ECON3120/4120 Mathematics 2, spring 2012

The Tuesday May 29 2012, 09:00–12:00 exam: on the solution and the requirements.

Note: this document is incomplete as a solution, and should not be used as a template for an exam paper!

Problem 1 Define for each value of u the matrix

$$\mathbf{A}_u = \begin{pmatrix} 0 & 1 & 2 \\ 3u & u+3 & u^3 \\ 0 & u-3 & u-4 \end{pmatrix}$$

(a) Show that $|\mathbf{A}_u| = 0$ if and only if $u = 0$ or $u = 2$.

(b) Consider the equation system (where $\mathbf{x} \in \mathbf{R}^3$ is the unknown)

$$\mathbf{A}_u \mathbf{x} = \begin{pmatrix} u \\ 2u \\ 3u \end{pmatrix}$$

Decide, for each value of u, whether the system has (i) no solution, (ii) one solution, or (iii) more than one solution.

On the solution:

(a) Cofactor expansion along the first column yields $|\mathbf{A}_u| = -3u(2-u)$, which is 0 if and only if $u = 0$ or $u = 2$.
Note: The candidates are expected to get the sign correct; if they apply cofactor expansion and fail to get the minus sign, they should be penalized even though it won't affect the answer. It was deliberate to give the problem in such a way that they would get the right answer even if they miss this part of the theory.

(b) For $u \notin \{0, 2\}$: unique solution.
For $u = 0$: Several solutions (existence because the RHS is null).
For $u = 2$: No solution, as first and last equation contradict each other. (Alternatively, a single step of Gaussian elimination yields the contradiction.)

Problem 2 In this problem, you shall take as given without proof that the equation system

$$
\begin{aligned}
x + v + e^{xu} - v^5 y &= 1 \\
\ln(1 + xuv) + x + ve^{uy} &= e
\end{aligned}
$$

defines u and v as continuously differentiable functions of x and y around the point P with coordinates $(x, y, u, v) = (0, 1, 1, 1)$.

(a) Differentiate the equation system (i.e., calculate differentials).

(b) Calculate $u'_x(0, 1)$ and $u'_y(0, 1)$.

On the solution:

(a) Differentiating term by term will yield

$$
\begin{aligned}
(1 + ue^{xu})\,\mathrm{d}x - v^5\,\mathrm{d}y + xe^{xu}\,\mathrm{d}u + (1 - 5v^4 y)\,\mathrm{d}v &= 0 \\
(\frac{uv}{1 + xuv} + 1)\,\mathrm{d}x + uve^{uy}\,\mathrm{d}y + (\frac{xv}{1 + xuv} + vye^{uy})\,\mathrm{d}u + (\frac{xu}{1 + xuv} + e^{uy})\,\mathrm{d}v &= 0
\end{aligned}
$$

(where I have gathered the terms).

(b) Since they are asked for the derivatives at P and only there, insert for the coordinates to get

$$
\begin{aligned}
2\,\mathrm{d}x - \mathrm{d}y + 0\,\mathrm{d}u - 4\,\mathrm{d}v &= 0 \\
2\,\mathrm{d}x + e\,\mathrm{d}y + e\,\mathrm{d}u + e\,\mathrm{d}v &= 0
\end{aligned}
$$

and eliminating $\mathrm{d}v$ by adding $e/4$ of the first equation to the second:

$$
2(1 + \frac{e}{4})\,\mathrm{d}x + e(1 - \frac{1}{4})\,\mathrm{d}y = -e\,\mathrm{d}u
$$

which yields $u'_x(0, 1) = -\frac{2}{e} - \frac{1}{2}$ $(= -(4 + e)/2e)$ and $u'_y(0, 1) = -\frac{3}{4}$.

Problem 3

(a) Evaluate the integrals

$$\text{(i)} \int x^3 e^x \, dx, \qquad \text{(ii)} \int_0^1 e^{y^{1/4}} \, dy, \qquad \text{(iii)} \int_2^e \frac{(\ln(\ln z))^3}{z} \, dz$$

(*Hint:* In (ii) and (iii), a substitution will lead to an integral where you can use (i).)

(b) Throughout this part, assume that $t \geq 0$. Show that

$$\int \frac{1}{(t+1)(t+2)^2} \, dt = \frac{1}{t+2} + \ln \frac{t+1}{t+2} + C$$

and use this to find the particular solution of the differential equation

$$\dot{x}(t) = \frac{x(t)}{(t+1)(t+2)^2} \qquad \text{with} \qquad x(0) = 1/\sqrt{e}.$$

On the solution:

(a) (i): By repeated use of integration by parts, the candidates should arrive at

$$e^x(x^3 - 3x^2 + 6x - 6) + C.$$

(ii): Subsitituting $u = y^{1/4}$ leads to $4 \int u^3 e^u \, du$. Now insert from (i) and evaluate at the bounds to obtain the answer

$$4(6 - 2e).$$

Note, for those candidates who substitute the limits as well: Substituting the limits will yield $\int_0^1 \ldots du$ as well. However, so will not be the case in (iii), which could serve as an indication on whether they got the theory correct.

(iii): The substitution $w = \ln \ln z$ will work. However, that might be too bold, and of course, substituting $v = \ln z$ and then $w = \ln v$ will do the same job. In any case, it leads to the integral (in the indefinite case) $\int w^3 e^w \, dw$. Now use (i) and evaluate at the bounds, to obtain the answer

$$-6 - \left[(\ln \ln 2)^3 - 3(\ln \ln 2)^2 + 6 \ln \ln 2 - 6\right] \ln 2$$

(b) To show the integral, differentiate the right-hand side (and rewrite until it matches). To solve the differential equation, note that it is separable. Since we are only asked for a particular solution, with $x(0) > 0$, we have

$$\int \frac{\mathrm{d}x}{x} = \text{[the integral asked for]}$$
$$\ln|x(t)| = \ln x(t) = \frac{1}{t+2} + \ln\frac{t+1}{t+2} + C$$

and at $t = 0$:

$$-\frac{1}{2} = \ln(e^{-1/2}) = \frac{1}{2} + \ln\frac{1}{2} + C$$
$$\text{which yields} \qquad C = \ln 2 - 1.$$

Solving for x:

$$x(t) = 2\,\frac{t+1}{t+2}\;e^{\frac{1}{t+2}-1}$$

(which equals $2\,\frac{t+1}{t+2}\,\exp\{-\frac{t+1}{t+2}\}$.)

Problem 4 Let $k > 0$ be a constant, and consider the function

$$g(x) = k\cdot(1 - e^{-kx}) + x\ln(1+x^2) - x^2$$

(a) Calculate

$$\lim_{x\to+\infty} \frac{\ln(1+x^2)}{x}$$

and use this to show that

$$\lim_{x\to+\infty} g(x) = -\infty$$

(*Hint:* $x\ln(1+x^2) - x^2 = x^2\cdot\left[\frac{\ln(1+x^2)}{x} - 1\right]$.)

(b) Use part (a) and the fact that $g'(0) > 0 = g(0)$ to show that g has (i) some zero $x_0 > 0$, and (ii) some stationary point $\bar{x} \in (0, x_0)$.
(*Note:* You are not supposed to compute neither x_0 nor $\bar{x}$, but you *shall* show that $x_0 > 0$ and $\bar{x} \in (0, x_0)$.)

(c) The stationary point $\bar{x}$ of part (b) will be a function $h(k)$. Put $G(k) = g(h(k))$. Find an expression for $G'(k)$.

On the solution

(a) The first limit follows from l'Hôpital's rule to obtain 0. For the second,

$$g(x) = k \cdot \underbrace{(1 - e^{-kx})}_{\to 1} + x^2 \cdot \Big[\underbrace{\frac{\ln(1+x^2)}{x}}_{\to 0} - 1\Big]$$

so that the first term is bounded and the bracketed term tends to -1. The limit is as the limit of $-x^2$.

(b) (i) We have:

- $g'(0) > g(0) = 0$ implies that for some $x_1 > 0$, we have $g(x_1) > g(x_0) = 0$.
- The limit implies that for some $x_2 > 0$, we have $g(x_2) < 0$.
- The intermediate value theorem implies an intersection.

(ii) The extreme value theorem implies existence of a maximum. This cannot be at 0 or x_0, since $g(x_1) > 0$. Hence it is interior, hence it is stationary.

(c) By the envelope theorem,

$$\begin{aligned} G'(k) &= \frac{\partial}{\partial k}\Big(k \cdot (1 - e^{-kx}) + x\ln(1+x^2) - x^2\Big)\Big|_{x=\bar{x}} \\ &= 1 - e^{-k\bar{x}} + k\bar{x}e^{-k\bar{x}} \end{aligned}$$

Note: The envelope theorem was the last thing reviewed, on popular request, and a not too different problem was used as an example. Now the candidates are not required to apply the envelope theorem (instead they can apply the argument which leads to it), but those who do not, are supposed to simplify away the $g'(\bar{x})h'(k)$ term.

`Note added 2014-11-15: a ’ was missing from the original document, and has been corrected in this version for the purpose of future use.`