Problem 1:

Screening: Informed observes uninformed's choice:

a) Game 1:

	L	R
U	0, 0	4, 2
D	2, 6	0, 8

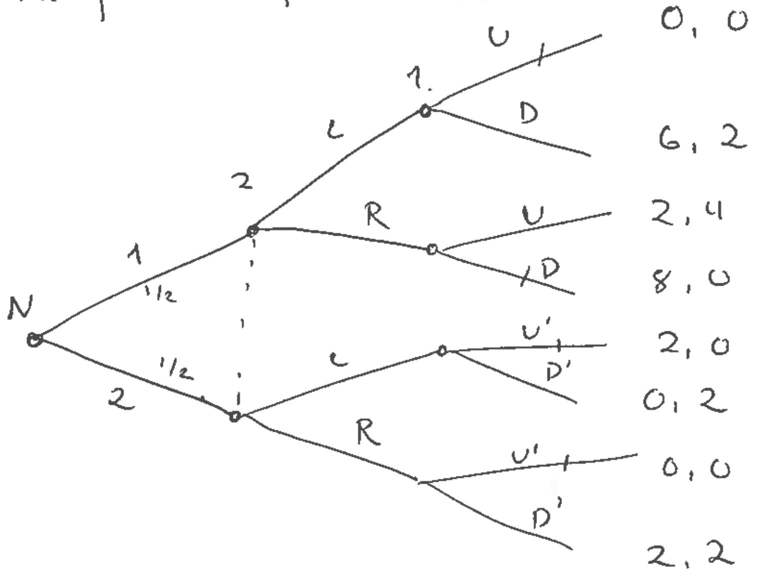
Game 2:

	L	R
U'	0, 2	0, 0
D'	2, 0	2, 2

Nature draws one of the games each has probability 0.5.

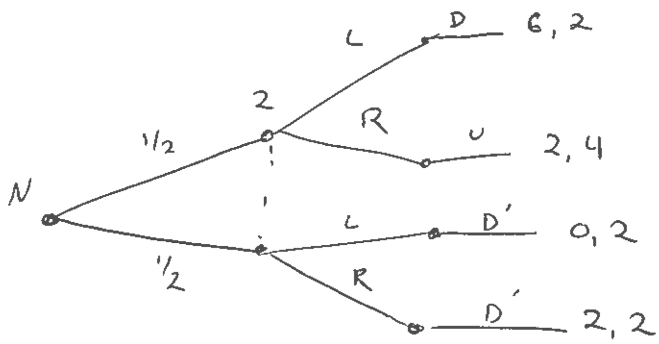
- Player 2 acts before player 1.
- Player 1 knows which game is played

Extensive form:
(One of the possible representations)



Note:
Reversed the pay-offs since player 2 acts first.
Not necessary, but then you need to keep in mind which payoffs are which.
Use backward induction which creates the smaller game:

Game given elimination of player 1's choices in end.
I.e. Backward induction.



Expected payoff of playing L for player 2:

$E(L) = \frac{1}{2} \cdot 6 + \frac{1}{2} \cdot 0 = 3$

$E(R) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 2 = 2.$

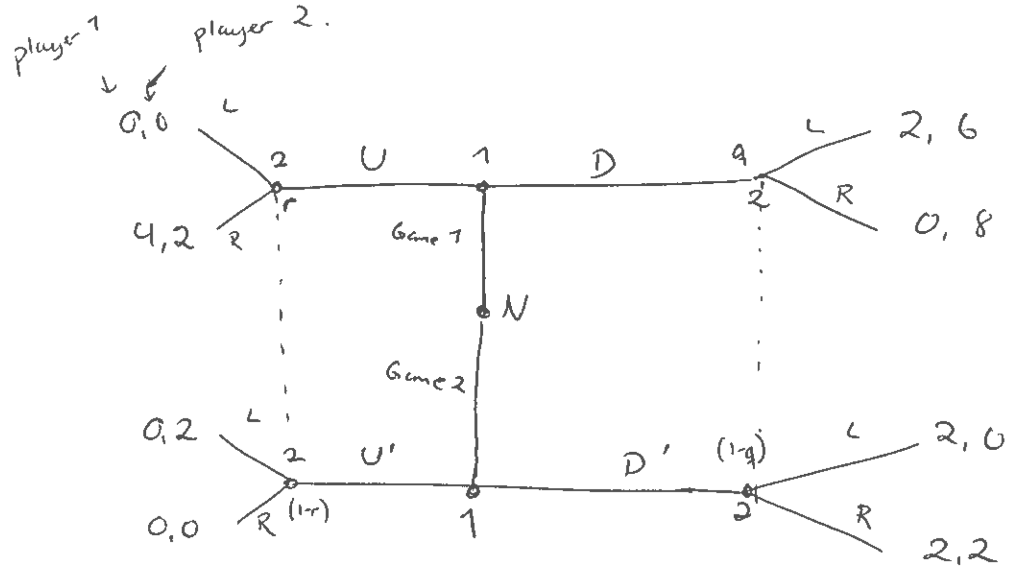
Player 2 will play L and player 1 will play:

$$S_1 = \left\{ \begin{array}{l} D \text{ if } L \text{ in game } 1 \\ D' \text{ if } L \text{ in game } 2 \\ U \text{ if } R \text{ in game } 1 \\ D' \text{ if } R \text{ in game } 2 \end{array} \right\}$$

$S_2 = \{L\}$

b) Signalling:

The uninformed observes the informed choice.



Separating equilibrium is if the types of a player behave differently.

How to calculate PBE:

1) Start with a strategy for player 1 (pooling or separating)

2) If possible, calculate updated beliefs by using Bayes' rule.

3) Given the updated beliefs calculate player 2's optimal action.

4) Check whether player 1's strategy is a best response to player 2's strategy.
If so, you have found a PBE.

Two possible separating equilibria:

1) Player 1 plays U in game 1 and D' in game 2.
2) Player 1 plays D in game 1 and U' in game 2.

* Is 1) and equilibrium:

First it must give beliefs that $r = 1$ and $q = 0$.
see problem 5 for details on how to calculate beliefs.

Given these beliefs player 2 plays R if U and R if D.

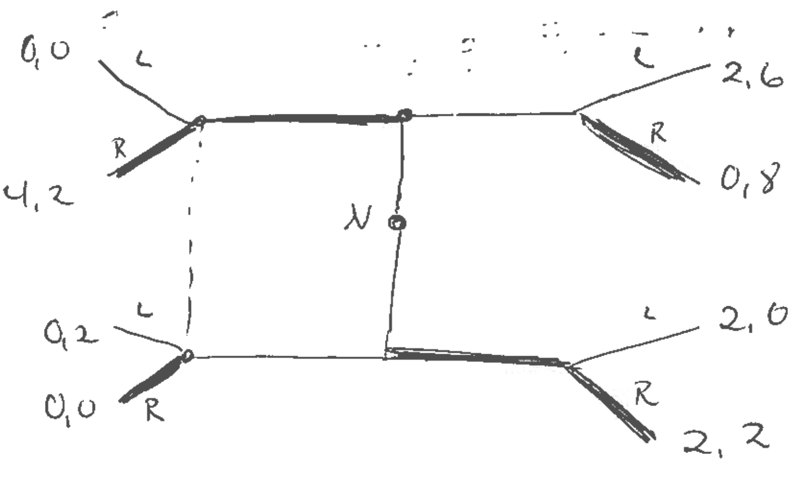
Given that player 2 plays R if U is observed Player 1 has no incentive to deviate from choosing D' in game 2.

Given that player 2 chooses R if D is observed Player 1 has no incentive to deviate to D in game 1. Eq $\{UD', RR, r=1, q=0\}$.

* Is 2) an equilibrium:

No. Since in game 2 D' dominates U' so independent of whether player 2 chooses L or R player 1 would want to deviate to choose D'.

Illustration of 1)



Heavy lines illustrates equilibrium strategies.

Is there a pooling equilibrium:

Two possibilities (UU') & (DD').

1) Hower as argued with separating U' cannot be part of equilibrium since U' is dominated by D'.

2) Then is DD' an equilibrium?

If player 2 finds himself in the right part of the game he cannot infer whether it is game 1 or game 2 that is played. Thus q must coincide with nature's probabilities.

$q = \frac{1}{2}$.

$E(L \mid q = \frac{1}{2}) = \frac{1}{2} \cdot 6 + \frac{1}{2} \cdot 0 = 3$

$E(R \mid q = \frac{1}{2}) = \frac{1}{2} \cdot 8 + \frac{1}{2} \cdot 2 = 5$.

So player 2 will play R after observing DD'.

Will player 1 want to deviate:

As mentioned will never want to deviate from D' to U'.

What about from D to U? That depends on player 2's response. Wouldn't deviate if Player 2 chooses L.

Player 2 chooses L if:

$E(L \mid r) > E(R \mid r)$

$0 \cdot r + 2 \cdot (1-r) > 2 \cdot r + (1-r) \cdot 0$

$2 - 2r > 2r$

$2 > 4r \qquad \frac{1}{2} > r$.

Thus: $S_1 = \{DD'\} \quad S_2 = \{R \text{ after } D,\ L \text{ after } U\},\ r < \frac{1}{2}$

$q = \frac{1}{2}$.

Note: Off equilibrium path any belief is possible so $r < \frac{1}{2}$ is possible even if U' is strictly dominated by D'.

Problem 2:

- $0 \leq s_i \leq 8$
- If $s_1 + s_2 \leq 8$ each get their demand

  If $s_1 + s_2 > 8$ no one gets anything.

a) Best response functions:

Assume that they like pizza and more is better than less.

$BR_i(s_j) = 8 - s_j$ for $0 \leq s_j < 8$.

$BR_i(s_j) = [0, 8]$ for $s_j = 8$.

where $i, j \in \{1, 2\}$ and $i \neq j$.

b) Find all pure NE:

- The players, BR, can be written as:

  $BR_1(y) = 8 - y$

  $BR_2(x) = 8 - x$

Pure strategy here implies only full slices of pizza, cannot divide them.

Then each combination that satisfies

$x = 8 - y \Rightarrow x + y = 8$

$y = 8 - x$

and $x \leq 8$ and $y \leq 8$ is an equilibrium.

Thus nine equilibria:

$\{0,8\}$ $\{1,7\}$ . . . . . $\{8,0\}$.

Finally $\{8,8\}$ is also an equilibria. To see it draw the matrix form. But BR function also show it Given that player 2 chooses 8 player 1 is indifferent between all his strategies including choosing 8. And given that player 1 chooses 8 player 2 is indifferent between all his strategies including playing 8. So $\{8,8\}$ is also a NE.

Example of matrix form. Only for illustrative purposes.

1\2	0	1	2	3	4	5	6	7	8
0	0,0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8
1	1,0	1,1	1,2	1,3	1,4	1,5	1,6	1,7	0,0
2	2,0	2,1	2,2	2,3	2,4	2,5	2,6	0,0	0,0
3	3,0	3,1	3,2	3,3	3,4	3,5	0,0	0,0	0,0
4	4,0	4,1	4,2	4,3	4,4	0,0	0,0	0,0	0,0
5	5,0	5,1	5,2	5,3	0,0	0,0	0,0	0,0	0,0
6	6,0	6,1	6,2	0,0	0,0	0,0	0,0	0,0	0,0
7	7,0	7,1	0,0	0,0	0,0	0,0	0,0	0,0	0,0
8	8,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0

So {8,0} and {0,8} and {8,8} are only weak equilibria as at least one of the players doesn't lose on deviating (nor gain).

Problem 3:

Pizza problem where player 1 makes demand before player 2.

a) A strategy for player 2 is then a strategy that specifies a demand given each possible demand that player 1 can make.

thus $S_2 = S_2(x(y=0), x(y=1), \dots, x(y=8))$.

b) The Nash outcomes are the same as in problem 2 as it assumes simultaneous moves.

c) Backward induction:

Step 2: Player 2 chooses y given choice of player 1 of x.
Gives BR:

$$y = 8 - x \quad \text{if} \quad 0 \leq x < 8$$
$$y = [0, 8] \quad \text{if} \quad x = 8.$$

Step 1: Player 1 maximize:

$$\max_x \quad x \quad \text{s.t } 1)\ x + y \leq 8 \ ;\ 2)\ y = 8 - x \ \text{ and } x \leq 8.$$

Put condition 2 into 1) gives

$$1')\quad x + 8 - x \leq 8 \implies 8 \leq 8.$$

Thus you are only maximizing based on $x \leq 8$ thus solution is $x = 8$.

NE :

$$S_1 = \{ y = 8 \}$$
$$S_2 = \left\{ \begin{array}{l} x = 8 - y \\ \text{s.t } x \geq 0 \end{array} \right\}$$

Solution assumes that player 2 will choose the weak equilibrium $\{8, 0\}$ over $\{8, 8\}$. Basically, think of playing 8 if 8 is observed as a threat. But players will only play the threat if it is in the players best interest to do so. And indifference is not enough.

Problem 4:

- 5 sizes of pizza with x slices
- $x \in \{4, 6, 8, 10, 12\}$

Player 1 observes x while player 2 only observes player 2's demand.

- Player 2 believes ex ante that each x happens with equal probability.

a) A strategy for player 1 is to specify a demand (z) for each possible size of pizza.

$$S_1 = S_1\left(z(x=4),\ z(x=6),\ z(x=8),\ z(x=10),\ z(x=12)\right).$$

b) Step 1:

Consider that player 2 has the following beliefs given player 1's strategy:

$\theta_2(s_1 \le 2) = 4$ with certainty.

$\theta_2(2 < s_1 \le 3) = 6$ with certainty.

$\theta_2(3 < s_1 \le 4) = 8$ "

$\theta_2(4 < s_1 \le 5) = 10$ "

$\theta_2(6 \le s_1 \le 12) = 12.$

Step 2: What is player 2's best responses given these beliefs?

$BR_2(s_1 \le 2) = 4 - s_1$

$BR_2(s_1 = 3) = 3$

$BR_2(s_1 = 4) = 4$

$BR_2(s_1 = 5) = 5$

$BR_2(6 \le s_1 < 12) = 12 - s_1$

$BR(s_1 = 12) =$ Anything.

Step 2 specifies a strategy s_2 for each observed s_1 and corresponding beliefs about X.

Step 3: Verify that player 1's strategy is optimal given player 2's strategy.

$BR_1(x=4, s_2) = 2$

For if player 1 chooses 3 or 4 then player 2 will believe that $x = 6$ or 8 and demand 3 or 4 and thus $s_1 + s_2 > 4$ if $s_1 = 3$ or $s_1 = 4$.

$BR_1(X = 6, s_2) = 3.$

Given $s_1 = 3$ player 2 chooses $s_2 = 3$.
If $s_1 < 3$ then player 2 chooses $s_2 = 4 - s_1$ which means that player 1 gets less than he can.
If $s_1 > 3$ player 2 chooses $s_2 = 4$ and then $s_1 + s_2 > 6$.

$BR_1(X = 8, s_2) = 4$

$BR_1(x = 10, s_2) = 5$

$BR_1(x = 12, s_2) = 11.$

Step 4: Check consistency of beliefs.
i) Full separating equilibrium thus decision is informative.
ii) Player 2's belief is consistent when observing these offers.

Beliefs off equilibrium path can be anything thus for instance $\theta_2(s_1 = 7) = 4$ with certainty is possible.

c) Are there other PBE in this game?

Yes! For example the same as in b except

$\theta_2 (s_1 = 12) = 12$ with certainty.

$BR_2 (s_1 = 12) = 0$

$s_1 (x = 12) = 12$.

Is there a pooling solution?

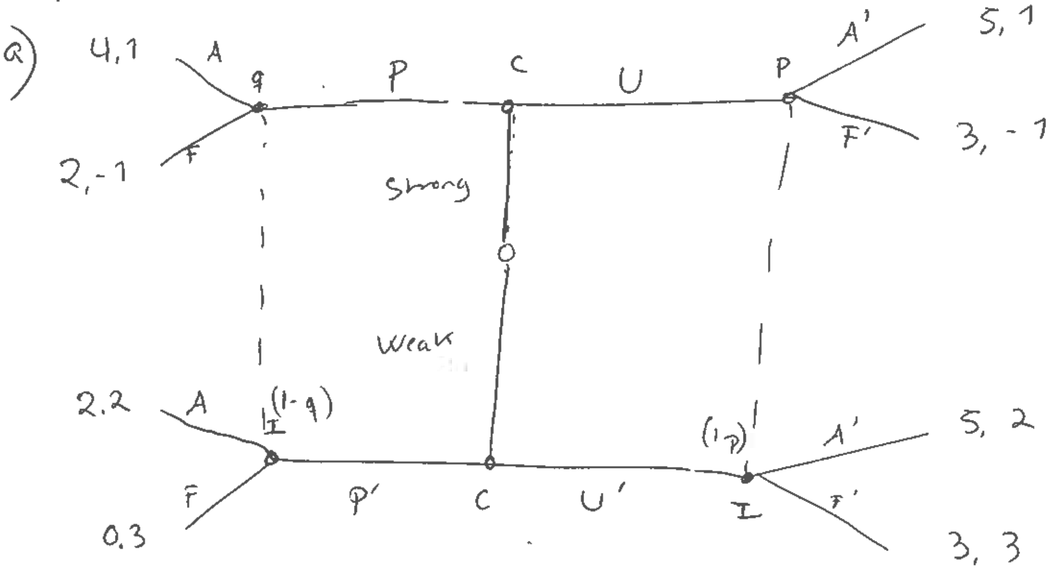
For it to be possible $s_1 \leq 4$ so that player 1 doesn't offer more than maximal in the lower case.

The strategy of player 1 will then not be informative to player 2 about the size of x thus player 2 has only one strategy to choose.

And since player 2 is weighting 5 sizes her strategy likely leads to $z + y > x$ for the low x's and thus player 1 would want to deviate and signal x.

Problem 5:

a)



The set of pure strategies for the challenger is

{Prepare, Prepare}

{Prepare, Not prepare}

{Not prepare, prepare}

{Not prepare, prepare}

Thus:

$$\underbrace{\{\text{Prepare}, \text{unprepared}\}}_{\text{if weak}} \times \underbrace{\{\text{Prepare}, \text{unprepared}\}}_{\text{if strong}}.$$

b) Chosing Prepared' for weak player gives 2 if A is chosen and 0 if F is chosen. While chosing unprepared' gives 5 if A' is chosen and 3 if F' is chosen. Thus Unprepared' strictly dominated prepared as he is better off chosing Unprepared' irrespective of the choice of the incumbat.

c) An equilibrium where the types are doing different strategies is called a separating equilibrium.

If q is the probability the incumbent puts on the player being a strong type conditioned on observing that the player is prepared.

And p is the probability the incumbent puts on the challenger being strong conditioned on observing unprepared

$$P(S \mid U) = p$$
$$P(S \mid P) = q.$$

Again we calculate PBE by first:

1) Start with a strategy of the challenger.
2) Update the beliefs of the incumbent using Bayes' rule if possible.
3). Calculate the incumbents optimal response given these beliefs.
4) Control whether the challengers strategy is a best response given the strategy of the incumbent.

Bayes rule: $P(A \mid B) = \dfrac{P(B \mid A) \cdot P(A)}{P(B)}$

<u>In words</u>:

In an equilibrium player 2's updated belief should be consistent with natures probability distribution and player 1's strategy.

So $P(S \mid U) = \dfrac{P(U \mid S) \cdot Prob(S)}{P(U)}$.

(challenger)

Thus probability that a player is strong conditional on reaching the node after the challenger choosing unprepared is the product of the probability that the strong type chooses unprepared (which is one in the strategy we are asked to evaluate) multiplied by the probability of being strong (given by nature to 1/2 in this case) divided by the probability of playing unprepared (which is 0.5 since strong plays prepared and weak choose unprepared)

Thus $P(S \mid U) = p = \dfrac{1 \cdot 0.5}{0.5} = 1$.

Same for

$P(S \mid P) = q = \dfrac{0 \cdot 0.5}{0.5} = 0$.

Thus updated beliefs are $p = 1$ $q = 0$.

Step 3 :

Incumbents BR :

- Given $q = 1$ A is BR to Prepared.

- Given $p = 0$ F' is BR to unprepared.

Step 4 :

Control whether the strategy is a best response for the challenger given results from step 3.

The weak player doesn't want to deviate as P'A gives 1 while U'F gives 2. So unprepared' is a BR for the weak.

The strong will face F' if deviate to unprepared which gives payoff −1, while he gets 4 if he doesn't deviate so no incentive to deviate.

Thus it is a NE.

d) Both choose 'unprepared' (pooling equilibrium).

Step 1: Analyse $\{UU'\}$

Step 2: Update beliefs:

$$p = P(S|U) = \frac{0,5 \cdot 0,5}{0,5} = 0,5$$

q = all possible things as strategies are not informative. We are off equilibrium path.

Step 3: Calculate incumbents BR:

BR (U):

$E(A') = 0.5 \cdot 1 + 0.5 \cdot 2 = 1.5$

$E(F') = 0.5 \cdot -1 + 0.5 \cdot 3 = 1.$

A' is best response.

Step 4: Do the challenger have a incentive to deviate?

The weak never wants to deviate irrespective of what the incumbent choose if 'prepared' is chosen since $5 > 3$ & $5 > 2$.

The strong doesn't 'want' to deviate either as $5 > 4$ & $5 > 2$ the other two possible outcomes.

So we have a NE.