.s.}$$

Solution

1. As $X \sim \mathcal{N}(\mu, \sigma^2)$ we know that $P_X \ll \lambda$ and its density is given by

$$\frac{dP_X}{d\lambda}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Hence, using the image measure theorem, we have that

$$\begin{aligned}\psi(\theta) &= \mathbb{E}[e^{\theta X}] = \int_\Omega e^{\theta X} dP = \int_{\mathbb{R}} e^{\theta x} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) d\lambda \\ &= \int_{-\infty}^{+\infty} e^{\theta x} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx.\end{aligned}$$

Note that

$$\begin{aligned}\theta x - \frac{(x-\mu)^2}{2\sigma^2} &= -\frac{-2\theta\sigma^2 x + (x-\mu)^2}{2\sigma^2} = -\frac{x^2 - 2\left(\mu+\theta\sigma^2\right)x + \mu^2}{2\sigma^2} \\ &= -\frac{x^2 - 2\left(\mu+\theta\sigma^2\right)x + \left(\mu+\theta\sigma^2\right)^2}{2\sigma^2} + \frac{\left(\mu+\theta\sigma^2\right)^2 - \mu^2}{2\sigma^2} \\ &= -\frac{\left(x - (\mu+\theta\sigma^2)\right)^2}{2\sigma^2} + \mu\theta + \frac{\theta^2\sigma^2}{2} \qquad (1)\end{aligned}$$

Therefore,

$$\psi(\theta) = \exp(\mu\theta + \frac{\theta^2\sigma^2}{2}) \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\left(x-(\mu+\theta\sigma^2)\right)^2}{2\sigma^2}\right) dx = \exp(\mu\theta + \frac{\theta^2\sigma^2}{2}),$$

because the integral is equal to one (we are integrating the density of a $\mathcal{N}(\mu+\theta\sigma^2, \sigma^2)$).

2. By construction $L(X;\theta) > 0$ and $\mathbb{E}[L(X;\theta)] = \frac{1}{\psi(\theta)}\mathbb{E}[e^{\theta X}] = \frac{\psi(\theta)}{\psi(\theta)} = 1$. Let's check that $Q_\theta(A) := \mathbb{E}[L(X;\theta)\mathbf{1}_A], \forall A \in \mathcal{F}$ defines a probability measure on $(\Omega, \mathcal{F})$, that is, we have to check that Q_θ is a measure (positivity, σ-additivity and $Q_\theta(\varnothing) = 0$) and $Q_\theta(\Omega) = 1$.

   (a) $Q_\theta(\Omega) = \mathbb{E}[L(X;\theta)\mathbf{1}_\Omega] = \mathbb{E}[L(X;\theta)] = 1$

   (b) (positivity) That $Q_\theta(A) \geq 0$ follows from the monotonicity of the expectation (if $Z \geq 0$, P-a.s then $\mathbb{E}[Z] \geq 0$) and the fact that $L(X;\theta)\mathbf{1}_A \geq 0$, P-a.s..

   (c) (σ-additivity) Let $\{A_n\}_{n\geq 1}$ pairwise disjoint. Note that $\mathbf{1}_{\cup_{n\geq 1} A_n} = \sum_{n\geq 1} \mathbf{1}_{A_n}$, because the events are pairwise disjoint. Moreover, $S_m = \sum_{n=1}^m L(X;\theta)\mathbf{1}_{A_n}$, $m \geq 1$ is a sequence converging P-a.s. to $\sum_{n=1}^{+\infty} L(X;\theta)\mathbf{1}_{A_n}$, which is dominated by $L(X;\theta) \in L^1(\Omega, \mathcal{F}, P)$, i.e.,

   $$\left|\sum_{n=1}^{+\infty} L(X;\theta)\mathbf{1}_{A_n}\right| \leq L(X;\theta), \qquad P\text{-a.s.}$$

   Therefore, we can apply the dominated convergence theorem to get

   $$\begin{aligned} Q_\theta\left(\cup_{n\geq 1} A_n\right) &= \mathbb{E}[L(X;\theta)\mathbf{1}_{\cup_{n\geq 1} A_n}] = \mathbb{E}[L(X;\theta)\sum_{n\geq 1}\mathbf{1}_{A_n}] = \mathbb{E}[\lim_{m\to\infty}\sum_{n=1}^m L(X;\theta)\mathbf{1}_{A_n}] \\ &= \lim_{m\to\infty}\sum_{n=1}^m \mathbb{E}[L(X;\theta)\mathbf{1}_{A_n}] = \lim_{m\to\infty}\sum_{n=1}^m Q_\theta(A_n) = \sum_{n=1}^\infty Q_\theta(A_n). \end{aligned}$$

   The σ-additivity can also be proved using the monotone convergence theorem or even the Tonelli-Hobson (Fubini) theorem.

   (d) As Q_θ is σ-additive then is additive and we have that $1 = Q_\theta(\Omega) = Q_\theta(\Omega \cup \varnothing) = Q_\theta(\Omega) + Q_\theta(\varnothing) = 1 + Q_\theta(\varnothing)$, which yields that $Q_\theta(\varnothing) = 0$. Alternatively, using that $L(X;\theta)\mathbf{1}_A = 0, P$-a.s. we get that $Q_\theta(\varnothing) = \mathbb{E}[L(X;\theta)\mathbf{1}_\varnothing] = \mathbb{E}[0] = 0$.

3. To prove that $Q_\theta \ll P$ we must show that : $\forall A \in \mathcal{F}$ with $P(A) = 0$ we have that $Q_\theta(A) = 0$. Therefore, assume that $A \in \mathcal{F}$ and $P(A) = 0$, by definition

   $$Q_\theta(A) = \mathbb{E}[L(X;\theta)\mathbf{1}_A] = \mathbb{E}[0] = 0,$$

   where we have used that $L(X;\theta)\mathbf{1}_A = 0, P$-a.s. and the integral of P-a.s. equal integrands coincide. Alternatively, we could have used the Radon-Nikodym theorem, which states that $Q_\theta \ll P$ iff there exists a random variable $\frac{dQ_\theta}{dP} \geq 0, P$-a.s. such that $Q_\theta(A) = \int_\Omega \frac{dQ_\theta}{dP} 1_A dP = \mathbb{E}[\frac{dQ_\theta}{dP} 1_A]$. Obviously,by construction, $L(X;\theta)$ is a version of $\frac{dQ_\theta}{dP}$.

4. The law of X under Q_θ is the image measure of Q_θ by X, which is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that we will denote by $Q_{\theta,X}$. For all $A \in \mathcal{B}(\mathbb{R})$, we have that

   $$\begin{aligned} Q_{\theta,X}(A) &= Q_\theta(X^{-1}(A)) = \int_\Omega L(X;\theta)\mathbf{1}_{X^{-1}(A)} dP = \int_\mathbb{R} L(x;\theta)\mathbf{1}_A dP_X \\ &= \int_A \frac{e^{\theta x}}{\psi(\theta)} dP_X = \int_A \frac{e^{\theta x}}{\psi(\theta)} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) d\lambda \\ &= \int_A \exp\left(\theta x - \mu\theta - \frac{\theta^2\sigma^2}{2}\right) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\ &= \int_A \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\left(x - \left(\mu + \theta\sigma^2\right)\right)^2}{2\sigma^2}\right) dx \end{aligned}$$

   where we have used the definition of image measure, the definition of Q_θ, the image measure theorem, the definition of $L(X;\theta)$, that $P_X \ll \lambda$ with density of a $\mathcal{N}(\mu, \sigma^2)$ and equality (1). Hence, for all $A \in \mathcal{B}(\mathbb{R})$, we have that

   $$Q_{\theta,X}(A) = \int_A \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\left(x - \left(\mu + \theta\sigma^2\right)\right)^2}{2\sigma^2}\right) dx,$$

which shows that $\frac{dQ_{\theta,X}}{d\lambda}$ is the density of a normal distribution of mean $\mu + \theta\sigma^2$ and variance σ^2 and we can conclude that, under Q_θ, $X \sim N(\mu + \theta\sigma^2, \sigma^2)$. This a very basic version of Girsanov's theorem.

5. By assumption $Y \in L^1(\Omega, \mathcal{F}, Q_\theta)$ and, therefore, $\mathbb{E}_{Q_\theta}[Y|\mathcal{G}]$ exists. Next, we must show that the right hand side of the equality is well defined. First note that $\mathbb{E}[|L(X;\theta)|] = \mathbb{E}[L(X;\theta)] = 1$, which yields that $L(X;\theta) \in L^1(\Omega, \mathcal{F}, P)$ and that $\mathbb{E}[L(X;\theta)|\mathcal{G}]$ exists. Moreover, $Y \in L^1(\Omega, \mathcal{F}, Q_\theta)$ iff $\mathbb{E}_{Q_\theta}[|Y|] = \mathbb{E}[|Y|L(X;\theta)] < \infty$, which yields that $YL(X;\theta) \in L^1(\Omega, \mathcal{F}, P)$ and that $\mathbb{E}[YL(X;\theta)|\mathcal{G}]$ also exists. To show that $\frac{\mathbb{E}[YL(X;\theta)|\mathcal{G}]}{\mathbb{E}[L(X;\theta)|\mathcal{G}]}$ is well defined we must prove that that $P(\mathbb{E}[L(X;\theta)|\mathcal{G}] = 0) = 0$, P-a.s. (as $Q \ll P$, this will also hold under Q_θ). Obviously, this follows from the fact that $L(X;\theta) > 0$, P-a.s.. Let's write the proof carefully. Consider the set $\{\mathbb{E}[L(X;\theta)|\mathcal{G}] = 0\} \in \mathcal{G}$, by the defining property of the conditional expectation we have that

$$\mathbb{E}[L(X;\theta)\mathbf{1}_{\{\mathbb{E}[L(X;\theta)|\mathcal{G}]=0\}}] = \mathbb{E}[\mathbb{E}[L(X;\theta)|\mathcal{G}]\mathbf{1}_{\{\mathbb{E}[L(X;\theta)|\mathcal{G}]=0\}}] = \mathbb{E}[0\mathbf{1}_{\{\mathbb{E}[L(X;\theta)|\mathcal{G}]=0\}}] = 0.$$

Note also that $L(X;\theta)\mathbf{1}_{\{\mathbb{E}[L(X;\theta)|\mathcal{G}]=0\}} \geq 0, P$-a.s. yields that $\mathbb{E}[L(X;\theta)\mathbf{1}_{\{\mathbb{E}[L(X;\theta)|\mathcal{G}]=0\}}] \geq 0$ by the monotonicity of the Lebesgue integral. By exercise 16., we know that this implies that $L(X;\theta)\mathbf{1}_{\{\mathbb{E}[L(X;\theta)|\mathcal{G}]=0\}} = 0$, P-a.s., but as $L(X;\theta) > 0$, P-a.s. we get that $\mathbf{1}_{\{\mathbb{E}[L(X;\theta)|\mathcal{G}]=0\}} = 0$, P-a.s., which is equivalent to say $P(\mathbb{E}[L(X;\theta)|\mathcal{G}] = 0) = 0$. Hence, $\frac{\mathbb{E}[YL(X;\theta)|\mathcal{G}]}{\mathbb{E}[L(X;\theta)|\mathcal{G}]}$ is well defined and it is obviously $\mathcal{G}$-measurable. Finally, we only need to prove the conditional expectation defining property. That is,

$$\mathbb{E}_{Q_\theta}[Y\mathbf{1}_B] = \mathbb{E}_{Q_\theta}\left[\frac{\mathbb{E}[YL(X;\theta)|\mathcal{G}]}{\mathbb{E}[L(X;\theta)|\mathcal{G}]}\mathbf{1}_B\right], \qquad \forall B \in \mathcal{G}.$$

For all $B \in \mathcal{G}$, we have that

$$\begin{aligned}
&\mathbb{E}_{Q_\theta}\left[\frac{\mathbb{E}[YL(X;\theta)|\mathcal{G}]}{\mathbb{E}[L(X;\theta)|\mathcal{G}]}\mathbf{1}_B\right] \\
&\underset{(1)}{=} \mathbb{E}\left[L(X;\theta)\frac{\mathbb{E}[YL(X;\theta)|\mathcal{G}]}{\mathbb{E}[L(X;\theta)|\mathcal{G}]}\mathbf{1}_B\right] \\
&\underset{(2)}{=} \mathbb{E}\left[\mathbb{E}\left[L(X;\theta)\frac{\mathbb{E}[YL(X;\theta)|\mathcal{G}]}{\mathbb{E}[L(X;\theta)|\mathcal{G}]}\mathbf{1}_B|\mathcal{G}\right]\right] \\
&\underset{(3)}{=} \mathbb{E}\left[\mathbb{E}\left[L(X;\theta)|\mathcal{G}\right]\frac{\mathbb{E}[YL(X;\theta)|\mathcal{G}]}{\mathbb{E}[L(X;\theta)|\mathcal{G}]}\mathbf{1}_B\right] \\
&\underset{(4)}{=} \mathbb{E}\left[\mathbb{E}[YL(X;\theta)|\mathcal{G}]\mathbf{1}_B\right] \\
&\underset{(5)}{=} \mathbb{E}\left[\mathbb{E}[YL(X;\theta)\mathbf{1}_B|\mathcal{G}]\right] \\
&\underset{(6)}{=} \mathbb{E}[YL(X;\theta)\mathbf{1}_B] \\
&\underset{(7)}{=} \mathbb{E}_{Q_\theta}[Y\mathbf{1}_B].
\end{aligned}$$

Where we have used:

(1) Proposition 55, Lecture 2, to express an expectation with respect to Q_θ as an expectation with respect to P using the Radon-Nikodym derivative $\frac{dQ_\theta}{dP} = L(X;\theta)$.

(2) Conservation of the expectation property.

(3) "What is measurable goes out" property.

(4) $\mathbb{E}\left[L(X;\theta)|\mathcal{G}\right]/\mathbb{E}\left[L(X;\theta)|\mathcal{G}\right] = 1$.

(5) "What is measurable goes out" property. In this case, it goes in.

(6) Conservation of the expectation property.

(7) Proposition 55, Lecture 2,.