NCF, Nov. 2014

The autumn 2011 exam in brief

* Not recommended to write this short on an exam (although it would suffice if everything correct)

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1 a) Cofactor expansion:

$$|A_t| = t(t^2-1) - t = \underline{\underline{t\cdot(t^2-2)}}$$

b) We have unique solution if and only if $t \notin \{-\sqrt{2}, 0, \sqrt{2}\}$

Gaussian elimination:

$$\left(\begin{array}{ccc|c} t & 1 & 0 & \sqrt{2} \\ 1 & t & 1 & 1 \\ 0 & 1 & t & \sqrt{2} \end{array}\right) \begin{array}{l} \\ -t \text{ eq} \\ -t \end{array} \sim$$

$$\left(\begin{array}{ccc|c} 0 & 1-t^2 & -t & \sqrt{2}-t \\ 1 & 0 & 1-t^2 & 1-t\sqrt{2} \\ 0 & 1 & t & \sqrt{2} \end{array}\right)$$

- If $t=0$: 1st and 3rd eq both say $x_2 = \sqrt{2}$. Choose x_3 free, then $x_1 = 1-x_3$. Infinitely many solutions
- If $t^2=2$, add 1st and 3rd eq to get $0 = 2\sqrt{2} - t$. No solution for $t = \pm\sqrt{2}$

*Note that for problem 2(b) you can use the formula!

Problem 2

(a)
$$El_x y = \frac{x}{y} y' = -\frac{x\, F_x'(x,y)}{y\, F_y'(x,y)} = -\frac{x(Aax^{a-1}+ky)}{y(Bbx^{b-1}+kx)} = -\frac{Aax^a + kxy}{Bby^b + kxy}$$

$\boxed{k=0}$

(b)
$$\sigma_{yx} = \frac{d \ln y/x}{d\ln(F_x'/F_y')} = \frac{d\ln y - d\ln x}{d\ln(Aax^{a-1}) - d\ln(Bby^{a-1})} = \frac{\left(\frac{y'}{y} - \frac{1}{x}\right)}{\frac{a-1}{x} - \frac{b-1}{y}y'}$$

$$= \frac{\frac{-1}{x}\left[\frac{Aax^a}{Bby^b}+1\right]}{\frac{1}{x}\left[a-1+(b-1)\frac{Aax^a}{Bby^b}\right]}\cdot\frac{Bby^b}{Bby^b} = \frac{Aax^a + Bby^b}{(1-a)Bby^b + (1-b)Aax^a}$$

$$= \frac{Aax^a + Bby^b}{Aax^a + Bby^b - ab\underbrace{[Ax^a + By^b]}_{=F(x,y)}}$$

☒

3 (a) $x^2+2y^2=k$ defines a nonempty, closed and bounded set. f continuous on R^2. Extreme value theorem $\Rightarrow$ solution exists.

With $L = f - \lambda(x^2+2y^2-k)$, conditions are

$$\begin{cases} 4x^3-2x-2\lambda x=0 & \text{I} \\ 2y-4\lambda y=0 & \text{II} \\ x^2+2y^2=k \end{cases}$$

(b) II yields $2y(1-2\lambda)=0$ or equivalently $y=0$ or $\lambda=\frac{1}{2}$.

- Case $y=0$. Then $x=\sqrt{k}$ ($x \geq 0$ assumed). Since the λ coeff. in I is $\neq 0$, a λ exists. Point: $(\sqrt{k},0)$
- Case $\lambda=\frac{1}{2}$. By I, $x=0$ and $y=\frac{1}{2}\sqrt{2k}$ or $2x^2=1+\lambda=\frac{3}{2}$ so $x=\frac{1}{2}\sqrt{3}$ and $y=\sqrt{\frac{k-3/4}{2}}=\frac{1}{2}\sqrt{2k-\frac{3}{2}}$ if $k \geq \frac{3}{4}$

Points: $(0,\frac{1}{2}\sqrt{2k})$ and if $k\geq\frac{3}{4}$: $\boxed{(\frac{1}{2}\sqrt{3},\ \frac{1}{2}\sqrt{2k-\frac{3}{2}})}$

(c) $f(\sqrt{k},0)=k^2-k$ ($=0$ for $k=1$)

$f(0,\sqrt{\frac{k}{2}})=\frac{k}{2}$ (>0)

(c3) $f(\sqrt{3/4},\sqrt{\frac{k-3/4}{2}})=\frac{9}{16}-\frac{3}{4}+\frac{k}{2}-\frac{3}{8}$ which for $k=1$, is <0: $\frac{9-12+8-6}{16}=-\frac{1}{16}$

So around $k=1$, will $V(k)$ be given by (c3) with derivative $\frac{d}{dk}\frac{k}{2}=\frac{1}{2}$

On the other hand, we know that $V'(k)=\lambda$ which for this point equals $\frac{1}{2}$

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4 No constant solution, so $\frac{2x\,dx}{x^2+k}=te^{2t}\,dt$

and $\ln(x^2+k)=\int t e^{2t}\,dt$

$=\frac{t}{2}e^{2t}-\frac{1}{2}\int e^{2t}\,dt$

$=C+\frac{1}{4}e^{2t}[2t-1]$

So, with $D>0$ arbitrary:

(Δ) $x^2+k=D\cdot e^{\frac{1}{4}e^{2t}[2t-1]}$

So $x(t)=\pm\sqrt{De^{\frac{1}{4}e^{2t}[2t-1]}-k}$

(b) Put $t=1$, $x=2$ in (Δ):

$D=(4+k)e^{-\frac{1}{4}e^2}$

so that

$$x(t)=\sqrt{(4+k)e^{\frac{1}{4}\{e^{2t}[2t-1]-e^2\}}-k}$$

("+" in front of $\sqrt{\ }$ because $x(1)>0$)