Microeconomics 3200/4200:

Part 1

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Outline

Inputs and Outputs

- Firms are the economic actors that produce and supply commodities to the market.
- The **technology** of a firm can then be defined as the set of production processes that a firm can perform.
- A production process is an (instantaneous) transformation of **inputs**–commodities that are consumed by production–into **outputs**–commodities that result from production.

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Examples 1

- What are the combinations of inputs and outputs that are feasible?
- Given a vector of inputs, what is the largest amoung of outputs the firm can produce?

- With 1 input and 1 output, a typical production function looks like:

$$y \leq f(x),$$

where y is output, x is input, and f is the **production function**.

- Examples: $f(x) = \alpha x$; $f(x) = \sqrt{x}$; $f(x) = x^2 + 1$.

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Examples 2

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Property 1.

Property 1. Impossibility of free production.

$f(0,0) \leq 0$

Property 2.

Property 2. Possibility of inaction.

$0 \leq f(0,0)$

Input requirement set and q-isoquant.

Define the "**input requirement set (for output y)**" as follows:

$$Z(y) \equiv \{(x_1, x_2) \,|\, y \leq f(x_1, x_2)\} \tag{1}$$

Formally, the **y-isoquant**:

$$\{(x_1, x_2) \,|\, y = f(x_1, x_2)\} \tag{2}$$

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Property 3.

Property 3. Free disposal.

For each $y \in \mathbb{R}_+$, if $x_1' \geq x_1$, $x_2' \geq x_2$, and $y \leq f(x_1, x_2)$, then $y \leq f(x_1', x_2')$.

Properties 4 and 5.

Property 4. Convexity of the input requirement set.

For each $y \in \mathbb{R}_+$, each pair $(x_1, x_2), (x_1', x_2') \in Z(y)$, and each $t \in [0,1]$, it holds that $t(x_1, x_2) + (1-t)(x_1', x_2') \in Z(y)$.

Property 5. Strict convexity of the input requirement set.

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Marginal product of input i.

- The **marginal product** of an input $i = 1, 2$ describes the marginal increase of $f(x_1, x_2)$ when marginally increasing x_i.

- Mathematically, this can be written as

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2) - f(x_1, x_2)}{\Delta x_1},$$

when $\Delta x_1 \to 0$. If ϕ is differentiable, the marginal product is the derivative of f w.r.t. x_i evaluated at (x_1, x_2) and is denoted by $MP_i(x_1, x_2)$.

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Technical rate of substitution.

- The **technical rate of substitution (TRS)** of input i for input j (at **z**) is defined as:

$$TRS(x_1, x_2) \equiv \frac{\Delta x_2}{\Delta x_1}, \tag{3}$$

such that production is unchanged.

- By first order approximation,

$$\Delta y \cong MP_1 \Delta x_1 + MP_2 \Delta x_2 = 0,$$

solving, this gives:

$$TRS(x_1, x_2) = -\frac{MP_1(x_1, x_2)}{MP_2(x_1, x_2)}$$

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Properties 6 and 7.

Property 6. Homotheticity.

For each (x_1, x_2) and each $t > 0$, it holds that $TRS(x_1, x_2) = TRS(tx_1, tx_2)$.

Property 7. Homogeneity of degree r.

For each (x_1, x_2) and each $t > 0$, it holds that $f(tx_1, tx_2) = t^r f(x_1, x_2)$.

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Properties 8, 9, and 10.

Property 8. Increasing returns to scale (IRTS).

For each (x_1, x_2) and each $t > 1$, it holds that $f(tx_1, tx_2) > tf(x_1, x_2)$.

Property 9. Decreasing returns to scale (DRTS).

For each (x_1, x_2) and each $t > 1$, it holds that $f(tx_1, tx_2) < tf(x_1, x_2)$.

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The optimization problem

- We split the optimization problem of the firm in two parts:

1. Cost minimization (choosing (x_1, x_2) for given y);
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The cost minimization problem

- Let quantity $y \in \mathbb{R}_+$ be the output that a firm wants to bring to the market.
- The firm wants to minimize the cost of producing y. How to do it?
- graphically....
- Algebraically. Solve the following minimization problem:

$$\begin{aligned} \min_{x_1, x_2} \quad & w_1 x_1 + w_2 x_2 \\ s.t. \quad & y \leq f(x_1, x_2) \end{aligned}$$

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The Lagrangian and FOCs

$$\mathscr{L}(x_1, x_2, \lambda; w_1, w_2, y) = w_1 x_1 + w_2 x_2 + \lambda (y - f(x_1, x_2)) \qquad (4)$$

- The FOCs (allowing for corner solutions!) require that:

$$\lambda^* MP_i(x_1^*, x_2^*) \leq w_i \qquad \text{for } i = 1, 2 \qquad (5)$$

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- Thus, if $x_i^* > 0$ (implying that $\lambda^* MP_i(x_1^*, x_2^*) = w_i$), a necessary condition for cost minimization is that:

$$\frac{MP_j(x_1^*, x_2^*)}{MP_i(x_1^*, x_2^*)} \leq \frac{w_j}{w_i} \tag{7}$$

- or (for interior solutions): **TRS** equals **input price ratio**.

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Conditional demand and cost function

- The **conditional demand function** for input i is:

$$x_i^* = H^i(w_1, w_2, y) \tag{8}$$

- Substituting these conditional demands in the cost minimization problem, we get the relationship between the total cost and the input prices w and the output choice q. This **cost function** is defined by:

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Exercise: cost minimization problem (1)

- Determine the cost function for the firm with production function $f(x_1, x_2) = (x_1 x_2)^{\frac{1}{3}}$.
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$$\mathscr{L}(x_1, x_2, \lambda; w_1, w_2, y) = w_1 x_1 + w_2 x_2 + \lambda \left(y - (x_1 x_2)^{\frac{1}{3}} \right)$$

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- Since f is increasing in x_1 and x_2 and $x_1, x_2 \neq 0$ (WHY?):

$$\begin{cases} \lambda^* \frac{1}{3} (x_1^*)^{-\frac{2}{3}} (x_2^*)^{\frac{1}{3}} = w_1 \\ \lambda^* \frac{1}{3} (x_1^*)^{\frac{1}{3}} (x_2^*)^{-\frac{2}{3}} = w_2 \\ y = (x_1^* x_2^*)^{\frac{1}{3}} \end{cases}$$

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Exercise: cost minimization problem (3)

- Dividing the first by the second FOC (and taking the cubic power of the third one), gives:

$$\begin{cases} \frac{x_2^*}{x_1^*} = \frac{w_1}{w_2} \\ y^3 = x_1^* x_2^* \end{cases}$$

- And, solving for x_2^*:

$$x_2^* = \frac{w_1}{w_2} x_1^* = \frac{w_1}{w_2} \frac{y^3}{x_2^*}$$

- Thus:

$$(x_2^*)^2 = y^3 \frac{w_1}{w_2}$$

- and the conditional demand function of input 2 is:

$$x_2^* = H^2(w_1, w_2, y) = y^{\frac{3}{2}} \sqrt{\frac{w_1}{w_2}}$$

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- The cost function is defined as:

$$C(w_1, w_2, y) \equiv w_1 x_1^* + w_2 x_2^* = w_1 H^1(w_1, w_2, y) + w_2 H^2(w_1, w_2, y)$$

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- And, simplifying,

$$C(w_1, w_2, y) = 2\sqrt{y^3 w_1 w_2}.$$

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Properties of the cost function

- Increasing in all input prices and strictly increasing in at least one; if f is continuous, then also strictly increasing in output y.
- The cost function is homogeneous of degree 1 in prices, i.e. changing all prices by 10% increases total cost by 10%.
- The cost function is concave in input prices.
- [Shephard's Lemma] $\frac{\partial C(w_1,w_2,y)}{\partial w_i} = x_i^* = H^i(w_1, w_2, q)$, i.e. the cost increase when marginally changing the input price is exactly the compensated input demand!

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- Increasing in all input prices and strictly increasing in at least one; if f is continuous, then also strictly increasing in output y.
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The output optimization problem

- Now that we know how a firm chooses inputs for production, we are left with the following problem:

$$\max_{y \in \mathbb{R}_+} py - C(w_1, w_2, y) \tag{10}$$

- The first order conditions are:

$$\begin{cases} p = C_y(w_1, w_2, y^*) & \text{if } y^* > 0 \\ p < C_y(w_1, w_2, y^*) & \text{if } y^* = 0 \end{cases} \tag{11}$$

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Furthermore...

- Our firm needs to be aware that even when profits are maximized, these might not be positive... so we should further require that $\Pi \geq 0$ or:

$$py - C\left(w_1, w_2, y\right) \geq 0 \tag{13}$$

or that average cost is lower than p ($\frac{C(w_1, w_2, y)}{y} \leq p$).

Demands and supply functions

- We can define the firm's **supply function** as the relationship between the optimal quantity produced and the market prices of inputs and output:

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- Remember that we already defined the *conditional demand function* for input i as:

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- We can now substitute (14) in (15) to obtain the **unconditional demand function** for input i:

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Outline

Slope of the supply function

- When $y^* > 0$, the FOC for the output optimization problem requires that:

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- Substituting the supply function for $y^* = S(w_1, w_2, p)$ gives:

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$$S_p(w_1, w_2, p) = \frac{1}{C_{yy}(w_1, w_2, S(w_1, w_2, p))} \geq 0 \qquad (17)$$

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- Consider the uncompensated demand for input $x_i^* = D^i(w_1, w_2, p)$ and take the derivative wrt **output** price p. **Remember** that $D^i(w_1, w_2, p) \equiv H^i(w_1, w_2, S(w_1, w_2, p))$.

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- Substitute to get

$$D_j^i(w_1, w_2, p) = H_j^i(w_1, w_2, y^*) - \frac{C_{iy}(w_1, w_2, y^*) C_{jy}(w_1, w_2, y^*)}{C_{yy}(w_1, w_2, y^*)} \quad (19)$$

- How does uncompensated demand change with the price of another input? Two effects: a **substitution effect** $H_j^i(w_1, w_2, y^*)$ and an **output effect** $\frac{C_{iy}(w_1,w_2,y^*) C_{jy}(w_1,w_2,y^*)}{C_{yy}(w_1,w_2,y^*)}$.

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- But by symmetry of the cross derivatives, $C_{ij}(w_1, w_2, y^*) = C_{ji}(w_1, w_2, y^*)$.
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Implication 2

- Look now at the effect of w_i on the demand of input i.

$$D_i^i(w_1, w_2, p) = H_i^i(w_1, w_2, q^*) - \frac{[C_{iy}(w_1, w_2, y^*)]^2}{C_{yy}(w_1, w_2, y^*)} \qquad (20)$$

- $H_i^i(w_1, w_2, y) = C_{ii}(w_1, w_2, y)$ (by Shephard's Lemma and taking the derivative).
- By concavity of the cost function (SOC for an optimum), $C_{ii}(w_1, w_2, y^*) \leq 0$. Thus, $H_i^i(w_1, w_2, y^*) \leq 0$.
- But $C_{yy}(w_1, w_2, y^*) \geq 0$ (again from the SOC) and also the squared term is larger than 0; thus:
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- Look now at the effect of w_i on the demand of input i.

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Many products, many inputs...

- Up to now, we have studied the case of a firm producing a single output y. What if the firm could produce many goods at the same time?
- Abstractly, all commodities (inputs or outputs) could be produced. So, let us write a (large) vector $\mathbf{y} \equiv (y_1, ..., y_n) \in \mathbb{R}^n$ of all commodities.
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Production technology and MRT

- We can now write the technology as an implicit inequality:

$$F(\mathbf{y}) \leq 0 \tag{21}$$

where the function F is non-decreasing in each of the y_i.

- We define the **marginal rate of transformation** of netput i into netput j by:

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- Our firm still wants to maximize profits (now much simplified):

$$\Pi = \sum_{i=1}^{n} p_i y_i \tag{23}$$

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Optimality conditions

- Deriving wrt each y_i and λ, we get the following FOCs:

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Properties of the profit function

- Non-decreasing in all net-put prices.
- The profit function is homogeneous of degree 1 in prices, i.e. changing all prices by 10% increases total cost by 10%.
- The profit function is convex in net-put prices.
- [Hotelling's Lemma] $\frac{\partial \Pi(\mathbf{p})}{\partial p_i} = y_i^*$, i.e. the marginal profit increase for marginally changing the netput price is exactly the optimal quantity of netput i!

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