

## ECON3120/4120 Mathematics 2: Problems for week 46 (2010)

- The entire problem set for the autumn 2006 exam
- Problem 4 from the autumn 2007 exam
- Problem 1 below.
- The below Problem 2 is again «experimental» and not exam relevant per se, but it is a nice exercise in retrieving information by fiddling back and forth with the Kuhn–Tucker conditions. If you take ECON4240, you will spend a couple of weeks on this, so of course it is too much to expect you to comprehend the entire problem from the brief sketch I gave at the lecture.  
*Homework:* Write down the steps, and see if you follow the reasoning.  
*Seminar coverage:* Only to the extent you request it.

### Assumptions for both problems

For both problems below, let  $p \in (0, 1)$  and  $k > 0$  be constants, and  $U$  and  $V$  given concave  $C^2$  (utility) functions defined on  $[0, \infty)$ . We assume that  $U'$  and  $V'$  take all real values (i.e.: both tend to infinity at zero and to zero at infinity), that  $U(0) = V(0) = 0$ , and that

$$U'(x) > V'(x), \quad \text{all } x.$$

### Problem 1

It can be shown – and in problem 2 you shall do precisely that – that the maximization of problem 2 reduces to

$$\max_{q, Q} p[U(Q) - U(q) + V(q)] + (1 - p)V(q) - k(pQ + (1 - p)q) \quad \text{subject to } Q \geq q. \quad (*)$$

- Assume that the problem has a solution. Solve in terms of the derivatives of the utility functions  $U$  and  $V$ .
- The interpretations are that  $U'$  and  $V'$  are marginal utilities, and  $k$  is marginal production cost. Comments?

## Problem 2

Let

$$f(t, q, T, Q) = pT + (1 - p)t - k(pQ + (1 - p)q),$$

and consider the maximization

$$\max_{(t,q),(T,Q)} f(t, q, T, Q) \quad \text{subject to}$$

$$U(Q) - T \geq 0 \quad (1)$$

$$V(q) - t \geq 0 \quad (2)$$

$$U(Q) - T \geq U(q) - t \quad (3)$$

$$V(q) - t \geq V(Q) - T \quad (4)$$

Problem 2 is about establishing that – under the above assumptions on  $U, V, k, p$  – this problem reduces to (\*) of problem 1. Proceed as follows:

- Show that (1) or (2) must be active:  
Assume for contradiction that they are both inactive. Then both  $t$  and  $T$  can be increased by the same small amount  $x$  without violating (1), (2); show that this does not affect (3), (4). What happens to  $f$ ?
- Recall that  $U' > V'$ , with  $U(0) = V(0)$ . Use this to show that the RHS of (3) is greater than 0 if (2) holds; hence the LHS of (3) is  $\geq 0$ , hence (1) holds *automatically* – and can therefore be dropped.
- Use the two previous bullet points to show that (2) must be active.
- Eliminating  $t$  by inserting for (2), yields

$$\max_{q,(T,Q)} pT + (1 - p)V(q) - k(pQ + (1 - p)q) \quad \text{subject to}$$

$$U(Q) - T \geq U(q) - V(q) \quad (5)$$

$$0 \geq V(Q) - T \quad (6)$$

Write down the Lagrangian (recall which way the inequalities are in our standard Kuhn-Tucker problem!) and use one of the first-order conditions to show that (5) is active (i.e.: can the respective Lagrange multiplier be zero?)

- With (5) being active, we can eliminate  $T = U(Q) - U(q) + V(q)$ :

$$\max_{q,Q} p[U(Q) - U(q) + V(q)] + (1 - p)V(q) - k(pQ + (1 - p)q) \quad \text{subject to}$$

$$U(Q) - V(Q) - (U(q) - V(q)) \geq 0 \quad (7)$$

- Now employ the following trick: Put  $\Delta = U - V$ . Then the right hand side of (7) equals

$$\Delta(Q) - \Delta(q) = \int_q^Q \Delta(x)dx = \int_q^Q (U'(x) - V'(x))dx. \quad (8)$$

Show that (8) is nonnegative – i.e. (7) holds – if and only if  $Q \geq q$ .

(It is possible to transform one of these problems into a concave one and hence apply sufficient conditions, but hey, let's not overdo this :-)