Microeconomics 3200/4200:
Part 1

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Outline

1. Technology
2. Cost minimization
3. Profit maximization
4. The firm supply
   - Comparative statics
5. Multiproduct firms
Inputs and Outputs

- Firms are the economic actors that produce and supply commodities to the market.

- The **technology** of a firm can then be defined as the set of production processes that a firm can perform.

- A production process is an (instantaneous) transformation of **inputs**—commodities that are consumed by production—into **outputs**—commodities that result from production.
Inputs and Outputs

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- A production process is an (instantaneous) transformation of **inputs**–commodities that are consumed by production–into **outputs**–commodities that result from production.
Examples 1

- What are the combinations of inputs and outputs that are feasible?
- Given a vector of inputs, what is the largest amount of outputs the firm can produce?

- With 1 input and 1 output, a typical production function looks like:

\[ y \leq f(x), \]

where \( y \) is output, \( x \) is input, and \( f \) is the production function.

- Examples: \( f(x) = \alpha x; \ f(x) = \sqrt{x}; \ f(x) = x^2 + 1. \)
Examples 2

- With 2 inputs and 1 output, a typical production function looks like:
  \[ y \leq f(x_1, x_2), \]
  which we can represent in the 2-dimensional input space (*isoquants*!).

- Examples: \( f(x_1, x_2) = \min\{x_1, x_2\} \); \( f(x_1, x_2) = x_1 + x_2 \);
  \( f(x_1, x_2) = A x_1^\alpha x_2^\beta \).
Property 1. Impossibility of free production.

\[ f(0,0) \leq 0 \]
Property 2. Possibility of inaction.

\[ 0 \leq f(0,0) \]
Define the "**input requirement set (for output y)**" as follows:

\[ Z(y) \equiv \{(x_1, x_2) \mid y \leq f(x_1, x_2)\} \]  

(1)

Formally, the **y-isoquant**:

\[ \{(x_1, x_2) \mid y = f(x_1, x_2)\} \]  

(2)
Property 3. Free disposal.

For each $y \in \mathbb{R}_+$, if $x'_1 \geq x_1$, $x'_2 \geq x_2$, and $y \leq f(x_1, x_2)$, then $y \leq f(x'_1, x'_2)$. 
Properties 4 and 5.

**Property 4. Convexity of the input requirement set.**

For each $y \in \mathbb{R}_+$, each pair $(x_1, x_2), (x_1', x_2') \in Z(y)$, and each $t \in [0, 1]$, it holds that $t(x_1, x_2) + (1 - t)(x_1', x_2') \in Z(y)$.

**Property 5. Strict convexity of the input requirement set.**

For each $y \in \mathbb{R}_+$, each pair $(x_1, x_2), (x_1', x_2') \in Z(y)$, and each $t \in (0, 1)$, it holds that $t(x_1, x_2) + (1 - t)(x_1', x_2') \in \text{Int}Z(y)$. 
Marginal product of input $i$.

- The **marginal product** of an input $i = 1, 2$ describes the marginal increase of $f(x_1, x_2)$ when marginally increasing $x_i$.

- Mathematically, this can be written as

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2) - f(x_1, x_2)}{\Delta x_1},$$

when $\Delta x_1 \to 0$. If $f$ is differentiable, the marginal product is the derivative of $f$ w.r.t. $x_i$ evaluated at $(x_1, x_2)$ and is denoted by $MP_i(x_1, x_2)$. 
Technical rate of substitution.

- The **technical rate of substitution** (TRS) of input $i$ for input $j$ (at $z$) is defined as:

$$TRS(x_1, x_2) \equiv \frac{\Delta x_2}{\Delta x_1},$$

such that production is unchanged.

- By first order approximation,

$$\Delta y \approx MP_1 \Delta x_1 + MP_2 \Delta x_2 = 0,$$

solving, this gives:

$$TRS(x_1, x_2) = -\frac{MP_1(x_1, x_2)}{MP_2(x_1, x_2)}$$

- It reflects the relative value of the inputs (in terms of production) and corresponds to the slope of the y-isoquant at $(x_1, x_2)$. 

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Properties 6 and 7.

Property 6. Homotheticity.
For each \((x_1, x_2)\) and each \(t > 0\), it holds that \(TRS(x_1, x_2) = TRS(tx_1, tx_2)\).

Property 7. Homogeneity of degree \(r\).
For each \((x_1, x_2)\) and each \(t > 0\), it holds that \(f(tx_1, tx_2) = t^r f(x_1, x_2)\).
Properties 8, 9, and 10.

Property 8. Increasing returns to scale (IRTS).
For each \((x_1, x_2)\) and each \(t > 1\), it holds that \(f(tx_1, tx_2) > tf(x_1, x_2)\).

Property 9. Decreasing returns to scale (DRTS).
For each \((x_1, x_2)\) and each \(t > 1\), it holds that \(f(tx_1, tx_2) < tf(x_1, x_2)\).

Property 10. Constant returns to scale (CRTS).
For each \((x_1, x_2)\) and each \(t > 0\), it holds that \(f(tx_1, tx_2) = tf(x_1, x_2)\).
The optimization problem

- We split the optimization problem of the firm in two parts:
  1. Cost minimization (choosing \((x_1, x_2)\) for given \(y\));
  2. Output optimization (choosing \(y\), given the cost-minimizing input choices).
The cost minimization problem

- Let quantity $y \in \mathbb{R}_+$ be the output that a firm wants to bring to the market.
- The firm wants to minimize the cost of producing $y$. How to do it?
- graphically....
- Algebraically. Solve the following minimization problem:

$$\min_{x_1, x_2} \quad w_1 x_1 + w_2 x_2$$

$$s.t. \quad y \leq f(x_1, x_2)$$
The Lagrangian and FOCs

\[ \mathcal{L}(x_1, x_2, \lambda; w_1, w_2, y) = w_1 x_1 + w_2 x_2 + \lambda (y - f(x_1, x_2)) \] (4)

- The FOCs (allowing for corner solutions!) require that:

\[ \lambda^* MP_i(x_1^*, x_2^*) \leq w_i \quad \text{for } i = 1, 2 \] (5)

\[ y \leq f(x_1^*, x_2^*) \] (6)
The Lagrangian and FOCs

- Thus, if $x_i^* > 0$ (implying that $\lambda^* MP_i(x_1^*, x_2^*) = w_i$), a necessary condition for cost minimization is that:

$$\frac{MP_j(x_1^*, x_2^*)}{MP_i(x_1^*, x_2^*)} \leq \frac{w_j}{w_i} \quad \text{(7)}$$

- or (for interior solutions): TRS equals input price ratio.
The conditional demand function for input $i$ is:

$$x_i^* = H^i(w_1, w_2, y)$$  \hfill (8)

Substituting these conditional demands in the cost minimization problem, we get the relationship between the total cost and the input prices $w$ and the output choice $q$. This cost function is defined by:

$$C(w_1, w_2, y) \equiv w_1 x_1^* + w_2 x_2^* = w_1 H^1(w_1, w_2, y) + w_2 H^2(w_1, w_2, y)$$  \hfill (9)
Exercise: cost minimization problem (1)

- Determine the cost function for the firm with production function
  \( f(x_1, x_2) = (x_1 x_2)^{\frac{1}{3}} \).

- The minimization problem is:
  \[
  \min_{x_1, x_2} w_1 x_1 + w_2 x_2 \\
  \text{s.t. } q \leq \phi(x_1, x_2) = (x_1 x_2)^{\frac{1}{3}}
  \]

- Write the Lagrangian:
  \[
  \mathcal{L}(x_1, x_2, \lambda; w_1, w_2, y) = w_1 x_1 + w_2 x_2 + \lambda \left( y - (x_1 x_2)^{\frac{1}{3}} \right)
  \]
Exercise: cost minimization problem (2)

- The FOCs are:
  \[
  \begin{align*}
  \lambda^* MP_1 (x_1^*, x_2^*) &\leq w_1 \\
  \lambda^* MP_2 (x_1^*, x_2^*) &\leq w_2 \\
  y &\leq (x_1^* x_2^*)^{\frac{1}{3}}
  \end{align*}
  \]

- Since $f$ is increasing in $x_1$ and $x_2$ and $x_1, x_2 \neq 0$ (WHY?):
  \[
  \begin{align*}
  \lambda^* \frac{1}{3} (x_1^*)^{-\frac{2}{3}} (x_2^*)^{\frac{1}{3}} &= w_1 \\
  \lambda^* \frac{1}{3} (x_1^*)^{\frac{1}{3}} (x_2^*)^{-\frac{2}{3}} &= w_2 \\
  y &= (x_1^* x_2^*)^{\frac{1}{3}}
  \end{align*}
  \]
Exercise: cost minimization problem (3)

- Dividing the first by the second FOC (and taking the cubic power of the third one), gives:

\[
\begin{align*}
\frac{x_2^*}{x_1^*} &= \frac{w_1}{w_2} \\
y^3 &= x_1^* x_2^*
\end{align*}
\]

- And, solving for \( x_2^* \):

\[
x_2^* = \frac{w_1}{w_2} x_1^* = \frac{w_1}{w_2} \frac{y^3}{x_2^*}
\]

- Thus:

\[
(x_2^*)^2 = y^3 \frac{w_1}{w_2}
\]

- and the conditional demand function of input 2 is:

\[
x_2^* = H^2 (w_1, w_2, y) = y^3 \sqrt{\frac{w_1}{w_2}}
\]
Exercise: cost minimization problem (4)

Since \( x_2^* = \frac{w_1}{w_2} x_1^* \), substituting \( x_2^* = y^{\frac{3}{2}} \sqrt{\frac{w_1}{w_2}} \) gives the conditional demand function of input 1:

\[
x_1^* = H^1(w_1, w_2, y) = y^{\frac{3}{2}} \sqrt{\frac{w_2}{w_1}}
\]

The cost function is defined as:

\[
C(w_1, w_2, y) \equiv w_1 x_1^* + w_2 x_2^* = w_1 H^1(w_1, w_2, y) + w_2 H^2(w_1, w_2, y)
\]

Thus, substituting:

\[
C(w_1, w_2, y) = w_1 y^{\frac{3}{2}} \sqrt{\frac{w_2}{w_1}} + w_2 y^{\frac{3}{2}} \sqrt{\frac{w_1}{w_2}}
\]

And, simplifying,

\[
C(w_1, w_2, y) = 2 \sqrt{y^3 w_1 w_2}.
\]
Properties of the cost function

- Increasing in all input prices and strictly increasing in at least one; if $f$ is continuous, then also strictly increasing in output $y$.

- The cost function is homogeneous of degree 1 in prices, i.e. changing all prices by 10% increases total cost by 10%.

- The cost function is concave in input prices.

- Shephard’s Lemma: $\frac{\partial C(w_1, w_2, y)}{\partial w_i} = x_i^* = H_i(w_1, w_2, q)$, i.e. the cost increase when marginally changing the input price is exactly the compensated input demand!
The output optimization problem

Now that we know how a firm chooses inputs for production, we are left with the following problem:

$$\max_{y \in \mathbb{R}_+} py - C(w_1, w_2, y)$$  \hspace{1cm} (10)

The first order conditions are:

$$p = C_y(w_1, w_2, y^*) \quad \text{if } y^* > 0$$
$$p < C_y(w_1, w_2, y^*) \quad \text{if } y^* = 0$$  \hspace{1cm} (11)

The second order condition is:

$$C_{yy}(w_1, w_2, y^*) \geq 0$$  \hspace{1cm} (12)
Furthermore...

- Our firm needs to be aware that even when profits are maximized, these might not be positive... so we should further require that $\Pi \geq 0$

or:

$$py - C(w_1, w_2, y) \geq 0$$  \hspace{1cm} (13)

or that average cost is lower than $p$ ($\frac{C(w_1, w_2, y)}{y} \leq p$).
Demands and supply functions

- We can define the firm’s supply function as the relationship between the optimal quantity produced and the market prices of inputs and output:

\[ y = S(w_1, w_2, p) \]  

(14)

- Remember that we already defined the conditional demand function for input \( i \) as:

\[ x_i = H^i(w_1, w_2, y) \]  

(15)

- We can now substitute (14) in (15) to obtain the unconditional demand function for input \( i \):

\[ x_i = D^i(w_1, w_2, p) \equiv H^i(w_1, w_2, S(w_1, w_2, p)) \]  

(16)
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Slope of the supply function

- When \( y^* > 0 \), the FOC for the output optimization problem requires that:

\[
p = C_y (w_1, w_2, y^*)\]

- Substituting the supply function for \( y^* = S (w_1, w_2, p) \) gives:

\[
p = C_y (w_1, w_2, S (w_1, w_2, p))\]

- Now take the derivative wrt \( p \):

\[
1 = C_{yy} (w_1, w_2, S (w_1, w_2, p)) S_p (w_1, w_2, p)\]

- Rearrange and obtain:

\[
S_p (w_1, w_2, p) = \frac{1}{C_{yy} (w_1, w_2, S (w_1, w_2, p))} \geq 0 \quad (17)\]

- Thus, the slope of the supply function is positive! Why? by the SOC...
Slope of the supply function

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S_p (w_1, w_2, p) = \frac{1}{C_{yy} (w_1, w_2, S (w_1, w_2, p))} \geq 0
  \]

- Thus, the slope of the supply function is positive! Why? by the SOC...
Output price effect on input demand

- Consider the uncompensated demand for input \( x_i^* = D_i(w_1, w_2, p) \) and take the derivative wrt output price \( p \). **Remember** that \( D_i(w_1, w_2, p) \equiv H^i(w_1, w_2, S(w_1, w_2, p)) \).

\[
D_p^i(w_1, w_2, p) = H^i_y(w_1, w_2, y^*) S_p(w_1, w_2, p)
\]

- By the Shephard’s Lemma, \( \frac{\partial C(w_1, w_2, y)}{\partial w_i} = H^i(w_1, w_2, y) \). Thus

\[
H^i_y(w_1, w_2, y) = \frac{\partial}{\partial y} \left( \frac{\partial C(w_1, w_2, y)}{\partial w_i} \right) = \frac{\partial C_y(w_1, w_2, y)}{\partial w_i} \quad \text{(cross derivatives are equal!)}.
\]

Substituting in the previous gives:

\[
D_p^i(w_1, w_2, p) = \frac{\partial C_y(w_1, w_2, y^*)}{\partial w_i} S_p(w_1, w_2, p) \quad \text{(18)}
\]

- How does uncompensated demand change with output price? If \( w_i \) increases the marginal cost of output, then an increase of the output price would imply a larger use of input \( i \).
Input price effect on input demand (1)

- Consider the uncompensated demand for input $x_i^* = D_i(w_1, w_2, p)$ and take the derivative wrt input price $w_j$. (Again, start from the identity $D_i(w_1, w_2, p) \equiv H_i(w_1, w_2, S(w_1, w_2, p))$).

$$D_j^i(w_1, w_2, p) = H_j^i(w_1, w_2, y^*) + H^i_y(w_1, w_2, y^*) S_j(w_1, w_2, p)$$

- As before, by the Shephard’s Lemma, $\frac{\partial C(w_1, w_2, y)}{\partial w_i} = H^i(w_1, w_2, y)$.

Thus $H^i_y(w_1, w_2, y) = \frac{\partial \left( \frac{\partial C(w_1, w_2, y)}{\partial w_i} \right)}{\partial y} = \frac{\partial C_y(w_1, w_2, y)}{\partial w_i}$ (cross derivatives are equal!).

- Furthermore, differentiate the FOC $p = C_y(w_1, w_2, S(w_1, w_2, p))$ wrt $w_j$ to obtain:

$$0 = \frac{\partial C_y(w_1, w_2, y^*)}{\partial w_j} + C_{yy}(w_1, w_2, y^*) S_j(w_1, w_2, p)$$
Input price effect on input demand (2)

- Substitute to get

\[ D^i_j (w_1, w_2, p) = H^i_j (w_1, w_2, y^*) - \frac{C^i_y (w_1, w_2, y^*) C^j_y (w_1, w_2, y^*)}{C^y_y (w_1, w_2, y^*)} \] (19)

- How does uncompensated demand change with the price of another input? Two effects: a **substitution effect** \( H^i_j (w_1, w_2, y^*) \) and an **output effect** \( \frac{C^i_y (w_1, w_2, y^*) C^j_y (w_1, w_2, y^*)}{C^y_y (w_1, w_2, y^*)} \).
Implication 2

- Look now at the effect of $w_i$ on the demand of input $i$.

$$D_i^i(w_1, w_2, p) = H_i^i(w_1, w_2, q^*) - \frac{[C_{iy}(w_1, w_2, y^*)]^2}{C_{yy}(w_1, w_2, y^*)}$$ (20)

- $H_i^i(w_1, w_2, y) = C_{ii}(w_1, w_2, y)$ (by Shephard’s Lemma and taking the derivative).

- By concavity of the cost function (SOC for an optimum), $C_{ii}(w_1, w_2, y^*) \leq 0$. Thus, $H_i^i(w_1, w_2, y^*) \leq 0$.

- But $C_{yy}(w_1, w_2, y^*) \geq 0$ (again from the SOC) and also the squared term is larger than 0; thus:

- $D_i^i(w_1, w_2, p) \leq 0$, i.e. the unconditional demand for input $i$ is decreasing in the own price.
Up to now, we have studied the case of a firm producing a single output $y$. What if the firm could produce many goods at the same time?

Abstractly, all commodities (inputs or outputs) could be produced. So, let us write a (large) vector $y \equiv (y_1, \ldots, y_n) \in \mathbb{R}^n$ of all commodities.

Then good $y_n$ is a net output if $y_n > 0$; it is net input if $y_n > 0$. 
We can now write the technology as an implicit inequality:

\[ F(y) \leq 0 \]  \hspace{1cm} (21)

where the function \( F \) is non-decreasing in each of the \( y_i \).

We define the marginal rate of transformation of netput \( i \) into netput \( j \) by:

\[ MRT_{ij} \equiv \frac{MF_j(y)}{MF_i(y)} \]  \hspace{1cm} (22)
Objective of the firm

- Our firm still wants to maximize profits (now much simplified):

\[ \Pi = \sum_{i=1}^{n} p_i y_i \]  

subject to \( F(y) \leq 0. \)

- Proceeding as before, we can write the Lagrangean of the maximization problem:

\[ \mathcal{L}(y, \lambda; p) \equiv \sum_{i=1}^{n} p_i y_i - \lambda F(y) \]
Optimality conditions

- Deriving wrt each $y_i$ and $\lambda$, we get the following FOCs:

$$ p_i \geq \lambda^* F_i \left( y^* \right) \quad \text{for each } i = 1, \ldots, n $$  \hspace{1cm} (25)

$$ F(y^*) \leq 0 $$  \hspace{1cm} (26)

- If $y_i^* > 0$, for each $j$ the following holds at the optimum:

$$ \frac{MF_j(y^*)}{MF_i(y^*)} \leq \frac{p_j}{p_i} $$  \hspace{1cm} (27)

- or, equivalently, MRT equals output price ratio.
The netput and profit functions

- As before we can write the optimal choice of $y_i$ as a function of the prices: $y_i^* \equiv y_i(p)$.

- Substituting these netput functions in the profit, we get the profit function:

$$\Pi(p) \equiv \sum_{i=1}^{n} p_i y_i^* = \sum_{i=1}^{n} p_i y_i(p)$$  \hspace{1cm} (28)
Properties of the profit function

- Non-decreasing in all net-put prices.

- The profit function is homogeneous of degree 1 in prices, i.e. changing all prices by 10% increases total cost by 10%.

- The profit function is convex in net-put prices.

- [Hotelling’s Lemma] \( \frac{\partial \Pi(p)}{\partial p_i} = y_i^* \), i.e. the marginal profit increase for marginally changing the netput price is exactly the optimal quantity of netput \( i \)! 

\[ \partial \]