ECON3120/ECON4120 Mathematics 2 (spring term 2008) solved

Problem 1

(a) *Note: Triangular matrices were treated in an example in a lecture, but should anyway follow easily from cofactor expansion. The candidates should also be familiar with «clever» choices of rows or columns with many zeroes.*
Cofactor expansion gives the product of the entries on the main diagonal: $|\mathbf{A}_d| = \underline{\underline{d^4(d^2+1)}}$.

$\mathbf{A}_d^n$ has an inverse if and only if its determinant is nonzero, and since $|\mathbf{A}^n| = |\mathbf{A}|^n$, this occurs if and only if $d^4(d^2+1) = 0$, that is, if and only if $d = 0$, Q.E.D.

(b) For part (i), we have a unique solution if and only if $\mathbf{A}_d^{2008}$ has an inverse, which from (a) occurs precisely when $\underline{\underline{d \neq 0}}$.

For part (ii), we know that there is a solution for all $d \neq 0$, so the only remaining d-value to check is $d = 0$, where $\mathbf{x} = \mathbf{0}$ is a solution. So there is always a solution, Q.E.D.
Note: Homogeneous equations were treated separately in a lecture.

Problem 2 Differentiation yields

$$\ln u\,dx + \frac{x}{u}\,du + \ln y\,dv + \frac{v}{y}\,dy = 0$$
$$-e^y\,dy + v\,du + u\,dv + e^{ux}(u\,dx + x\,du) = 0.$$

To calculate the derivatives at the point, we insert for the point to get

$$a\,du + \ln a\,dv = 0$$
$$-e^a\,dy + dv + e^a(dx + a\,du) = 0.$$

Inserting for $a\,du$ in the second equation yields

$$(1 - e^a \ln a)\,dv = e^a(dy - dx)$$

so that

$$\frac{\partial v}{\partial x} = (\ln a - e^{-a})^{-1}$$
$$\frac{\partial v}{\partial y} = (e^{-a} - \ln a)^{-1}.$$

Problem 3

(a) From the formula, we have

$$z = e^t(C + \int e^{-t}e^{-bt}\,dt) = Ce^t - e^t\frac{1}{b+1}e^{-(b+1)t} = \underline{\underline{Ce^t - \frac{1}{b+1}e^{-bt}}}$$

(b) Integrating $\dot{y} = z$ we get

$$y = \int \left[Ce^t - \frac{1}{b+1}e^{-bt}\right]dt = D + Ce^t + \frac{1}{b(b+1)}e^{-bt}.$$

In order for y to converge to 1 as $t \to \infty$, we must have $C = 0$ and $D = 1$, so the answer is $\underline{\underline{y = 1 + \frac{1}{b(b+1)}e^{-bt}}}$.

Note: The candidates can use the facts $b \neq 0 \neq b+1$ without any comments.

Problem 4

(a) The Lagrangian becomes $L(x, y) = e^{xy^2} - Ax^2y - 1 + \lambda_x x + \lambda_y y - \lambda_A(x + y)$, and the Kuhn-Tucker conditions become

$$0 = y^2e^{xy^2} - 2Axy + \lambda_x - \lambda_A \tag{1}$$
$$0 = 2xye^{xy^2} - Ax^2 + \lambda_y - \lambda_A \tag{2}$$
$$\lambda_x \geq 0 \quad (= 0 \text{ if } x > 0) \tag{3}$$
$$\lambda_y \geq 0 \quad (= 0 \text{ if } y > 0) \tag{4}$$
$$\lambda_A \geq 0 \quad (= 0 \text{ if } x + y < A^2). \tag{5}$$

Note: The candidates are allowed to – and have to some extent been encouraged to – include the feasibility conditions in the Kuhn-Tucker conditions.
To show that the Kuhn-Tucker conditions are satisfied at the origin, one inserts $x = y = 0$ and observes that (1)–(5) are satisfied with all multipliers equal to zero.
Note: The candidates have to point out that (3)–(5) can indeed be satisfied for some values of the multipliers.

(b) We are given that the solution is not on the axes, hence $\lambda_x = \lambda_y = 0$. We therefore have to prove that $\lambda_A \neq 0$. Assuming for contradiction that $\lambda_A = 0$ also, we have a stationary point; by the assumption $xy \neq 0$, (1) will imply $ye^{xy^2} = 2Ax$ while (2) will imply $2ye^{xy^2} = Ax$. Together they imply $4Ax = Ax$, a contradiction.
Notes: Proof by contradiction should be well-known and has been stressed both at lectures and in a compulsory term paper. Some candidates will most probably divide by x and/or y without pointing out that they are nonzero; this is OK, as the assumed positivity is done on purpose.