University of Oslo / Department of Economics **English**

ECON3120/4120 Mathematics 2

Tuesday December 11 2012, 09:00–12:00

There are 2 pages of problems to be solved.

All printed and written material may be used, as well as pocket calculators.

Grades given run from A (best) to E for passes, and F for fail.

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.

Problem 1 Define for each real number t the matrix $\mathbf{A}_t$ by

$$\mathbf{A}_t = \begin{pmatrix} 1 & 1 & t^2 \\ 3t & 2t+18 & 9t \\ 3 & 2 & 0 \end{pmatrix}$$

and the matrix $\mathbf{B}_t = e^t \mathbf{A}_t$.

(a) The determinants of $\mathbf{A}_t$ and $\mathbf{B}_t$ are of the form $|\mathbf{A}_t| = a(t) \cdot (1 - 6t)$, resp. $|\mathbf{B}_t| = b(t) \cdot (1 - 6t)$. Find $a(t)$ and $b(t)$.

(b) For which value(s) of t does the equation system (where $\mathbf{x}$ is the unknown)

$$\mathbf{A}_t\, \mathbf{x} = \begin{pmatrix} 6t - 1 \\ 12t - 2 \\ 18t - 3 \end{pmatrix}$$

have (i) unique solution, (ii) no solution, (iii) several solutions?

Problem 2 Evaluate the following integrals (hint: in both, you have to perform a substitution):

$$\text{(i)} \quad \int (x+1)^3\, e^{x^2+2x+1}\, \mathrm{d}x \qquad\qquad \text{(ii)} \quad \int_{e^e}^{e^{e^e}} \frac{\mathrm{d}y}{y \cdot \ln y \cdot \ln(\ln y)}$$

Problem 3 Define for $k > 0,\ \ell > 0$ the function

$$f(k,\ell) = \frac{1}{2}k^{1/2}\ell^{1/3} + \frac{1}{3}k^{1/3}\ell^{1/2} - (pk + q\ell)$$

Notice that the equation system (*) below, states the first-order condition for f to have stationary point at $(k,\ell) = (u,v)$.

(a) It is a fact (and you shall not prove) that the sum of concave functions, is concave. Show that f is concave.

The equation system

$$\begin{aligned} \tfrac{1}{4}u^{-1/2}v^{1/3} + \tfrac{1}{9}u^{-2/3}v^{1/2} &= p \\ \tfrac{1}{6}u^{1/2}v^{-2/3} + \tfrac{1}{6}u^{1/3}v^{-1/2} &= q \end{aligned} \tag{*}$$

defines u and v as continuously differentiable functions of p and q for $p > 0,\ q > 0$.
(You are not supposed to prove this.)

(b) Differentiate the equation system (*) (i.e., calculate differentials).

(c) Calculate $\partial u/\partial p$ at the point where $p = 13/18$, $q = 2/3$ and $u = v = 1/64$.

(d) f has a maximum for $(k,\ell) = (u,v)$. Approximately how much does $f(u,v)$ change if p is increased by $\Delta p = 1/125$?

Problem 4 Consider the Lagrange problem (L), and the nonlinear programming problem (N):

$$\max\ e^{x-1} + e^{y-2} + e^{z-3} \quad \text{subject to} \quad 15x^2 + 12y^2 + 10z^2 = 900 \tag{L}$$

$$\max\ e^{x-1} + e^{y-2} + e^{z-3} \quad \text{subject to} \quad \begin{cases} 15x^2 + 12y^2 + 10z^2 = 900 \\ x \geq 0, \quad y \geq 0, \quad z \geq 0 \end{cases} \tag{N}$$

(a) State the Lagrange conditions associated with problem (L), and show that they are satisfied for the point $(x,y,z) = (4,5,6)$.

(b) Does $(x,y,z) = (4,5,6)$ satisfy the Kuhn–Tucker conditions associated to problem (N)?

(c) For each of the problems (L) and (N): Is it clear that a solution exists?

(d) You can assume without proof that $(x,y,z) = (4,5,6)$ solves problem (L). If problem (L) is modified by replacing 900 by 898, how much, approximately, does the optimal value change?